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π in the Sky magazine is primarily aimed at high-school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

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Cover Page: The Silent Echo. One in a series of Hubble images of "light echo" images of the star V838 Monocerotis and its environs. In 2002, the red supergiant star at the middle of the image gave off a flashbulb-like pulse of light. As the halo of light expands, different parts of the surrounding dust are gradually illuminated unveiling never-before-seen patterns. Nature's own piece of performance art, this structure will continue to change its appearance for many years to come. The image is associated with the article *The Art of Physics: Visualizing the Universe, Seeing the Unseen*, by Anna Czolpinski and Arif Babul. Photo kindly provided by NASA, the Hubble Heritage Team (AURA/STScI) and ESA.

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The Emperor's New Art

When he died at age 97, Bertrand Russell had lived as full a life as any Earl could wish: married four times, elected to the Royal Society of London for his brilliant intellect, kicked out of Trinity College at Cambridge for his stubborn pacifism, twice jailed for anti-war activities, and (in between) awarded the Nobel Prize for literature. He had burst onto the world stage in 1901, when, like the child in *The Emperor's New Clothes*, he pointed to a simple paradox, now named after him, which brought the mightiest edifice of 19th century logic crashing down. As he recalls the sundry sources of his passion for truth, he writes:

At age eleven, I began Euclid, with my brother as my tutor. This was one of the greatest events of my life, as dazzling as first love. I had not imagined that there was anything as delicious in the world. From that moment until I was thirty-eight, mathematics was my chief interest and my chief source of happiness.

This is not the voice of a nerd, but of a man notorious for his many love affairs—which must have brought him *some* happiness before the age of thirty-eight. What could such a man, even in retrospect, find so incredibly "delicious" in Euclid's thirteen arid books? After all, the grand finale of this painstaking slog through nine dozen propositions in Books I to IV is nothing more glorious than the construction of a paltry pentagram (cf. the lower Figure (a) on p. 11). An edifying and useful figure, to be sure—but "*dazzling*"?

All creative people hate mathematics. It's the most uncreative subject you can study... said Sir Alec Issigonis*, who created the epoch-making Morris Minor, a car presumably built by rule of thumb. As the Third Earl Russell made no significant contribution to automobile design, Sir Alec naturally missed this potential counterexample, but he deserves full credit for boldly expressing a view that must be widespread, judging only by what goes on in schools. Honest folk rightly refuse to be rail-roaded into pretending to admire beauty where they see just plain fog.

Yet, what if there's something behind the fog? How would we know? Ask a well-placed source? Richard Feynman must have done *something* right to receive the Nobel Prize for Physics in 1965. Moreover, his popular books (e.g., *You must be joking, Mr.Feynman*) quickly show that he was no fawning hypocrite. However, when he declared that *to those who do not know mathematics it is difficult to get across a real feeling...[for] the deepest beauty of nature*, he could have been pulling his audience's legs, as he was known to do on occasion. The suspicion of such mischievous humour, however, would wither next to the sincerity of the eccentric Oxbridge don G.H. Hardy (seen by mathematicians as one of their finest), who wrote in 1941 that he was *interested in mathematics only as a creative art.*

Where, then, *is* this "creative art," ostensibly deep and delicious enough for an emperor—but as invisible, inaudible, and intangible as it is devoid of taste and odour? *It's in your head, stupid*, a former US president might have said, without the slightest intent of thereby declaring it unreal: any well-educated hedonist would know that all his pleasures (including artistic ones) must willy-nilly squeeze through that mysterious mollusc in his skull. Isn't it plausible that "turning it on" directly would be sheer bliss—for those who find the right vein, with fraternal help or otherwise?

One could imagine young Bertie spell-bound by the sheer cleverness of Euclid's arguments, his mind set ablaze by the crackling of his own synapses, as his brain tracked the ancient thinker's dazzling twists and turns. What art—worming its way through eye or ear—could do better?

"But it's *ugly!*" some will say, looking at the stodgy text and scrawny diagrams in Euclid's magnum opus. "It has no colour." Charlie Brown's pal Schroeder might assure them that his favourite musician, who wrote symphonies though he was stone deaf, found coloured scores rather distracting. If they objected that Beethoven wasn't *born* deaf, he would only sigh. Alright, allegories are never perfect. Lord Russell was guided by his brother instead of his own ears, and began with a "score" from 23 centuries ago, when mathematics was very young. An innocent look at a page of contemporary theorems is no doubt less likely to evoke feelings of "first love."

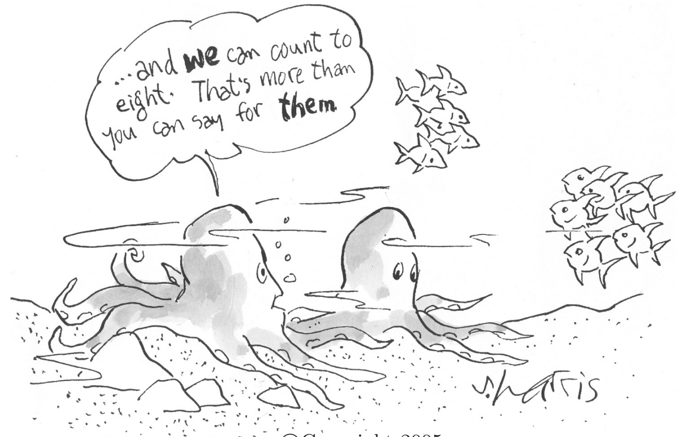
No wonder, then, that mathematics has a tiny audience and no star performers. Most of its adepts busy themselves as composers and explorers, restless seekers of that nugget that will make their day or year or life—often spending sleepless nights caused by a glimpse of fool's gold. Nevertheless this is their chief source of happiness: they cannot imagine anything more delicious in the world. Listen to Andrew Wiles, who in 1994 captured the elusive unicorn first sighted by Pierre de Fermat in 1637:

...the first seven years I had worked on this problem I loved every minute of it however hard it had been. [Then came a whole year in which he was stuck, apparently defeated, almost ready to throw in the towel, when] *...suddenly, totally unexpectedly, I had this incredible revelation. It was the most important moment of my working life. ...it was so indescribably beautiful, it was so simple and so elegant, and I just stared in disbelief for twenty minutes...*

A *revelation?* Isn't that the very antithesis of the rigour that is the back-bone of science? Sure, but remember that such revelation must withstand the most merciless interrogation by a famously fearsome rigour. The crucial role of mathematics in science has therefore never been questioned—and we shall come back to it another time. For now, let us give the last word to the late Richard Feynman. After offering various excuses for using mathematics in the study of nature, he writes: *But the real reason is that the subject is enjoyable, and although we humans cut nature up in different ways, and we have different courses in different departments, such compartmentalization is really artificial, and we should take our intellectual pleasures where we find them.*

We hope that the reader will find a few such pleasures in the following pages.

K.H.



* quoted in The Independent (London), July 26, 2005.

The Art of Physics: Visualizing the Universe, Seeing the Unseen

Anna Czolpinski† and Arif Babul*

In 1905, Albert Einstein penned three watershed articles that engendered a revolution in physics, and laid the foundations for Relativity and Quantum Mechanics.

In commemoration of this "Miraculous Year," the United Nations declared 2005 the International Year of Physics. Throughout the year, the worldwide physics community organized events to mark the pioneering contributions of Einstein, by highlighting the vitality of physics, and bringing the excitement of discovery to the public. At the University of Victoria, this physics-fest has been marked by lectures and a special exhibition, "The Art of Physics: Visualizing the Universe, Seeing the Unseen." (An online version can be explored at http://maltwood.uvic.ca/physics/.)

Organized jointly by particle physicist, Dr. Margret Fincke-Keeler, who studies the basic building blocks of matter and the forces that hold them together, and one of us (A. Babul), a cosmologist who studies the origin, evolution and ultimate fate of the Universe, this exhibition draws together a series of striking visual images and video installations from areas as diverse as stellar astronomy and medical physics. The images were contributed by scientists and institutions from around the world. The aim of the exhibition is two-fold: first to highlight the relatively unknown, though central, role of visualization in science and second, to draw attention to the deep connection between art and the aesthetics of scientific imagery. The images provide a rare glimpse into the arcana of the scientists' efforts to render the physical world comprehensible.

Of the 33 images, about half feature cosmic phenomena. Visual imagery has always been integral to astronomy. Early on, the images seen through telescopes were sketched on paper. Later photographic film was used, and nowadays, the images are recorded in digital format, allowing them to be easily manipulated. Many of the images shown are not as they would appear to the eye. Instead, they are comprised of different data digitally combined to provide insight about processes underlying the phenomena.

On the theoretical side, contemporary astrophysicists take advantage of powerful supercomputers to understand how the universe, having emerged from the fires of the Big Bang in an exceedingly smooth and homogeneous state, has evolved into today's richly structured system where galaxies trace out web-like chains woven about giant voids millions of light-years across. Astrophysicists use sophisticated image analysis and visualization tools to turn billions of bytes generated by the supercomputers into meaningful information.

Visualization is also central to particle physics. Particle physicists use it to make sense of the interactions between tiny ghost-like particles that are too small to be directly seen. In close analogy with woodsmen who can identify animals

Stellar Sprite In The Eagle Nebula: Appearing like a winged fairy-tale creature poised on a pedestal, this object is actually a billowing tower of cold gas and dust rising from a stellar nursery called the Eagle Nebula. A torrent of energy in the form of ultraviolet light from young stars is eroding the pillar, sculpting fantasy-like landscapes in the gas. The starlight is also responsible for illuminating the tower's rough surface. The column is silhouetted against the background glow of more distant gas. The colours in the image are artificial in that they have been chosen to enhance specific features of interest to astronomers and astrophysicists. Photo kindly provided by NASA/ESA, Space Telescope Science Institute, and the Hubble Heritage Team.

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Gargamelle Event: Bubbles forming in the wake of charged subatomic particles streaking through a CERN bubble chamber called "Gargamelle." The bubble chambers are filled with a superheated liquid. The wakes induced by the particles cause cavitation. The resulting lines of bubbles can then be photographed and analyzed. This image is the first observation of "neutral currents" in the Gargamelle chamber where a neutrino interacts with a nucleon and emerges as a neutrino. Photo kindly provided by CERN, Geneva, Switzerland.

by their tracks, particle physicists are able to deduce the presence of different particles and elucidate their properties by the "tracks" they create as they pass through sensitive detectors. The resulting images are invaluable jigsaw pieces in the grand puzzle of matter and energy.

The exhibition also includes visualizations of "atoms" in different arrangements from the world of solid-state physics. Until the invention of the scanning tunnelling microscope two decades ago, the very idea of trapping, imaging and moving about individual atoms was a "pipe dream." This scanner maps out corrugations on a surface due to individual atoms via a finely sharpened, atom-wide tip. The atoms are detected by tiny electrons that fly off the probe and "tunnel" into the electron shells of individual atoms. The resulting measurements are given a visual form using digital image processing. Today, these stunning images are considered contemporary scientific visual icons, in the same category as images taken by NASA's Hubble Space Telescope. They not only provide new insights that promise breakthrough technological applications but also present beautiful evidence of the validity of the theory of quantum mechanics.

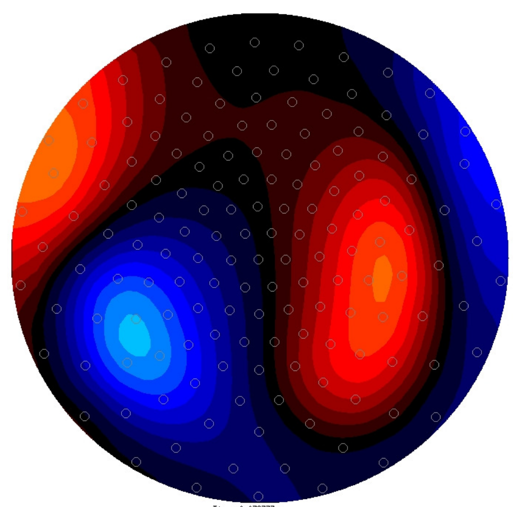
Finally, the exhibition hosts several instalments contributed by medical physicists. Medical physics—dating back to the discovery that X-rays could be used to view the insides of the human body—has been firmly rooted in the art and science of scientific imagery. And while X-ray imaging is still the basis for many diagnostic approaches, the replacement of the film by electronic detectors and the ability to manipulate the data digitally has led to a host of techniques, including the ability to generate three-dimensional views of the human organs without resorting to surgery. Moreover, sophisticated instruments, such as the functional MRI and the Magnetoencephalograph, are opening new vistas into previously inaccessible regions like the brain. It is now possible to collect remarkably accurate spatial and temporal information about neural activity in the brain in a non-invasive fashion, literally making it possible to image "thinking." The exhibition includes a video clip of brain activity associated with, appropriately enough, "seeing."

In short, the exhibition, with its array of visually stunning images and video installations, offers a unique opportunity to be awed by the beauty and harmony in areas of nature not normally associated with sense experience. Additionally, it also provides a glimpse into the world of scientific research, highlighting how heavily scientists rely on creative scientific visualization to uncover and understand the subtle mechanisms that underlie the workings of nature.

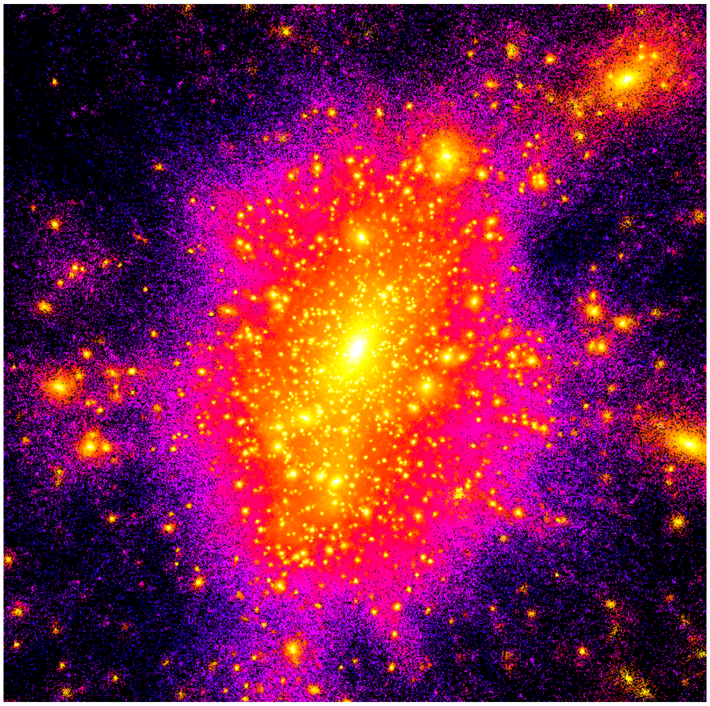
Scientific Imagery, Intuition, and Insight

Though often mistakenly taken to mean merely the design of presentation graphics, the phrase "scientific visualization" has a much broader definition. It is the art of transforming the abstract, be it in the form of reams of meticulous measurements or streams of computer-generated numbers, into geometric or symbolic representations. Visualization, to quote from the 1987 National Science Foundation (USA) panel report on Visualization in Scientific Computing, *"offers a method for seeing the unseen. It enriches the process of scientific discovery and fosters profound and unexpected insights."* Or, in the words of the theoretical chemist Primas, *"There is no insight without internal images!"* (Primas, as quoted by Euler 2001). Insight here, as Euler elaborates, refers to the ability to see a problem or a natural phenomenon clearly in one's mind, and understand its essence intuitively in spite of the fact that it cannot be directly perceived.

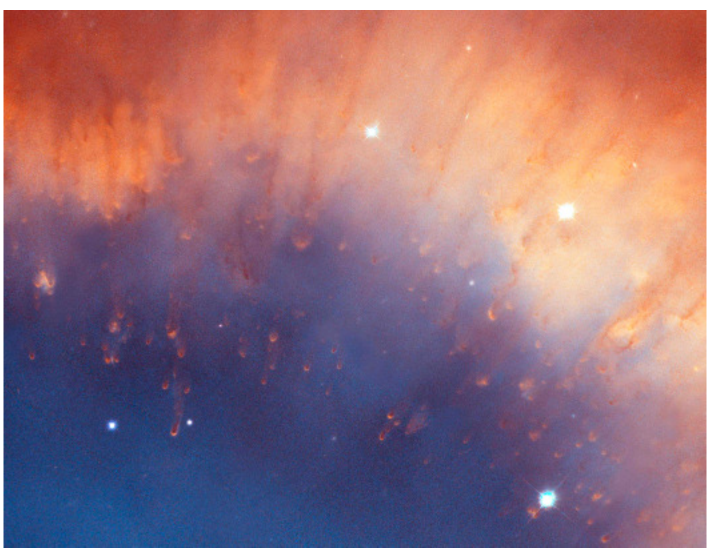
From this perspective, scientific imagery is especially important today as scientists probe levels of reality that cannot be directly accessed by human senses. Moreover, real processes in nature are complex, with the underlying organizing principles shrouded in a confusing cacophony of details. Through scientific imagery, scientists can de-emphasize, or even abstract away, the non-essential aspects of a phenomenon, so they can explore it more easily and ultimately "see" through to its essence.



Brain-seeing: A two-dimensional projection showing magnetic field patterns on the surface of a human brain during a task involving "seeing." This image was taken using a Magnetoencephalograph (MEG). As the brain takes in and processes inputs, waves of electrical impulses associated with nerve activity ripple about. These electrical impulses give rise to minute magnetic fields. The MEG utilizes exquisitely sensitive superconducting quantum interference devices, cooled to −269 degrees Celsius, to pick up these tiny magnetic fields literally through the skull and the scalp, allowing for non-invasive study of a live brain in action. Photo kindly provided by T. Cheung and N. Virji-Babul, Down Syndrome Research Foundation (DSRF).



Virtual Cluster of Galaxies: An image showing the distribution of "unseen" dark matter in a numerically simulated cluster of galaxies. Clusters of galaxies are among the most gravitationally bound systems in the universe. They are populated by swarms of hundreds of galaxies and are filled with very hot X-ray emitting gas. To understand how such structures have arisen, cosmologists use powerful supercomputers to simulate the evolution of the universe over its 14 billion year history. Photo kindly provided by T. R. Quinn, *N*-Body Shop, Astronomy Dept., U. Washington.



Cosmic Tadpoles: This impressionistic image details the outcome of collisions between streamers of gas in the Helix Nebula, the closest planetary nebula to the sun. Astronomers have dubbed the tadpole-like objects in these images "cometary knots" because their glowing heads and gossamer tails resemble comets, and there are thousands of such knots. Each gaseous head is at least twice the size of our solar system; each tail stretches 100 billion miles, about 1000 times the Earth's distance to the Sun. Photo kindly provided by NASA, ESA, STScI, NOAO, the Hubble Helix Nebula Team, M. Meixner (STScI), and T. A. Rector (NRAO).

Recent findings in neuroscience and psychology may explain why imagery plays such a fundamental role. The human brain processes visual information much more efficiently than textual, numerical or even diagrammatic data. It is primed for accepting visual inputs. It devotes a significant fraction of its resources to the processing of these inputs, transforming them into mental representations that allow for easy recognition of patterns and anomalies otherwise concealed in a jumble of numbers. It is especially fine-tuned for identifying the unexpected. Increasingly research into the role of visual imagery in science suggests that there is a close connection between the creation and manipulation of visual imagery, cognition, and "creative thinking."

The use of scientific imagery is a centuries old tradition. From Ptolemy to Tusi, Copernicus to Kepler, Newton to Feynman, imagery has been at the root of historical breakthroughs. Even Einstein, the man whose monumental insights of a century ago are the focus of this International Year of Physics, relied heavily on visuals. His biography and personal letters indicate that visualizations were the foundation of many of his ideas, including his greatest legacy: the Theory of General Relativity.

In the future, visualization will become even more important to the scientific endeavour. From a scientist's perspective, the march of progress—driven by the advance in technology—that has brought us to the current epoch of discovery and comprehension now threatens to overwhelm us with tsunamis of data. In the last two decades, the rate of scientific data generation has leapt from tens of megabits per day to just under a terabit per day, with no limits to growth in sight. This explosion reflects not only the improvement in the resolution of observations and numerical simulations, but also the increase in the dimensionality of the data. This colossal volume of data must be processed and catalogued. Most importantly, it must be explored, analyzed and understood.

Consequently, scientists are now compelled to transgress the imaginary boundary between the arts and sciences in order to foster transdisciplinary collaborations. Such collaborations, between scientists and visual artists—who have an intuitive understanding of colour, form, shape, and representation—will become increasingly vital in terms of giving complicated datasets meaningful visual form. Several astronomy departments in universities across North America have in-house visual artists. And many astronomy graduates have developed such strong skills in visualization that it is not uncommon for those who do not pursue the field professionally to be recruited by video and animation companies like DreamWorks.

Of course, while imagery and the design of creative representations of abstract phenomena is central to the scientific endeavour, it is only one part of the process. In the words of Primas, *"What is intuitively seen must be critically questioned and confirmed by rational reconstruction.... An adequate interplay between intuition and rational reconstruction is crucial not only for doing physics but also for learning physics."* (Primas, as quoted in Euler 2001).

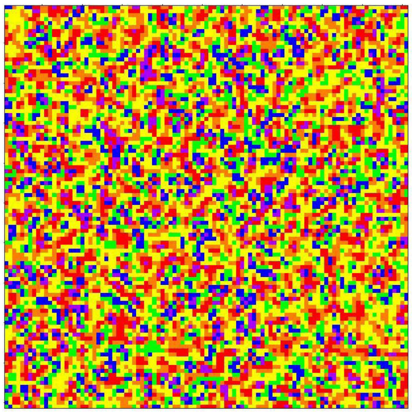
Scientific Imagery and Science Education

Given the fundamental role of scientific imagery, one would expect that the construction and manipulation of such imagery would be a crucial part of the science experience in the elementary and secondary years. This, unfortunately, does not appear to be so.

Various studies[1] have shown that today's science educa-

[1] OECD Programme for International Student Assessment [PISA]; Third International Mathematics and Science Study [TIMSS]; Connecting Research in Physics Education with Teacher Education: An International Commission on Physics Education (ICPE) Publication.

tion falls far short of teaching students how to look at nature with the passion of an explorer and how to make sense of what they discover. Instead, science education tends to focus on the transmission of established facts and principles, sometimes supplemented with simple mathematical exercises and demonstrative experiments. This goal is indeed attained, but at the expense of a much more important ideal of portraying science as a grand and dynamic, human endeavour to comprehend the natural world.



Noise II: Just like a television or radio, that emits static "noise," if not tuned to any station, a small amount of noise is always present in medical images. This picture depicts the intrinsic noise in a Computed Tomography (CT) scan image. Photo kindly provided by M. Hilts, BC Cancer Agency Vancouver Island Centre, and A. Jirasek, U. Victoria.

The purpose of teaching science ought to be to introduce students to the broad structures that gird scientific endeavour and to create opportunities for the students to experience the excitement of exploration and discovery that is at the root of science. Most importantly, the aim should be to teach the students how to convert their concrete observations into imagery that can be creatively manipulated to reveal the order and harmony underlying natural phenomena.

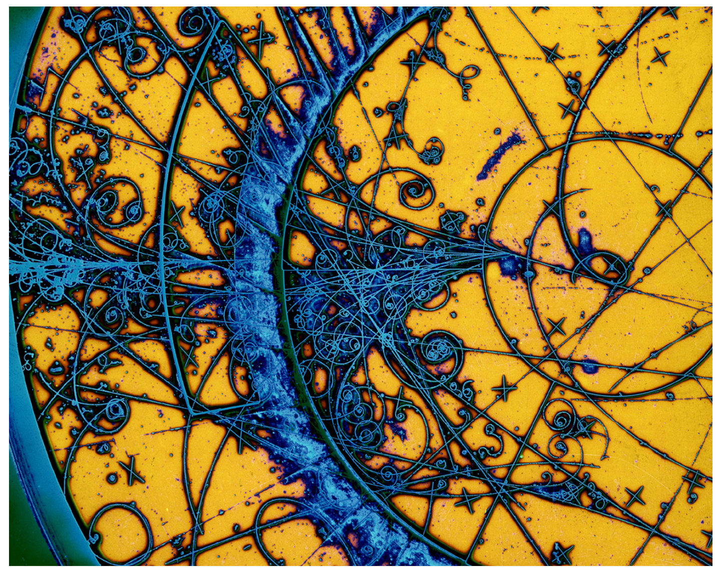
Both aspects are poignantly emphasized by Hirschbach, a Nobel Laureate in chemistry: *"In our science courses, the students typically have the impression—certainly in the elementary or beginning courses—that it's a question of mastering a body of knowledge that's all been developed by their ancestors.... Particularly...they get the impression that what matters is being right or wrong—in science above all.... I like to stress to my students that they're very much like the research scientists: that we don't know how to get the right answer; we're working in areas where we don't know what we're doing...I think any way we can encourage our students to see that, in science, it's not so important whether you are right or wrong...because the truth is going to wait for you."* (Hirschbach, as quoted in White and Gunstone 1998).

One consequence of limiting the teaching of science to the memorization of facts is that today's students are not able to operate between the concrete and the abstract with ease. They commonly confuse the symbols used to describe objects and the objects themselves. This hinders them from being able to translate their knowledge to different contexts and from using their knowledge creatively (Euler 2001). In other words, at the very time when our society is become increasingly knowledge-based, there is a growing concern that the present-day educational system does not provide for the level of scientific literacy and scientific skills necessary to meet the challenges of the future. Mechanically running through a series of prescribed problem solving steps does not engender insight and genuine understanding.

The studies mentioned previously have collectively identified a number of factors that are at the root of the problem. Many teachers have not had adequate exposure to science, and either lack the confidence to teach it or do not fully appreciate its very nature and goals (White and Gunstone 1998). Apprehensions and misperceptions have a direct impact on how teachers speak of science and the way they teach it.

Alternatively, teachers often cite the lack of easily accessible resources that would allow them to introduce science as an exploration. Today, the combination of easy access to computers, Internet connections that bring a growing number of online scientific archives within easy reach, and readily available data manipulation and imaging software, offers a unique opportunity to bring new dimension to science education.

Of course, technology, in and of itself, is not a panacea. The focus must be on teaching "formal thinking." The construction of symbolic descriptions, a process that is at the heart of the methodology of physics, is not a generic mode of mental activity. Euler (2001) argues that is the main stumbling block that makes the learning of physics a challenge.



Big European Bubble Chamber (BEBC)—Colour Treated Image: The European centre for subatomic research (CERN) often provides artists the opportunity to use the research environment as a stimulus for artistic endeavour. Depicted here is an artistically enhanced picture of particle tracks in the Big European Bubble Chamber (BEBC). Photo kindly provided by CERN, Geneva, Switzerland.

Visualization, however, is an exciting foil for introducing and incorporating formal thinking within science education. The ability to "see the data" and manipulate it visually carries an immediate appeal, a cache that tables and graphs simply do not have. Interactive visualization offers a unique opportunity to promote the active creation of mental images corresponding to the visual ones, to encourage the fostering of an intuitive understanding of the images, and to stimulate efforts at active mental transformations of these images to make "educated guesses" of what one would expect under different conditions. Inherent in the ability to experiment interactively with different visual renderings of data is the potential for seeing the data in new and unique ways. These are the very abilities that are critical for the successful doing of science.

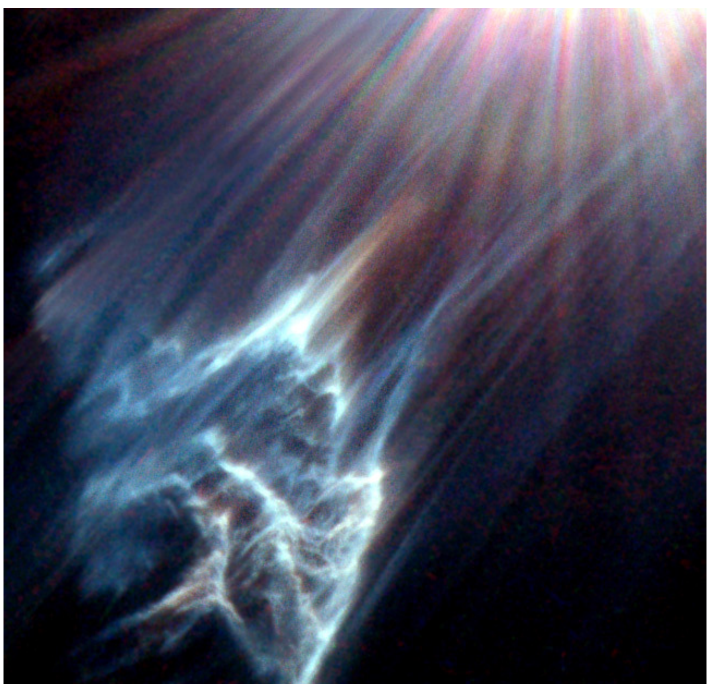
More generally, the above skills are essential not only for budding scientists, but are a prerequisite for any form of advanced abstract thinking, be it deconstructing Shakespeare, searching for patterns and predictability in the stock market, critically analyzing the historical terrain of a people or events, taking advantage of the digital revolution to choreograph powerful new visual art installations, or designing the next hit software or hardware application.

Insightful, okay! But is it really Art?

The discussion of insight and understanding aside, the exhibition has been a resounding success. The general reaction is best summarized by the following quote: *"I just saw the 'Art of Physics' exhibition. It was quite a powerful and intriguing experience. I was caught between responding to the beauty of the images without thinking about them as information data, on the one hand, while responding just as strongly, on the other hand, to learning about what was actually being represented."*

This is not to say that there weren't any dissenting voices. Of these, the typical challenge was *"Is this really art? After all, aren't the images just showing natural phenomena?"* Well, yes and no!

While it is true that the images shown at the exhibition have their origins in measurements, they are far from being simple straightforward depictions. Typically, the phenomena cannot be "seen" and even when they can, the "seen" often masks the more important "unseen." The scientists' task is to consider all the available properties—whether it is something visible or just measurable, whether it is an observable or a more abstract deduced quantity—and seek to represent these creatively using colour, forms and shapes in juxtaposition in order to tease out clues about the underlying phenomena. In seeking the most meaningful representation, each scientist is guided by both his/her own individual sense of the aesthetics as well as the understanding that the construction must be consistent with the general framework of science.



Ghostly Reflections: In this image, the Hubble Space Telescope has caught the play of light reflecting off the ripples and wispy tendrils extending from a pitch black cloud of cold interstellar gas laced with dust, much like moonbeams reflecting off gentle waves on a dark ocean surface at night. The source of the light is the star Merope just outside the frame on the upper right. The colourful rays of light at the upper right, pointing back to the star, are an optical phenomenon produced within the telescope, and are not real. However, the remarkable parallel wisps extending from lower left to upper right are real features. They were caused by ripples on the cloud surface when the star began to shred the cloud. Photo kindly provided by NASA/ESA, STScI, the Hubble Heritage Team, G. Herbig and T. Simon (U. Hawaii).

In the words of Michelle Miller, an abstract artist living in Victoria (BC), *"This is no different than how I teach and what I look for in abstract art. I have a basis of rules that exist.... For instance, if I have some large shapes on the canvas...everything that happens around those shapes will change the way those shapes look. Every brush stroke influences every other brush stroke. It becomes a chain reaction. You cannot clearly anticipate all of the variables. Sometimes you need to look and 'listen' to what the painting is saying to you. By this, I mean relinquish control and just try to understand by observing what happens. If you have...a visual grasp on when things 'work', then you're on your way to the creation of something incredible."*

The problem of the creation of imagery in the physical sciences is very similar to that faced by artists in their work. Attempting to find appropriate symbols to represent concrete objects and natural phenomena in the physical world is no different from the problems an artist faces in choosing signs and symbols, colours and shape, form and allegories to represent his/her internal world. Although the two disciplines of art and science speak different languages, they have a similar aim: the investigation and representation of the world in which they live. From this perspective, imagery of the physical sciences truly straddles the boundary between Science and Art. It seeks to give expression to 'what is there' and 'what it might mean.' It seeks to unveil the aesthetics of the physical world. One can argue that a scientist is a medium through whom nature makes her works known.

While artists attempt to decipher their place in the world viewed from the prism of their experiences, the scientists attempt to decipher the underlying order and harmony of the physical world from the prism of their limited perspective. Both approaches reveal previously hidden relations, and both are investigations into the nature of reality that defines humanity.

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[6] White, R. and R. Gunstone 1998: *Teacher's attitudes about physics classroom practice* in reference [5] above.

[2] Also see: Beaton, A.E., M. O. Martin, I. V. Mullis et al., 1996: *Science achievement in the middle school years: IEA's Third International Mathematics and Science Study (TIMSS)* and Beaton, A.E., I. V. Mullis et al., 1996: *Mathematics achievement in middle school years: IEA's Third International Mathematics and Science Study (TIMSS).* Both published by Chestnut Hill, Mass., TIMSS International Study Center, Boston College.

Mathematical Connections in Art

Reza Sarhangi[†]

Symmetry is a manifestation of structural harmony and transformations of geometric structures, and it lies at the very foundation of intuitive geometric reasoning. The manifestation of symmetry in nature is a recurrent, unifying theme in all areas of human endeavour, from art to nuclear physics.

Mathematics has periodically been employed not only to interpret and analyze art and architecture, but also to integrate directly with artistic products. There are periods in the civilizations of numerous cultures around the world in which artists have been fascinated by mathematics, encouraged and even forced to become mathematicians, as happened in antiquity, during the era of Islamic art, and in the Renaissance.

"The School of Athens" by Renaissance painter Raphael.

During the European Renaissance, art, mathematics, architecture, science, and music flourished side by side. This is no longer the case, and although many artists and scientists are calling for ways to regain the lost mutual understanding, appreciation and exchange, it has been hard to know how to create environments in which this can happen in a meaningful way.

The Bridges Conferences

No less a divide exists between mathematics and the general public. Human beings are fluent in recognizing and appreciating patterns, and are able to deal effortlessly with the abstractions of language, music, visual art, and theatre. Yet most people think that they have a latent aversion to mathematics and are largely unaware of how deeply embedded it is in the world around them. Still, we have seen over and over again how fascinated and excited people become when mathematical connections are presented in ways that relate to their experiences and trigger their natural curiosities and aesthetic sensibilities.

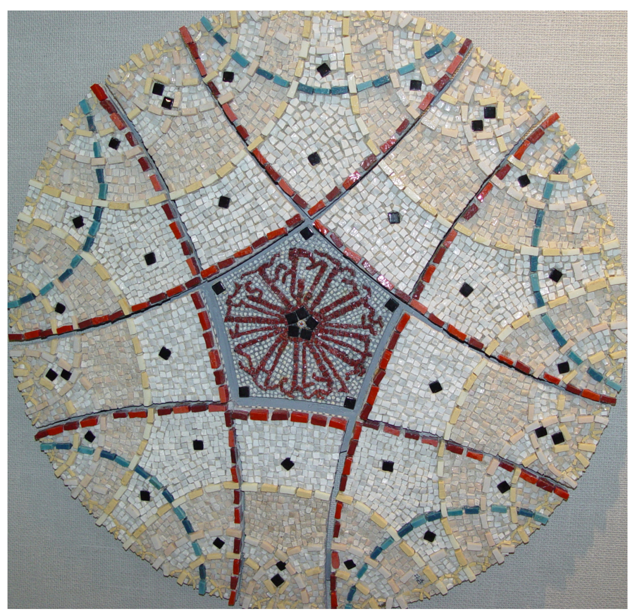
The Bridges Conferences, created in 1998 and running annually since, have provided a remarkable model of how these divides can be crossed. Here practicing mathematicians, scientists, artists, educators, musicians, writers, computer scientists, sculptors, dancers, weavers, and model builders have come together in a lively and highly charged atmosphere of mutual exchange and encouragement. Important components of these conferences, apart from formal presentations, are gallery displays of visual art, working sessions with practitioners and artists who are crossing the mathematics-arts boundaries, and musical or theatrical events. Furthermore, a lasting record of each Bridges Conference is its Proceedings—a beautiful resource book of the papers and the visual presentations of the meeting.

The 2005 Bridges conference was held at Banff International Research Station (BIRS) and the Banff Centre. It was co-sponsored by PIMS, the Banff Centre, and the Canadian Mathematical Society. The four-day workshop/conference was titled "Renaissance Banff," to indicate its innovation. It was the first time a mathematics/arts event of this magnitude has been brought to Canada, and in particular to the western Canadian community.

The Renaissance Banff conference consisted of two parts: A three-day Bridges Conference, and a Coxeter Day. H. S. M. (Donald) Coxeter was one of the foremost geometers of the 20th century. His work and writing not only played a significant role in mathematics, but also touched innumerable people in the arts and other areas of science.

A tribute to the late Donald Coxeter from the exhibition at the Renaissance Banff Conference.

A Selection of Images from the 2005 Bridges Visual Art Exhibit



"Hyperbolic Diminution–White," by Irene Rousseau (artist living in New Jersey). This mosaic is inspired by hyperbolic tessellations using the Poincaré disk.

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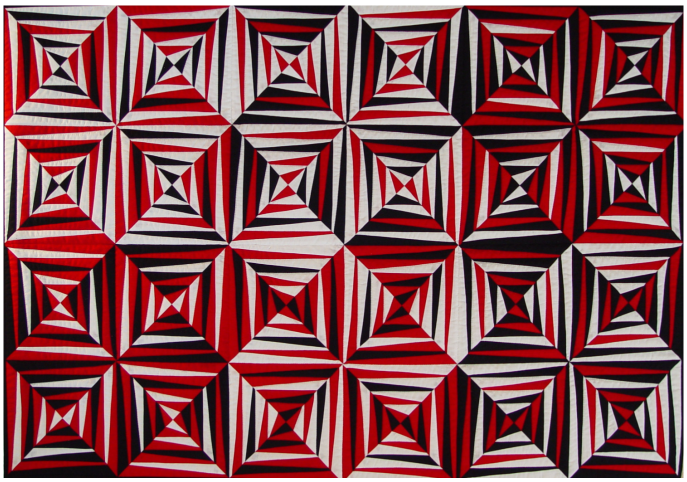
Wood sculpture of a trefoil knot by Susan Greene (sculptor and retired research chemist living in Virginia).

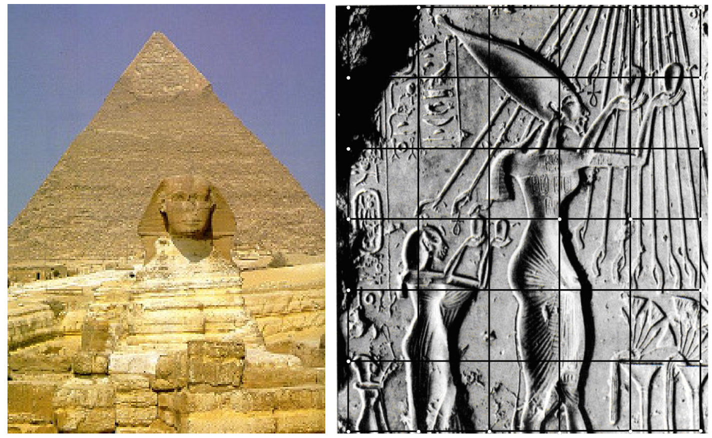


“Cyclic Permutations,” a quilt by Gerda de Vries (Professor of Mathematics, University of Alberta). This quilt explores permutations of the colouring of a geometric design based on right isosceles triangles.

“Looking for the Order” by Dick Termes (artist living in South Dakota). Termes’ painted spheres make use of six-point perspective. This work pays homage to Albert Einstein. The sphere demonstrates a statement by Einstein that if you could look far enough in one direction, you would see the back of your own head.

Mathematical Connections in Art

For students and educators who are interested in mathematical connections provided by different cultures in different time periods, we present some examples here.



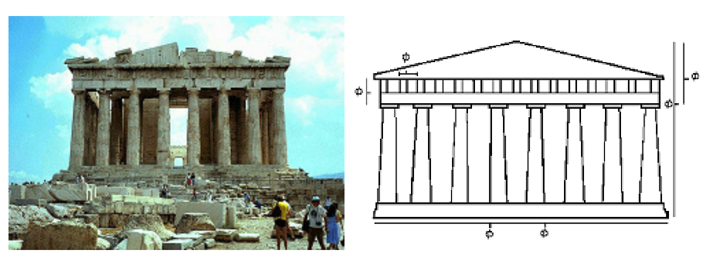
An Egyptian pyramid (left) and grid (right).

In addition to the masterpieces of mathematics and art in ancient Egypt, the pyramids, we can mention another use of mathematics in art: Egyptians have been credited as the first to employ grids for replications and enlargements of their artworks. A grid consists of a system of equally spaced parallel and perpendicular lines that yields a convenient framework for spatial organization.

Perhaps the most prominent, identifiable individual found in the history of mathematics in Western civilization is Pythagoras. He was born on the Greek island of Samos, off the coast of what is now Turkey, around the year 580 BC. It is believed that he coined the words “mathematics” and “philosophy.” “Mathematics” means “that which is learned” and “philosopher” means a person who loves knowledge. Pythagoras has also been credited as the first person to investigate connections between numbers and musical sounds. He established the first system of music, which is called the *Pythagorean Diatonic Scale*, based on rational numbers that are created by 1, 2, 3 and 4, and their multiplications.

One of the most famous piece of architecture in the ancient Western world is the *Parthenon*. This temple is located on the highest part of the acropolis in Athens, Greece. It was built during the Golden Age of Greece (450–400 BC) under the aegis of Pericles. One of the artists responsible for the aesthetics of the work was Phidias.

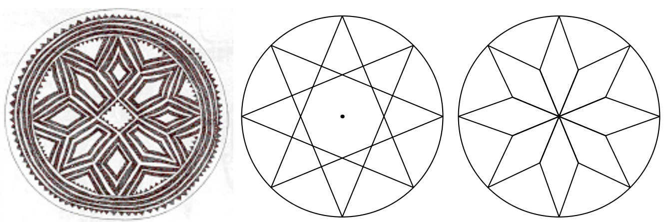
The analysis of the façade of the Parthenon reveals the recurrence of a number of proportions, which are derived based on compass and straight edge, such as the Golden Ratio $\phi = (1+\sqrt{5})/2$, $\sqrt{2}$, and $\theta = 1+\sqrt{2}$.



The Parthenon Temple in Athens (left) and the analysis of the façade (right).

On a different note, the decoration in (a) of the following figure is on a calabash from the Hona tribe in Nigeria, Africa.

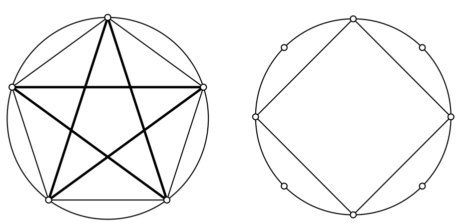
This design has been constructed based on a (8, 3) star polygon (b)—a star based on dividing a circle into 8 equal arcs and joining a vertex to another one with 3 arcs distance and continuing the process to meet all the vertices in a single stroke—and then shows the underlying star of the decoration (c).



(a) (b) (c)

The decoration from a figure on a calabash from the Hona tribe in Nigeria (a). The basis of the design is shown in (b) and (c).

Star polygons have been used by various cultures around the world. The star polygon based on a regular pentagon, referred to as a *pentagram*, was selected as the sacred symbol of the Pythagorean Society (figure (a) below). There is a relationship between the regular pentagon and the Golden Ratio. It is also interesting to know that this ratio appears many times when we compare certain segments of a pentagram. This star polygon can be expressed as (5, 2). We notice that it can be expressed as (5, 3) as well. The reason for it is if we begin at a point from five equally spaced points on a circle and go around this circle in one direction and join every second point by a segment, then the result will be the same as if we joined every third point in the opposite direction. In general then we can conclude that an $(n,\ k)$ star polygon is the same as an $(n,\ n-k)$ star polygon.



(a) (b)

A pentagram star polygon based on a regular pentagon (a). A $(4, 1)$ star polygon is shown in (b). Note that (b) is also a $(4, 3)$ star polygon.

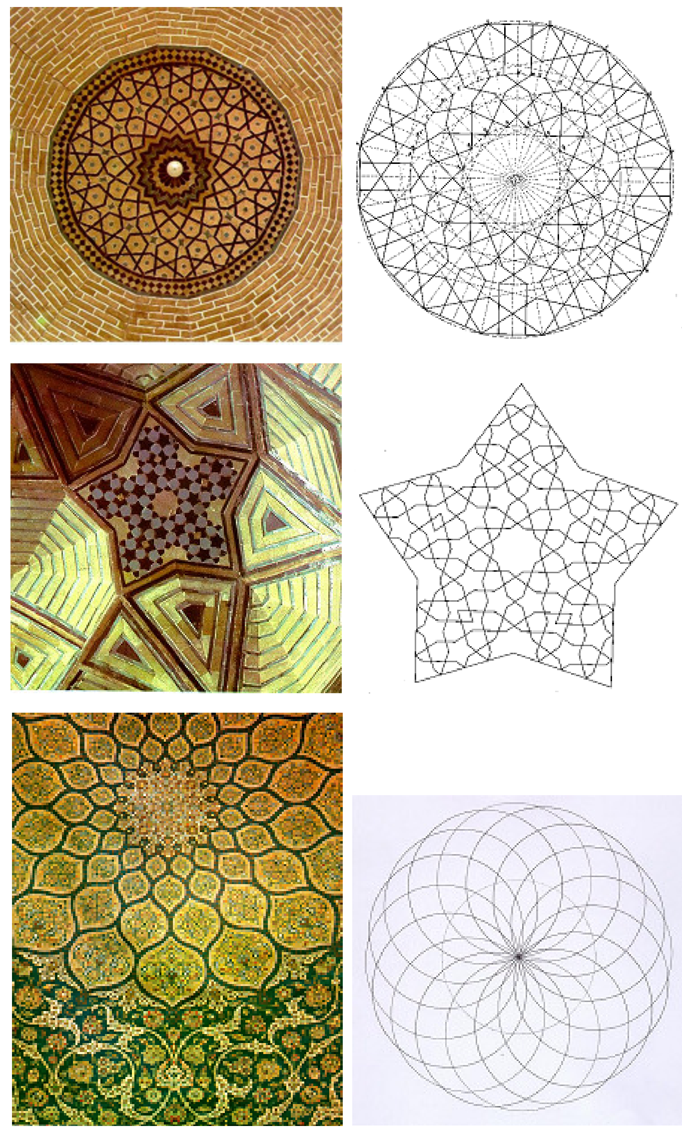
Let us try to construct all possible star polygons that can be constructed based on a set of eight equally spaced points on a circle, i.e., to construct all $(8,\ k)$ star polygons, $1 \leq k \leq 8$.

We notice that (8, 1)—which is the same as (8, 7)—is a regular polygon with 8 sides, an octagon, and not a star polygon. Now, if we begin at a vertex and go around the circle and join every second point by a segment we only join four points in a single stroke and the process will stop before joining the rest of the points (b). So an (8, 2)—as well as an (8, 6)—star polygon does not exist. We have already constructed an (8, 3) star polygon, which is the same as an (8, 5) star polygon. For constructing an (8, 4) star polygon, we will have the same problem as we experienced with the (8, 2) case. Therefore, for the case of eight equally spaced points on a circle, we will only have one star polygon, the (8, 3) star polygon.

Numbers 8 and 2 have a common divisor other than 1. The same is true for 8 and 4. However, the only common divisor for 8 and 3 is 1. In the case that 1 is the only common divisor of two numbers, they are called *relatively prime* numbers. An example of a pair of relatively prime numbers is 15 and 16 (yet neither 15 nor 16 are prime!).

The following theorem generalizes what we discovered for the specific case of an $(8,\ k)$ star polygon, $1 \leq k \leq 8$.

Theorem: Let n be a number of equally spaced points on a circle. Begin at any point and go around in one direction, joining every kth point. An (n, k) star polygon joining all vertices exists if and only if $k \neq 1$, $k \neq (n-1)$, and n and k are relatively prime.



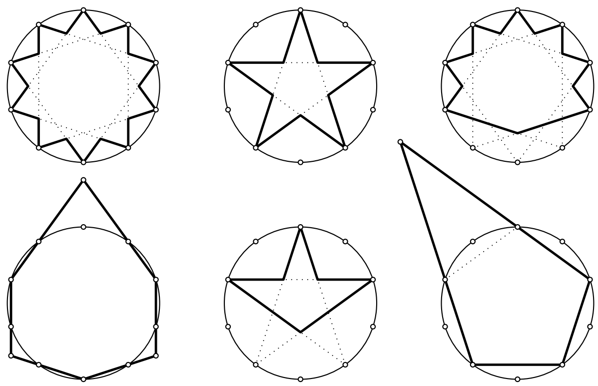
Mosaic patterns ornamenting monuments from the Medieval Islamic World. The designs are based on the division of a circle and constructing regular polygons.

Mosaic patterns ornamenting monuments from the Medieval Islamic World period bear witness to the predominance of geometry in Islamic art. These designs and patterns were normally gathered by stucco makers and other artist-constructors, who would pass them along to the next generation. The designs were graphed on a scroll. Ink pens were used for major lines, however, all circles were sketched with a compass without lead. Both end points of the compass were sharp metal. The metal etched barely visible grids onto the scroll. Then using a straight edge, they drew the design with ink. What follows are some Islamic Art designs based on the division of a circle and regular polygon constructions.

The figure on the left illustrates a Persian ceramic design, which includes carpet-like details made from solidly coloured, small, curved tiles. The division of space in the layout creates geometrical pieces. The construction of the design is based on the pentagon (and thus the decagon as well).

The figure below illustrates the geometrical construction of the pieces in the previous figure.

A Persian ceramic design.

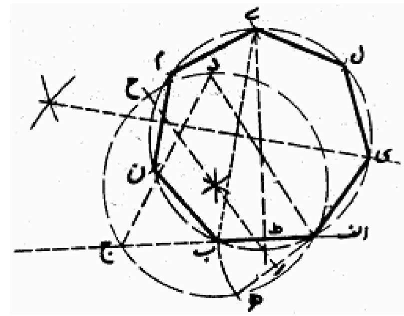


The geometrical constructions of pieces in the Persian ceramic design shown above.

When we study Islamic Art we notice that we rarely see designs that incorporate regular heptagon or regular nonagons (One such an exception is the following design, which is based on the regular heptagon). The reason for it may very well be related to the idea of constructible regular polygons.

Ancient mathematicians discovered how to construct regular polygons of 3, 4, 5, 6, 8, and 10 sides using only a compass and straight edge. The list of other constructible regular polygons known to them included 15-gons and any polygon with twice the sides as a given constructible polygon. Despite their efforts, mathematicians, until 1796, were not successful in constructing a regular heptagon by compass and straight edge or proving that such a construction is impossible. After a period of more than 2000 years, Gauss, as a young student, proved its impossibility. In fact, he proved that in general, a construction of a regular polygon having an odd number of sides is possible when, and only when, that number is either a prime Fermat number, a prime of the form $2^k + 1$, where $k = 2^n$, or is made up by multiplying together different Fermat primes. Such a construction is not possible for 7 or 9. Gauss, at first showed that a regular 17-gon is constructible, and after a short period he completely solved the problem. It was this discovery, announced on June 1, 1796, but made on March 30th, which induced the young man to choose mathematics instead of philology as his life's work. He requested that a regular 17-sided polygon be engraved on his tombstone.

The geometric construction to create an approximation for a regular heptagon (shown on the right) is by Buzjani (Iran, born 940 A.D., died 997/998 A.D.). He was given the title *Mohandes*, which meant "the most skillful and knowledgeable professional geometer" by the mathematicians, scientists, and artisans of his time. The design below is also by Buzjani.

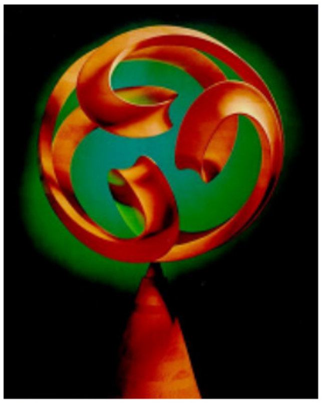


A geometric construction for a regular heptagon created by Abul Wafa al-Buzjani.

A star based on a regular heptagon constructed in a dome interior of a medieval Persian building.

Michele Emmer, a professor of mathematics at the University of Rome, a 2000 Bridges Conference speaker, and one of the first in the world in recent years to call for a gathering of mathematicians and artists under one roof, writes: *"Renaissance painters turned to mathematics not only because they had the problem of depicting the natural world realistically on canvas, of producing scenes in three dimensions with depth, but also, as Morris Kline has pointed out in his important book on mathematics in western culture, they were profoundly influenced by the rediscovery of Greek philosophy. They were wholly convinced that mathematics was the true essence of the physical world and that the universe was ordered and explainable in geometric terms. This great interest forced Renaissance painters to become—as Kline defined them—the best applied mathematicians of the period. Since the professional mathematicians of that time did not have the geometric instruments that the artists needed, they themselves also had to become learned and active theoretical mathematicians."*

Artists and designers around the world have used and are using ideas from mathematics to express themselves and advance their arts: Hyperbolic geometry, the four dimensional cube and hypercube, fractals, tessellation, Möbius bands, solids, and minimal surfaces are only a few that are employed by artists/mathematicians in today's art.





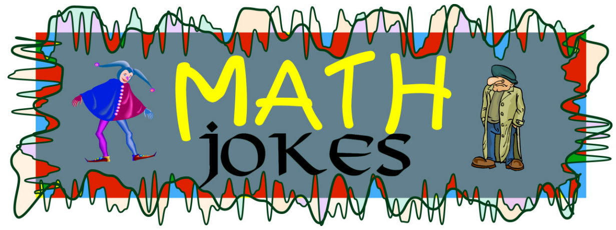

As one can see from the images above, artists and designers around the world have used and are using ideas from mathematics to express themselves and advance their arts.

M. C. Escher, perhaps the most famous visual mathematics artist in today's world, writes: *"The ideas that are basic to them [mathematicians] often bear witness to my amazement and wonder at the laws of nature which operate in the world around us. He who wonders discovers that this is in itself a wonder. By keenly confronting the enigmas that surround us, and by considering and analyzing the observations that I had made, I ended up in the domain of mathematics. Although I am absolutely without training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow artists."*

The above statement by Escher summarizes perfectly the privileged relationship that an artist may establish with the scientific community.

The International Conference of Bridges: *Mathematical Connection in Art, Music, and Science* is an annual conference. For information about this conference you may visit the Bridges website at `www.sckans.edu/~bridges`.

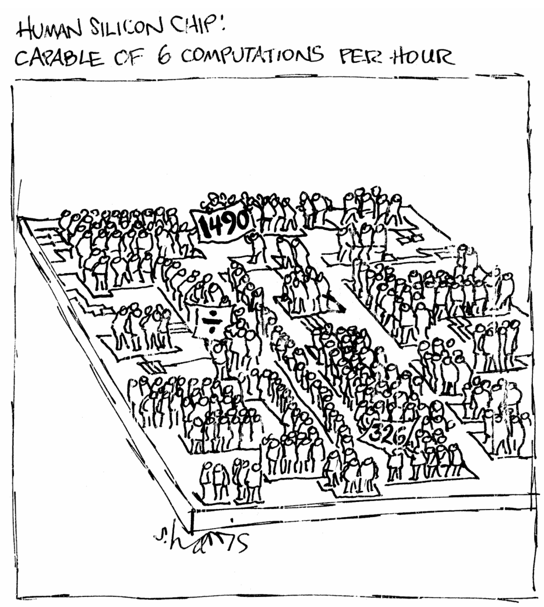
PIMS has copies of the 2005 Bridges Proceedings available for members of academic departments at the PIMS universities, and students and teachers in Alberta or British Columbia. If you would like a complimentary copy please contact pi@pims.math.ca. Postage costs will apply. For one copy of the proceedings the postage costs are approximately $10 within Canada and $17 to the USA.



Theorem: A cat has nine tails.
Proof: No cat has eight tails. Since one cat has one more tail than no cat, it must have nine tails.

Q: What is non-orientable and lives in the ocean?
A: Möbius Dick...

"Students nowadays are so clueless," the math professor complains to a colleague. "Yesterday, a student came to my office hours and wanted to know if General Calculus was a Roman war hero..."



Symmetry and Order in Turbulence

Bruce R. Sutherland[†]

The study of fluids is pervasive in such scientific disciplines as mathematics, physics, chemistry, engineering and medicine. To name but a few examples, fluid dynamics researchers might examine methods for extracting oil from Alberta's tar sands and transporting it through pipes, or they might study how medicine is distributed though the body's cardiovascular system, or they may try to predict how the cold waters in the Equatorial Pacific will affect Canada's weather during a winter La Niña. Although the equations describing the motion of fluids were derived two centuries ago, exact solutions have been found for only a few special cases. The challenge of finding exact or approximate solutions have continually pushed the frontiers of mathematics, most recently through devising efficient and reliable computer codes and through the development of new fields of mathematics, including chaos theory and pattern formation, about which many popular science books have been written.

Much of my work examines mixing and waves in fluids with varying density. Such fluids are said to be "stratified" because they act as if they are composed of slabs of fluid layered one on top of the other. Oceans, lakes and the atmosphere are stratified fluids. (Indeed, the stratosphere gets its name because its density decreases relatively rapidly with altitude.) The air in the room where you are sitting is a stratified fluid: hot, less dense air floats near the ceiling and cooler air is closer to the floor.

I have been drawn to study fluid dynamics not only because its applications are of such practical importance, but because of the intuition and breadth of knowledge required by the discipline. Furthermore, because of the remarkable growth in the speed and memory size of computers, many fundamental problems in fluid dynamics that were previously unsolvable can now be modelled numerically and studied in laboratory experiments using lasers and digital image processing.

The following describes an experiment that can be done in the kitchen and that demonstrates some of the beauty and surprising complexity of stratified fluid motion. You will need the following:

- 9" x 11" glass baking dish (or similarly large glass dish)
- a 4 cup measurer or bowl of at least this volume
- 8" x 8" piece aluminum foil
- $\frac{1}{2}$ cup sugar
- food colouring (two colours)
- water

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Figure 1: What you need to make a stratified fluid.

Figure 2: After filling the bottom half of the pan with sugar water, pour tap water, dyed red, into the aluminum boat. This will inhibit mixing and most of the red water will end up floating on top of the sugar water solution.

Figure 3: Put in a few drops of blue food colouring and watch the patch evolve into spirals.

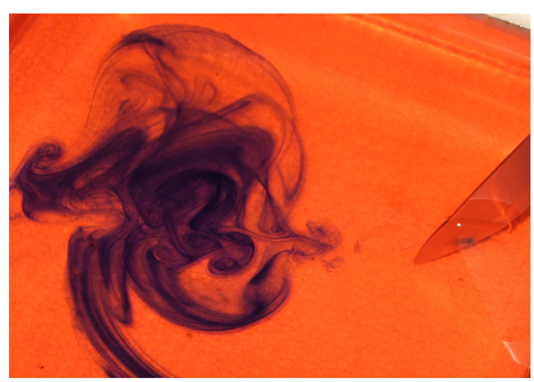

Figure 4: Dragging a knife through the patch makes more complex patterns of vortices.

To make a stratified fluid, add 4 cups of water and 1/2 cup of sugar to the glass dish. (You may wish to put the dish on a white cloth to observe the fluid motions more easily later on.) Mix these together to form a strong sugar solution (the density of the solution should be about $1.1\,\mathrm{g\,cm^{-3}}$, compared with fresh water, which has a density of about $1.0\,\mathrm{g\,cm^{-3}}$). Now measure out another four cups of water and add a light coloured dye to it (four drops of red food colouring should be enough). We want to layer this dyed fresh water on top of the dense sugar water. The following is a crude but effective way to do this. Make a "boat" from the aluminum foil with a flat bottom and sides as high as the sugar water in the baking dish (about 1 cm). Float the boat on the sugar water and slowly pour in the dyed water. The boat will lower into the sugar water and the dyed water will eventually overflow spilling over the sugar water. You will notice that as the dyed water spills out, it floats over the sugar water. Continue to pour all four cups of the dyed water into the overflowing boat, pouring at a rate so that it takes about a minute to do this. When you are done, you should still be able to see some clear, undyed sugar water at the bottom of the dish.

Congratulations! You have made a stratified fluid!

At this point it is easy to see an astounding property of stratified fluids: add a single drop of dark coloured dye (such as green or blue) to the centre of the dish and observe what happens. If you do this within a few minutes of making the stratified fluid, it is likely you will see the dye stretched out into a spiral-like vortex looking not unlike a nebula in various Star Trek movies. A coherent slowly swirling vortex such as this typically does not occur in unstratified, homogeneous fluids. To see this, just add a drop of dye to fresh water in a bowl or another baking dish. You will likely find that the dye in this case gets pulled into filaments of ever finer structure in a motion that is typically chaotic and progressively less predictable.

To emphasize further the difference between the behaviours of stratified and unstratified fluids, take a knife and slowly drag it through both fluids creating a wake about 1 mm wide. In both cases, you will see small-scale turbulence in the knife's wake. But what happens over time as the turbulence decays? In the unstratified fluid, mixing occurs near the wake and the resulting motion dies down after a minute or so. In the stratified fluid, from the small-scale mixing emerge large, slow-moving vortices that grow in size as they combine with other vortices and that continue to evolve for many minutes.

The collapse and decay of turbulence in a stratified fluid involves many complicated processes that are the subject of active research today. How might a mathematician approach this problem? The first step is to write the exact equations of fluid motion appropriate to this problem. Although it requires an understanding of calculus to make any sense, the equations describing the motion of sugar water are given below (with an English translation of their meaning in red below them):

$\rho \frac{D}{Dt}\vec{u}$	$=$	$-\nabla p$	$-\rho g\hat{z}$	$+\nu\nabla^2\vec{u}.$
A fluid moves	because of	pressure changes,	buoyancy forces acting downward,	and viscosity slowing it down.

Here ρ is the density, $\vec{u}$ the velocity, and p the pressure, all three of which are functions of space and time. The constant g is the acceleration due to gravity and ν is the kinematic viscosity (which is a measure of friction within a fluid). The symbols $\frac{D}{Dt}$, ∇ and ∇^2 are convenient notations involving derivatives, which are used to describe infinitesimal changes in time and space. Similar equations also exist describing how the density changes in time.

The equations describing the motion of unstratified, fresh water are the same as those for stratified water but with the $\rho g\hat{z}$ term removed.

Although, with experience, it is a relatively simple matter to write down the equations, at present they cannot be solved to describe the turbulent motions in the above experiment. Indeed, they may never be solved; one would be hard pressed to think of a function that could encompass such complexity of evolution in time and space.

Nonetheless, mathematical progress has been made. From experiments it was realized that unstratified (homogeneous) turbulence exhibits a special kind of symmetry, which today we describe as being fractal: a close up view of turbulence looks almost identical to turbulence seen from farther away. For example, the turbulent plume formed by pouring cream in your coffee is similar in many respects to the turbulent plume from a chimney or an exploding volcano. Using scaling theory, scientists have been able to estimate how quickly energy is dissipated and how fast pollutants are mixed in turbulence.

Stratified turbulence is much more difficult to model in this way, however. Mathematically, this is due to the presence of the $\rho g\hat{z}$ term in the equations of motion for stratified fluids, which represents buoyancy forces acting vertically to carry relatively heavy fluid downward and light fluid upward. Physically it means that fluid, loosely speaking, "prefers" to move horizontally when it is stratified. You can see this in the experiments. When the knife is dragged through the stratified fluid, vertical motions are suppressed in its turbulent wake and only horizontal motions persist. Effectively, the motion evolves from one that is three dimensional (moving horizontally and vertically), to one that can be thought of as two dimensional (moving strictly horizontally). Although the transition from three-dimensional to two-dimensional motion is not yet well understood, scaling theory can be applied to "two-dimensional turbulence" to predict that large-scale, slowly evolving vortices should develop, as observed.

Progress is being made in understanding turbulence with the aid of computer models. To this end, scientists are reformulating equations, like that above, into a form that can be calculated numerically. Such methods can only approximate the exact solution because one must ultimately impose a restriction on the smallest sizes of motion that can be resolved by the computer. Computers with greater speed and memory are providing ever more accurate solutions that are only now capable of reproducing some of the observations in laboratory experiments. It is not unreasonable to hope that with improvement in computers and laboratory measurements, further mathematical breakthroughs are just around the corner.

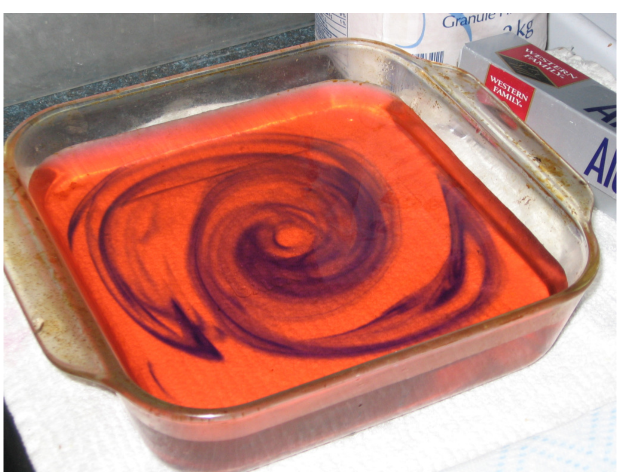

Figure 5: But if you leave the fluid undisturbed for a long time, it will form one large spiral. This spiral took 5 minutes to form.

Finding Meteorites with Mathematics

Jeremy Tatum†

Everyone knows that the equation

$$ax + by + c = 0 \tag{1}$$

represents a straight line. But what if you have a three-dimensional problem? Presumably a straight line in three dimensions is represented by the equation

$$ax + by + cz + d = 0. \tag{2}$$

Wrong! Equation (2) is the equation, in three dimensions, of a *plane*. To represent a *line* in three dimensions, we need *two* equations of the form of Equation (2); a line is the intersection of two planes and we need to give the equations of two planes to specify it.

Three non-collinear points ought to define a plane, so, if we know the coordinates of three points, (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), how can we determine the equation of the plane containing these three points? Well, we can write down the equations

$$ax_1 + by_1 + cz_1 + d = 0, \tag{3}$$

$$ax_2 + by_2 + cz_2 + d = 0, \tag{4}$$

and

$$ax_3 + by_3 + cz_3 + d = 0. \tag{5}$$

That doesn't seem to be enough to solve for the four unknowns, a, b, c and d. If I want a condition for a point (x, y, z) to be in the plane, I combine Equation (2) with Equations (3), (4), and (5) to give me four equations in the three unknowns,

$$\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

This system has a solution if and only if the matrix has a zero determinant, that is,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \tag{7}$$

and (7) is the equation of the plane containing the three points.

Having disposed of that little bit of mathematics, we can now get down to the topic announced in the title of this article.

Until recently, I served on the Meteorites and Impacts Advisory Committee to the Canadian Space Agency. One of our duties was to investigate reports of *fireballs* and to see if we could find an associated meteorite. Most of the faint *shooting stars* or *meteors* that we can see in the sky on any night are tiny particles of dust of cometary origin. Occasionally, however, a much more spectacular phenomenon is reported. A huge ball of light streaks across the sky, illuminating the countryside for hundreds of miles around, perhaps accompanied by thunderous noise, attracting widespread public attention or even alarm. This is a *fireball*, and it is a large chunk of stone or iron of asteroidal origin. While in orbit around the Sun, it was a *meteoroid*. If any of it survives the fiery plunge through the atmosphere, the specimen that reaches the ground is a *meteorite*. Figure 1 shows a fragment of the Canyon Diablo meteorite that fell in Arizona about 50,000 years ago. The speed of the fireball is several tens of kilometres per second and its path through the atmosphere, which lasts for just a few seconds, is nearly a straight line. I wondered if I could use a little mathematics to help track the fireball through the atmosphere.

I found that talking to eyewitnesses by telephone was interesting, but did not produce much in the way of quantitative information. So, I bought myself a compass and a clinometer (the latter measures angular height above the horizon) and I decided that the best way to proceed was to interview each witness *in situ*, within a few days at most from the event. You ask the witness to re-enact exactly what he or she was doing when the fireball appeared and to point to two points on its track through the sky. The directions to these two points together with the geographical position of the witness are sufficient to define a plane that contains the path of the fireball. Then you visit another witness, maybe 50 km or so away, and ask him or her to indicate the directions to two points. This gives a second plane, and, where the planes intersect is the path of the fireball.

One thing that investigators commonly find is that witnesses very commonly believe that the object they saw was only a few hundred yards away and many of them will swear that they saw it land in the next field. In reality, the object is several tens or even hundreds of kilometres away.

Here's how the geometry works. In the first place I assume a Flat Earth. This is not because I am a member of the Flat Earth Society. It is justified (within the limited precision of eyewitness accounts) by the circumstances that the height of a witness above sea level is very much smaller than the height of the fireball above the ground, and the height of the fireball above the ground is very much smaller than the radius of Earth. I set up a rectangular coordinate system with the origin at some arbitrary point on Earth (usually for convenience

Figure 1: A fragment of the Canyon Diablo iron meteorite from the author's collection.

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a little bit southwest of all witnesses), with the x axis pointing east, the y axis north, and the z axis straight up. My southwest choice of origin means that all witnesses are in the first quadrant, so I don't have to deal with minus signs!

For each witness I record his or her x and y coordinates, and I measure the spherical coordinates θ and ϕ of two points on the sky track. These angles are, respectively, the angular distance of a point from the zenith, and the azimuth or bearing, measured counterclockwise from the x axis (east) in the usual manner for spherical coordinates. If you refer to Figure 2 you will see that the fireball is in a plane containing the following three points:

$$W = (x_0, y_0, 0), \tag{8}$$

$$A = (x_0 + r_1 \sin\theta_1 \cos\phi_1, y_0 + r_1 \sin\theta_1 \sin\phi_1, r_1 \cos\theta_1), \tag{9}$$

and

$$B = (x_0 + r_2 \sin\theta_2 \cos\phi_2, y_0 + r_2 \sin\theta_2 \sin\phi_2, r_2 \cos\theta_2). \tag{10}$$

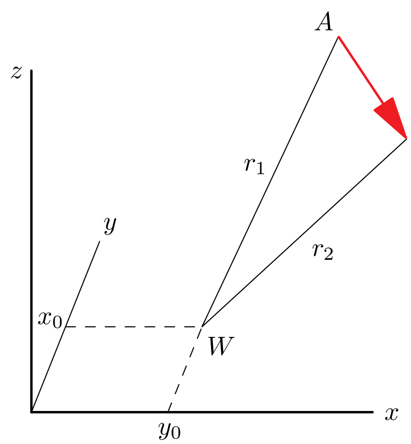

Figure 2: A witness at W sees a fireball go from A to B. The spherical coordinate angles θ and ϕ are measured for both points. What is the equation of the plane WAB?

Of course we don't know either of the distances r_1 or r_2, but if you use Equation (7) to find the equation of the plane containing the path of the fireball, you'll find (as you would expect) that it doesn't depend on these distances.

Then, as mentioned, we interview a second witness, and define a second plane. These two planes define the path of the fireball, which in practice is essentially a straight line. In practice we have many witnesses, all inconsistent with each other, and we have to handle that as best we can!

Perhaps a numerical example is in order. I give below the coordinates (in kilometres) of two hypothetical observers and the spherical coordinates (in degrees) of two points on the sky track as seen by each of the witnesses:

x_0	y_0	θ_1	ϕ_1	θ_2	ϕ_2
15	5	25.5	54.5	36.7	16.7
30	15	29.5	202.9	33.6	242.9

Perhaps you can use these data to calculate the two planes whose intersection gives you the atmospheric trajectory of the fireball. I obtain

$$0.1260x + 0.3162y - 0.1577z - 3.4712 = 0 \tag{11}$$

and

$$0.2683x + 0.1598y + 0.1757z - 10.4445 = 0. \tag{12}$$

These two planes, then, define the path—but can you visualize it? You will be able to visualize it if

(a) you can calculate the point where the path intersects the ground;

(b) you can determine the *ground track*, which is the vertical projection of the atmospheric track on the ground;

(c) you can determine the angle that the path makes with the vertical.

For (a), just put $z = 0$ in each of the equations. This will give you two equations in x and y. Solve for these and they give you the coordinates of the *extrapolated ground-level point*. Can you perform this calculation? I compute $(42.4, -6.0)$; that is, 42.4 km east and 6.0 km south of your origin. Usually the meteorite will fall somewhat short of this point.

For (b), eliminate z between the two equations. This will give you a single equation in x and y, and this is the equation of the ground track. I find

$$y = -0.79796x + 35.02; \tag{13}$$

the meteorite lies somewhere along this line, and a little before the extrapolated ground-level point.

For (c), you already know the coordinates of one point (the extrapolated ground level point) on the path. Calculate the coordinates of any other point on the path, and this will enable you to determine the angle it makes with the vertical. This I leave to you.

So far I haven't managed the ultimate goal of finding the prize—a meteorite—from the relatively crude angle estimates made by eyewitnesses, although four meteorites have been recovered from the much more precise measurements that can be made from photographic records. A photograph allows very precise measurements to be made with a microscope, but we then have to be much less cavalier with the mathematics. A Flat Earth approximation just won't do! Figures 3 and 4 show two photographs, obtained by Barry Burgess and Michael Boschat on November 19, 2002, from two sites in Nova Scotia about 45 km apart. You can see that the starry background is different in the two photographs. From measurements we were able to calculate that the height of the

Figure 3: A meteor photographed from Nova Scotia by Barry Burgess.

Figure 4: The same meteor as is shown in Figure 3. The point of observation was 45 km away from that of Figure 3. Photograph taken by Michael Boschat.

meteor was 112.16 km when it was first detected, with an error of only 20 metres.

Another interesting aspect to this work is that the flight of a fireball through the atmosphere is often accompanied by thunderous noise. The fireball is usually so high in the atmosphere that the sound may take several minutes to reach the ground, and an eyewitness may not always associate the noise with the fireball that was seen. What is exciting is that it has been recognized in recent years that sound from a fireball can be detected by seismographs, and these can record the exact time of arrival of the sound signal, thereby giving scope for more mathematics. Figure 5 shows a seismic record of a fireball.

When a meteoroid streaks through the atmosphere at many times the speed of sound, it generates a conical shock front of very small (often less than a degree) semivertical angle, and this shock front can be recorded on seismographs. During this stage of the flight through the atmosphere, the surface of the meteorite becomes extremely hot, and much of the surface vaporizes. The flight through the atmosphere during this supersonic phase lasts only a few seconds, and there isn't much time for the heat to penetrate to the interior. Because of this, a tremendous temperature gradient and consequent thermal stress may be set up in the stone, and it may suddenly disintegrate in a violent, explosive *terminal burst*. This generates another shock front (initially spherical), which can also be detected by seismographs. If there is no violent terminal burst (iron meteorites are stronger than stony meteorites), a substantial chunk may survive. It will slow down to a speed low enough that its glow can no longer been seen (this probably happens while its speed is still supersonic, though it will eventually reach subsonic speed), and it may subsequently fall to Earth as a relatively cold stone. (We often get reports of meteorites landing and starting a fire, but it is very doubtful whether this ever happens!) This slow fall may take a few minutes, and the path is no longer a straight line, which is why it will fall short of the extrapolated ground-level point. The impact may also be heard by seismographs. In that case, the sound travels through the ground, much faster than through the air, so that a seismograph may record the impact first, and the atmospheric shock fronts later, which can be confusing. A meteorite typically has a thin black fusion crust to indicate how the surface (but not the interior) has been subject to great heat.

The entire seismic phenomena can be quite complicated because of the three separate events, so, for the purpose of this article, let's keep it simple and concentrate just on the *terminal burst*, which we regard as a point source of sound somewhere in the atmosphere. It generates a spherical shock front, which is heard at a number of seismographic stations. If it is recorded at four stations, it should be possible to find the position (x_0, y_0, z_0) and the time t_0 of the explosion. This is not too hard, because at time t the radius of the spherical shock front will be $v(t - t_0)$, where v is the speed of sound. The equation of the spherical shock front at time t is therefore just

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = v^2(t - t_0)^2. \qquad (14)$$

If you know the positions (x, y, z) of four stations and the arrival times t at each of them, you can set up four equations like this, and solve them for (x_0, y_0, z_0, t_0). The four equations are each quadratic in the four unknowns, but, provided you know how to solve four simultaneous quadratic equations in four unknowns (!), there will be no difficulty. (If there are more than four stations, we have to perform a least squares solution, a technique from statistics.)

An example is in order. Let's suppose that the coordinates of the four stations in kilometres and the arrival times in seconds are

x	y	z	t
15	4	0.1	92
5	38	0.3	81
36	20	0.4	82
20	33	0.5	59

Suppose that the speed of sound is 0.33 km s^{-1}. Can you set up the four equations and then solve them? Quite a challenge! I make out the answer to be $x_0 = 18.50$ km, $y_0 = 26.06$ km, $z_0 = 14.45$ km, and $t_0 = 11.55$ s.

That was relatively painless, because we made the assumption that the temperature of the atmosphere (and hence the speed of sound) is the same at all heights, and consequently sound travels in straight lines. But this is far from the case in the real atmosphere, and sound does not travel in straight lines. I therefore looked in several textbooks, and they told me that the path of a sound wave in the atmosphere is an arc of a circle. This was promising information, but I needed to know exactly how big a circle, and where the centre was, so I needed to try and prove for myself that the path is a circle. This took me a little while, but I eventually managed it by making the assumption that the speed of sound in the atmosphere decreases linearly with height. Then indeed, sound rays are arcs of circles.

I should have been pleased with this—but in fact I was puzzled. I knew that in the lower 11 km or so of the atmosphere—the part known as the *troposphere*—the *temperature*, to a good approximation falls off linearly with height at a rate of about 6.5 °C/km, which is called the *temperature lapse rate*. (Above 11 km, in the *stratosphere*, the lapse rate changes.) Since the speed of sound depends on the square root of the temperature, the sound speed also falls off as the square root of the height, not linearly. After struggling with that for a while, I found that the path of a

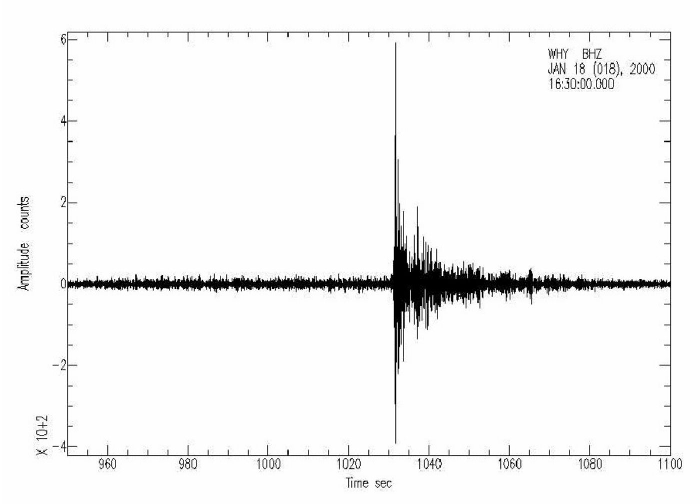

Figure 5: Seismogram of the terminal burst of a fireball. This was obtained at the Whitehorse (Yukon) seismic station on 18th January, 2000. The height of the terminal burst was about 37 km, and the seismic station was distance about 30 km from the sub-burst point. A meteorite from this fireball was subsequently recovered from the frozen surface of Tagish Lake in the extreme northwest corner of British Columbia, and turned out to be one of the most primitive meteorites known, very little changed from the early stages of the formation of the solar system. The seismogram was kindly supplied by the Geological Survey of Canada, courtesy of Dr. John Cassidy, Pacific Geoscience Centre.

sound wave in the atmosphere is not along an arc of a circle at all—it is along the arc of a curve I remember studying in math class long ago, namely a *cycloid*! So, apparently you can't necessarily believe everything you read in books all of the time—not even scientific ones!

However, the geometry of the cycloid, while great fun, is not quite so easy as that of a circle, so, in order to prevent this article from becoming too complicated, let's go back to the supposition that the sound speed falls off linearly with height. This will not change any major conclusions, though the details may not be quite accurate. Specifically, we'll suppose that the sound speed v at height z is given by

$$v = v_0 - kz. \tag{15}$$

Here v_0 is the sound speed at ground level, and k is a constant that shows how fast the sound speed decreases with height. What I found was that, if a sound wave at any given level makes an angle ψ with the horizontal, it subsequently moves along the path

$$\left(x - \frac{v_0 \tan\psi}{k}\right)^2 + \left(z - \frac{v_0}{k}\right)^2 = \left(\frac{v_0 \sec\psi}{k}\right)^2. \tag{16}$$

You will probably recognize this as a circle, and you can probably say what its radius is, and where the centre is. If we express the horizontal distance x and height z in units of v_0/k, the equation looks a little easier:

$$(x - \tan\psi)^2 + (z - 1)^2 = \sec^2\psi. \tag{17}$$

Now let us imagine that a terminal burst takes place when a meteoroid is at a horizontal distance $x = 1.5$ from some origin on the ground, and at a height $z = 0.5$ above ground level. Sound is emitted in all directions, and, in Figure 6 we see the paths of several sound rays at different starting angles ψ_0. The continuous curves are for $\psi_0 = 30°, 40°, 50°, 60°, 70°$ and $80°$. The dashed path is for $\psi_0 = 48.2°$ and it is seen that it just scrapes the ground at $x = 0.38$. Anyone to the left of this position—i.e. anyone more distant than 1.12 from the sub-burst point—will not hear the burst. This is not because he or she is too far from the explosion, but because the sound never reaches the ground; it moves instead in a circular arc.

Of course calculating the atmospheric trajectory from the signal arrival times now becomes much more complicated if the speed of sound varies with height and if the sound moves in arcs of circles, or, worse, of cycloids, but the principles are the same, and you just have to fill a few more sheets of paper with equations and tear out a few more handfuls of hair. We haven't yet actually recovered a meteorite from seismograph records, but, mark my words—we will, we will!

In a previous article in this magazine I showed how mathematics is useful if you are interested in moths. This time I have shown that mathematics is useful if your interest is meteorites. It seems that, however obscure or esoteric one's interests, mathematics seems always to have a role to play.

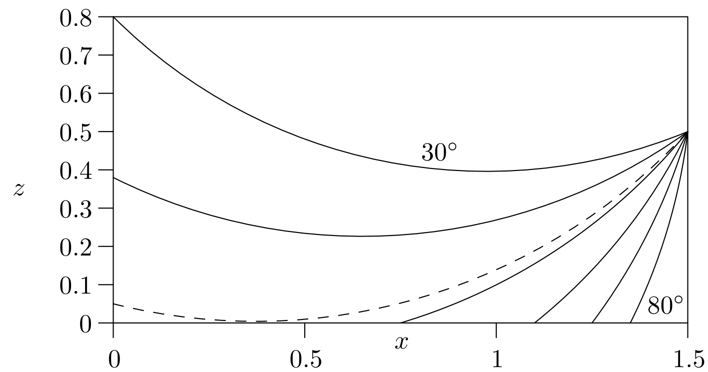

Figure 6: Sound from an explosion at the right of the figure travels in circular paths and never reaches the ground to the left of $x = 0.38$.

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Much of the material in this article has been adapted from a number of previously-published technical papers by the author, as follows.

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Symmetry of the Modified Mandelbrot Set

Valerij Rozouvan†

The Mandelbrot set [1] is the best known fractal; it is defined as the set of all points C_0 in the complex plane, such that the infinite sequence $C_0, C_1, \ldots, C_n, \ldots$ remains bounded, where:

$$C_{n+1} = C_n^2 + C_0 \quad \text{for } n = 0, 1, 2, 3, \ldots \tag{1}$$

The pictures of the Mandelbrot set in this article were created using software written by the author. The Mandelbrot set is shown in Figure 1.

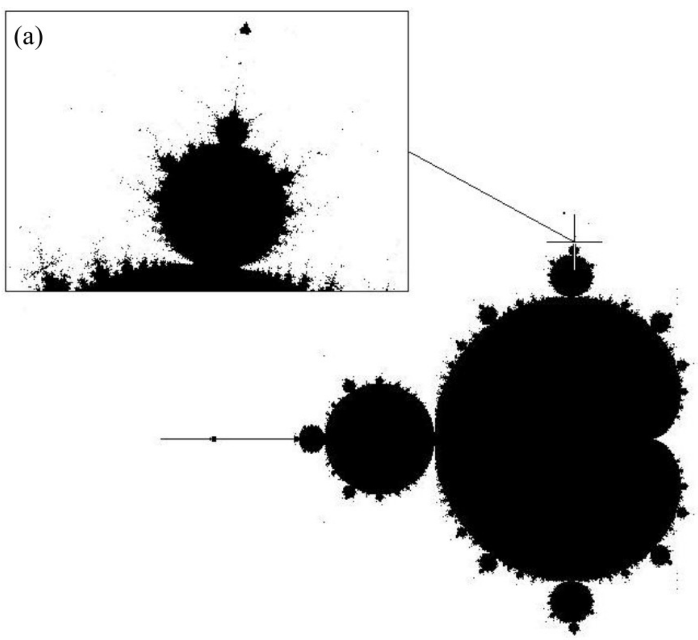

Figure 1: The Mandelbrot set. Inset (a) is centred around $C \approx -0.1078 + 0.8966i$ with a magnification of ≈ 25.

This set has a very complicated theory that shows the Mandelbrot set has a fractal character and actually contains small copies of itself. The complete theory of the Mandelbrot set is very sophisticated and is based on the behaviour of polynomials in the complex plane [2]. The goal of this work is to study the symmetry of sets that are produced by slightly modifying the Mandelbrot set definition. In particular, the modified Mandelbrot set can be defined by changing the recursive definition above to:

$$C_{n+1} = C_n^a + bC_0, \quad a \in \mathbb{N}, b \in \mathbb{C}. \tag{2}$$

Here, a is a natural number, and b is a complex number. Let us study two particular cases of Equation 2.

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Case 1. $b = 1$

In our first case, we keep $b = 1$ (as it is in the basic Mandelbrot set), and vary the exponent a. The results for $a = 4$, and $a = 14$ are shown in Figure 2 and Figure 3.

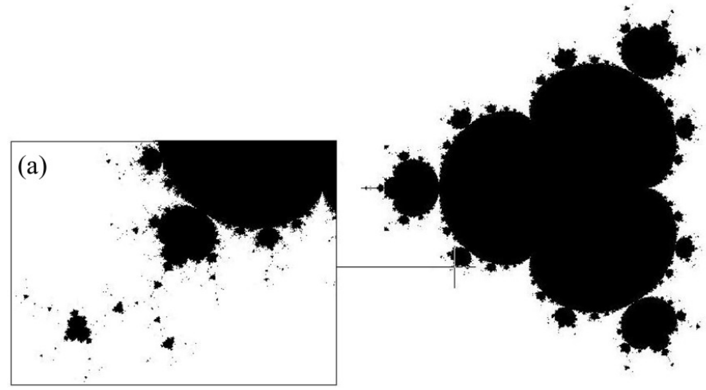

Figure 2: The modified Mandelbrot set with $a = 4$, $b = 1$. The area in (a) is centred around $C \approx -0.696 + 0.4805i$ with a magnification of ≈ 50.

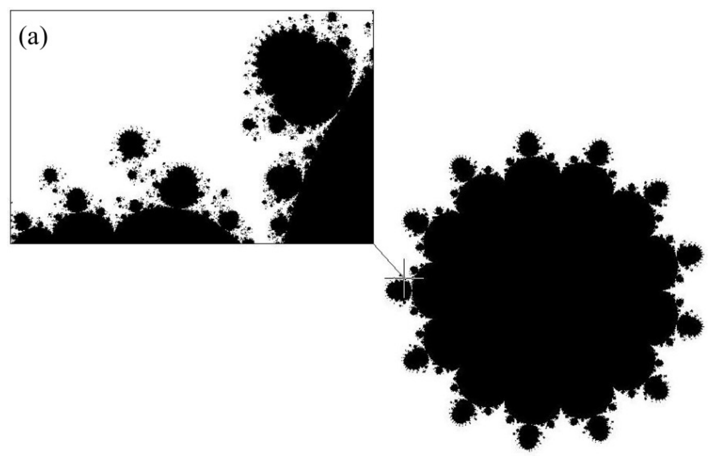

Figure 3: The modified Mandelbrot set with $a = 14$, $b = 1$. The blown up area (a) is centred around $C \approx -0.9294 + 0.0976i$ with a magnification of ≈ 25.

In each case we seem to have symmetry about any angle

$$\varphi = \frac{2\pi k}{a-1}, \quad k \in \mathbb{N}. \tag{3}$$

Thus when $a = 4$ we appear to have rotational symmetry through any multiple of $2\pi/3$, and when $a = 14$, we seem to get symmetry through rotation by multiples of $2\pi/13$. In what follows we shall prove this. A complex number defined [3] as $x + iy$ (where $i^2 = -1$, and x and y are real numbers) can be represented, due to Euler's identity [4], as $r(\cos\theta + i\sin\theta)$ or $r\exp(i\theta)$ (r is the length of the complex vector on the complex plane where the vertical axis represents imaginary parts of complex numbers and the horizontal axis represents real parts of complex numbers). The multiplication of a complex vector C and a complex vector $\exp(i\theta)$ can be represented as rotation of vector C counterclockwise about the origin by an angle of θ.

To prove the symmetry of the modified set, we assume that C_0 is in the set, and that K_0 is formed by rotating C_0 about the origin by one of the angles φ given in (3). That is,

$$K_0 = C_0 e^{i\varphi}. \tag{4}$$

We will show that the infinite sequence $K_0, K_1, K_2, \ldots$ where

$$K_{n+1} = K_n^a + K_0 \tag{5}$$

is bounded and that K_0 is thus in the modified set. This will prove the symmetry of the modified set. Because of (4) above, we can rewrite $K_1 = K_0^a + K_0$ as:

$$K_1 = C_0^a e^{ai\varphi} + C_0 e^{i\varphi}. \tag{6}$$

Since

$$a\varphi = 2\pi k \frac{a}{a-1} = 2\pi k \left(1 + \frac{1}{a-1}\right) = 2\pi k + \varphi, \tag{7}$$

a rotation by $a\varphi$ is equivalent to a rotation by φ, and we can simplify (6) to get

$$\begin{aligned} K_1 &= C_0^a e^{i(2\pi k+\varphi)} + C_0 e^{i\varphi} \\ &= C_0^a e^{i\varphi} + C_0 e^{i\varphi} \\ &= (C_0^a + C_0) e^{i\varphi} \\ &= C_1 e^{i\varphi}. \end{aligned} \tag{8}$$

This shows that K_1 is just a rotation of C_1 about the origin by an angle of φ. It is easy to extend this argument to show that

$$K_n = C_n e^{i\varphi} \quad \text{for } n = 1, 2, 3, \ldots \tag{9}$$

and thus that every K_n is a rotation of C_n about the origin by an angle of φ. This implies that the sequence $K_0, K_1, K_2, \ldots$ is bounded and that therefore the modified set has rotational symmetry about any angle φ.

Case 2. $b = \exp(i\varphi)$

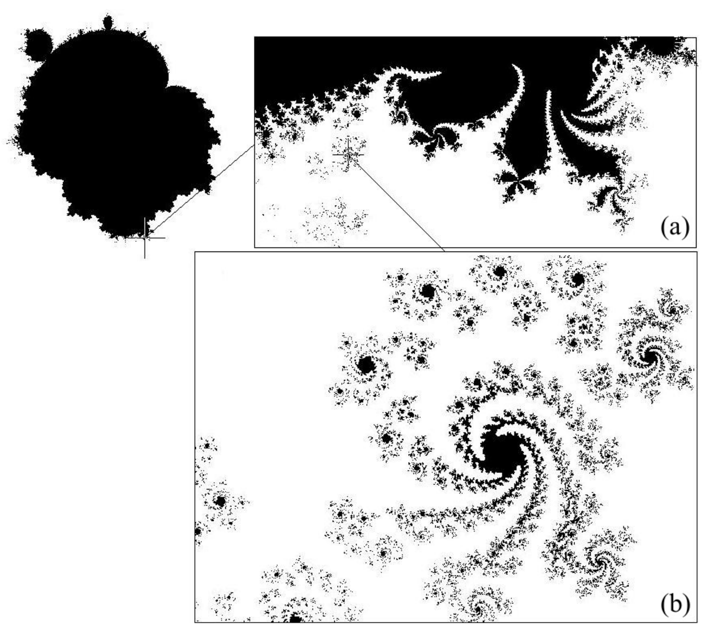

Figure 4: The Modified Mandelbrot set with $a = 2$, $b = \exp(i\pi/4)$. Area (a) is centred around $C \approx 0.0694 - 0.7748i$ with a magnification of ≈ 25. Area (b) is centred around $C \approx -0.00782 - 0.74128i$ with a magnification of ≈ 250.

The example of the calculation with $a = 2$ and $\varphi = \pi/4$ is presented in Figure 4. The set here is rotated and distorted, compared with the original Mandelbrot fractal. One can see that the angle of the rotation depends on the argument φ of the complex number b. The rotation angle of the set in Figure 4 is determined by b and is equal to $\pi/4$. One can also see that smaller parts of the set demonstrate higher order rotation symmetry. For example, the spiral structures in Figure 4(b) have rotational symmetry about the angle $2\pi/5$.

The modified Mandelbrot set, proposed in this work, demonstrates interesting properties of rotational symmetry. One can easily construct a fractal set with a desired rotational symmetry by changing the parameter a in Equation 2. This approach also suggests another means of analyzing the original Mandelbrot set: changing the parameter b in the equation. For example one can choose a b parameter slightly different from 1, and by observation of the rotational distortions in different areas of the Mandelbrot set, one may identify powers in the expanded members of the polynomial that determine the Mandelbrot set shape in those areas.

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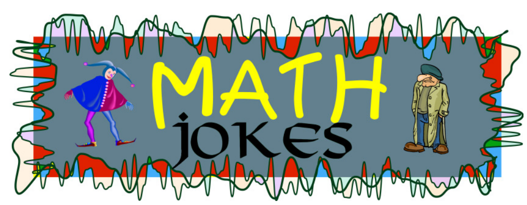
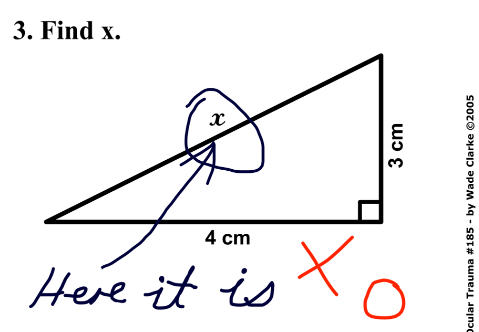


Q: What does the zero say to the eight?
A: Nice belt!

Teacher: What is $2k + k$?
Student: 3000!



Posters of this image can be bought at the Ocular Trauma online store http://www.cafepress.com/oculartrauma

Q: What do you get if you divide the circumference of a jack-o-lantern by its diameter?
A: Pumpkin Pi!

Little Bits of Geometry

Spira Mirabilis

Klaus Hoechsmann†

The December 2001 issue of this magazine began a "series" called *The Rose and the Nautilus*, which has so far meandered through rabbits and golden rectangles to pentagrams and roses, then got side-tracked into ratios, and finally crashed into the Platonic Solids, spilling algebra all over the hill-side. If you carefully look at this sequence, the one common thread you might detect is that all of it has to do with proportion, and makes a consistent effort to reason around pictures.

We shall continue in this vein as we approach the long-awaited *Nautilus* (a tropical sea-shell of the distant past), shown at the top of Figure 1, conveniently cut open to reveal its structure. The super-imposed geometric scaffolding can be drawn into any non-square rectangle: a lopsided cross involving a diagonal and a perpendicular, with a kind of rectangular spiral wrapped around it, tiling the given rectangle by an infinite sequence of ever smaller ones. Various curves could be drawn into this pattern—for instance, a quarter-ellipse in each "tile" (whose sides would be the major and minor half-axes). At first glance, the ancient shell seems to fit this description—but a second look shows a small but clear discrepancy. In reality its contour is a good approximation of an *equiangular* spiral, because the resident mollusc built itself a (sickle-shaped) chamber of the *same shape* every year, while it was growing steadily fatter.

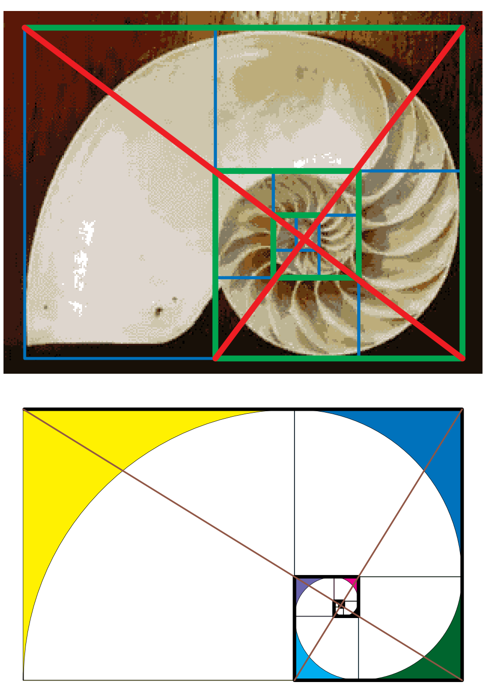


Figure 1: A *Nautilus* shell cut open and a Golden Spiral.

The lower rectangle in Figure 1 is *"golden"*, which means that the tiling just described is done by *squares*, into each of which a quarter-circle can be drawn to make a beautiful round spiral. Because of its pedigree, we call this a *golden spiral*. Many people say, and probably believe, that it is equiangular (or "logarithmic" as they prefer to say). The main point of the present article is to *prove that this is not true*, and to explore just how close it does come to the truth. Of course, we must first say what we mean by "equiangular."

Figure 2 shows the ghost of an equiangular spiral: twenty points of its arc marked in blue. It owes this name to the following remarkable property: take any two triangles, each with one vertex attached to the centre of the spiral and the other two on the rim; then *if their central angles are equal, so are the other ones* (taken in order). The three orange triangles in the top diagram illustrate this: their central angles are all equal to 27 degrees; and in the bottom diagram all twenty of the little slivers are similar. At the rim, each of them has a larger angle followed counter-clockwise by a smaller one (a bit farther from the centre), and all these larger angles are equal—and so, of course, are the smaller ones.

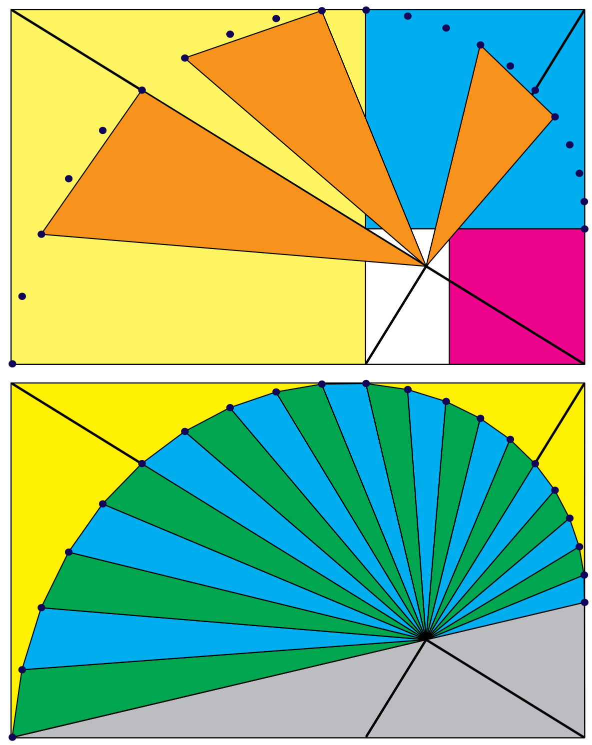


Figure 2: Ghost of an equiangular spiral holding up three fat similar triangles (above) and twenty skinny ones (below).

As the central angles get smaller and smaller (the blue points becoming more numerous), while always remaining equal to one another, the two rim angles approach those made on either side of a tangent to the spiral, as it crosses a *polar ray*, i.e., one coming from the centre, also known as the *pole*, of the spiral. Conclusion:

An equiangular spiral cuts every polar ray by the same angle.

If you imagine a point on the spiral moving from left to right in the preceding diagram, you would see something like the path of a low-flying plane gradually circling toward the north pole while keeping a steady course of about 17° north of west.

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The Golden Spiral is not Equiangular

The Golden Spiral does *not* "see" all polar rays equally. Take, for instance the rays, PS and PD in Figure 3, with S lying on the diagonal AC. The tangent for S goes through some point T, the one for D passes through C, but the angles TSP and CDP, we claim, are not equal: the first equals the right angle TSO minus the blue angle PSO, while the second one equals the right angle CDO minus the grey angle PDO—so our claim amounts to the blue angle at S being different from the grey one at D.

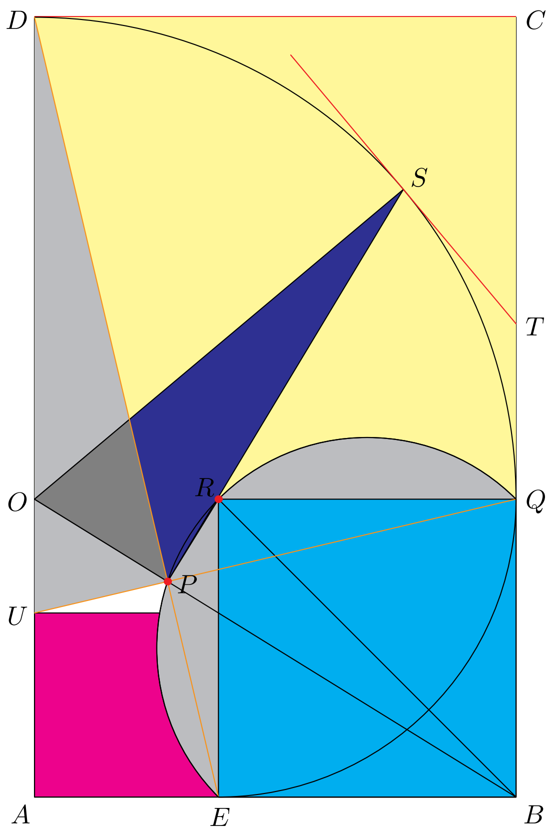

Figure 3: Polar rays PS and PD crossed at different angles.

This will be derived from the following observations:

- P is farther from O than from U,
- QPE and OPS are right angles.

Once these two items are established, you can prove our claim by comparing the hypotenuses and the short sides of the right triangles OPS and UPD. *Please do it.*

With a little help from Pythagoras, the first item boils down to P being closer to the extension of the top edge of the purple square than to the line RO, just as it is closer to AD than to BC (a fact about Golden Rectangles—see?). For the second one, note that OPS and BPR are part of the perpendicular crossing of AC and OB. That does it for OPS, but what about QPE? Here, the trick is to observe that P lies on the *circumcircle* of the turquoise square, because BPR is a right angle and BR is a diagonal. This yields the angle equalities $QRB = QPB$ and $BRE = BPE$, hence $QPE = QPB + BPE = QRE$.

As a by-product, *a second rectangular cross through P is formed by DE and QU. This argument does not depend on $EBQR$ being a square*—it works in any rectangular setting, such as shown in Figure 4.

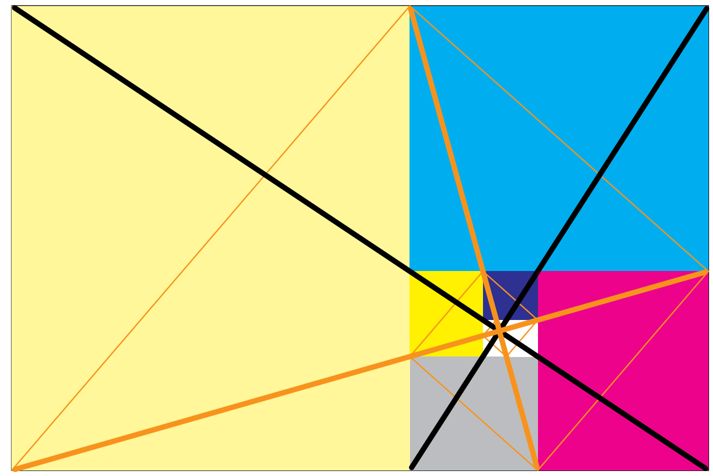

Figure 4: Rectangular scaffolding with the two crosses.

Polygonal Bernoulli Spirals

Now that we have disqualified the Golden Spiral, you might wonder whether equiangular spirals exist at all. Well, if they didn't, how could Jacob Bernoulli have studied them so enthusiastically that he had one carved on his tombstone in Basel, Switzerland?

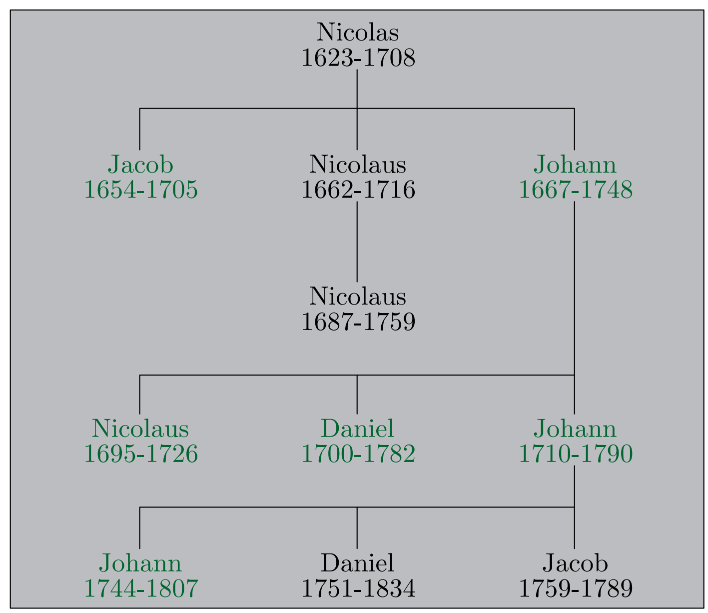

Figure 5: The family's mathematicians shown in green.

The name Bernoulli pops up so often in connection with theorems, laws, principles, curves (!), and numbers, that you might marvel at that fellow's productivity and longevity. In fact, Jacob blazed the trail for a whole dynasty of brilliant mathematicians. He and his brother Johann were among the first to get their hands on the new "calculus" toy, freshly crafted by Newton and Leibniz, and they quickly became virtuoso players.

Of course, this involved coordinates and formulas, and since we're still on a low-algebra diet, we'll try another approach to the "spira mirabilis" (so named by Jacob B.). We shall construct sequences of triangles akin to those in Figure 2: all of them similar and spiralling around a point P (the *pole*) where they have the same angle—and call such a thing a *polygonal Bernoulli spriral*. Our strategy will be to double the number of these wedges successively, until we have zillions of them, so that the rim will certainly *look* like a continuous curve, which, moreover, crosses all polar rays by the same angles.

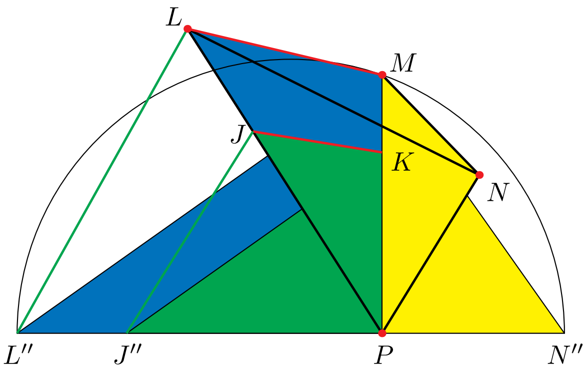

Figure 6: A Bernoulli spoke midway between two given ones.

This doubling will be done by finding the appropriate point on the polar ray bisecting each of the given central angles—a fairly easy task, as we'll presently see.

Problem: Given a triangle PNL, find a point M on the angle bisector of LPN such that the triangles LPM and MPN are similar.

Solution: Swing L and N outward by identical angles (where they are renamed L'' and N'') until P lies straight between L'' and N'', and construct a semi-circle over the segment $L''N''$; then M is the point where this semi-circle meets the perpendicular to $L''N''$ drawn from P.

This manoeuvre creates a right triangle $N''ML''$ with altitude MP. Its parts $L''PM$ and MPN'' are obviously similar, and this similarity is then inherited by LPM and MPN as follows. Choose J, J'', K so as to make $PM = PJ = PJ''$ and $PN = PK$. That makes JPK and MPN into congruent triangles and does the same for $J''PK$ and MPN''. Now Monsieur Desargues steps into the breach (just as in September 2002) assuring us that the two red lines are parallel, whence the similarity of JPK to LPM. Why? Because $L''L$ and $J''J$ are parallel (being the bases of isosceles triangles), and $L''M$ is parallel to $J''K$ on account of the congruency $J''PK$ and MPN''.

Now that we know how to *thicken* the plot, the question remains: how do we *start* it? Why not with the kind of rectangular scaffolding shown in the top diagram of Figure 1 and again in Figure 4? In the latter, the second cross is clearly seen to produce a sequence of mutually similar right triangles spiralling around the pole (ignore the first cross: it only served to define that pole), and they could form the initial configuration. Thereafter, bisect, bisect, bisect, bisect.

That is exactly what was done to obtain Figure 7. It displays the same old Golden Spiral with thirty-three red vertices of a polygonal Bernoulli spiral weaving around it through the turquoise and the yellow squares, like flowers on a wreath. Despite appearances, they do *not* lie *on* the Golden Spiral, except at the beginning and the end, as well as at Q, where they cross from *inside to outside*, and some point X (which seems to lie on the diagonal AC), where they cross in the opposite sense. Creatures from two different galaxies—Bernoulli's spiral with the disarmingly simple formula $\log r = c\,\theta$ in polar coordinates (also invented by Jacob), and the Golden Spiral with its easy recipe for drawing but no unifying formula at all—they are so far apart in theory, and yet so close in practice.

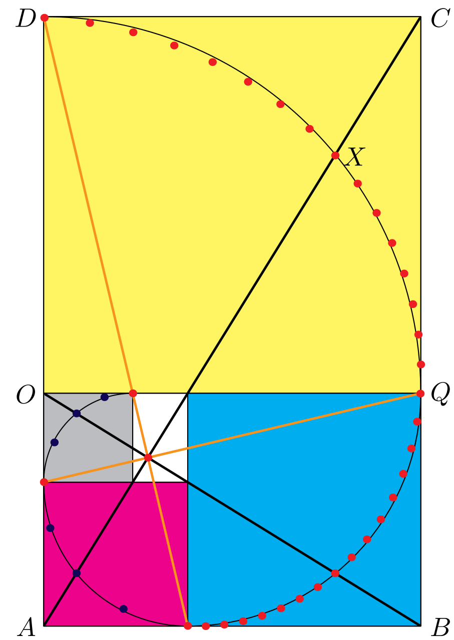

Figure 7: Bernoulli polygon weaving through Golden Spiral.

What most amazed Jacob, however, was the perseverance of this spiral under all the standard treatments poor innocent curves were subjected to back in those days: evolutes and involutes, pedals and caustics, changing circles to nephroids, cissoids to cardioids, etc., while his spiral kept returning to itself through all these attempted mutations—a fitting symbol of his own immortal soul: *Eadem mutata resurgo*.

Figure 8: Inscription on the tomb in the Münster of Basel.

The figures on pages 11, 12, 17, 19 (Figure 6 only), 22–29, and 31 were drawn by Steven Melenchuk with `Asymptote`, a powerful new descriptive vector graphics language for technical drawing developed at the University of Alberta. The term *vector graphics* refers to a method for producing figures that retain their high quality even at arbitrarily large magnifications.

The authors of `Asymptote` (Andy Hammerlindl, John C. Bowman, and Tom Prince) would like to thank the Natural Sciences and Engineering Research Council of Canada, the Pacific Institute for Mathematical Sciences, and the University of Alberta Faculty of Science for their generous financial support. `Asymptote` is freely available, under the GNU General Public License, from the web site `http://asymptote.sourceforge.net`, which includes a gallery of example `Asymptote` code and output.

The next issue of π *in the Sky* will include an article about `Asymptote`.

From Stability to Chaos —A Mathematical Journey

Aidan Chatwin-Davies†

In 1667, Sir Isaac Newton introduced three laws of dynamics, thus creating a deterministic model of the physical world. It would later be discovered that often determinism is not sufficient for making predictions because of a phenomenon called *chaos*. But before examining chaos, we must understand determinism. Let us take an example.

To describe the motion of a dropped ball, we need two variables: its height off the ground h and its vertical velocity v at a given instant in time. Since they change with the time t these variables can be written as $h(t)$ and $v(t)$. The rate at which $h(t)$ changes is $v(t)$, and we write:

$$\frac{dh}{dt} = v. \quad (1)$$

The rate at which $v(t)$ changes is the acceleration, which is due to the Earth's gravity. This increases the ball's velocity by 9.8 $\mathrm{m\,s^{-1}}$ every second:

$$\frac{dv}{dt} = -9.8. \qquad \text{(The negative sign shows it is accelerating downwards.)} \quad (2)$$

To explain these so-called *differential equations*, we will carry on geometrically. Instead of observing the ball in three dimensions, we will look at it in the *phase space* of its variables. This is a plane with two axes, h and v. Say the ball is dropped from height x_0. The velocity increases at a constant rate while the height decreases quadratically until the ball reaches the ground. This produces a curve as in Figure 1. The evolution of the functions h and v appears in Figure 2 and depends on the initial height and velocity, which are represented by a point in phase space.

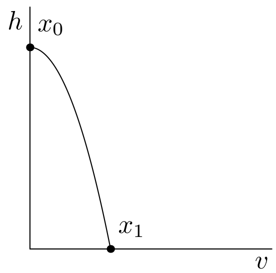

Figure 1: The path followed by a ball dropped from height x_0 in phase space.

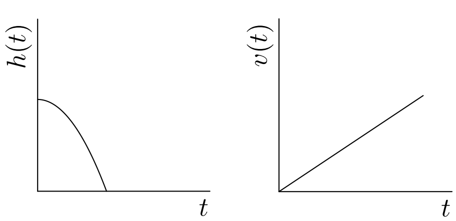

Figure 2: Height and velocity as functions of time.

For each such point the differential equation attaches an arrow directing the outcome. By filling in the phase space with arrows, we obtain a *vector field*, which describes all possible motions of the ball. By starting at one point and following the flow, a curve is obtained that is always tangent to the vector field. Each of these curves is a *solution* (also called *orbit* or *trajectory*) of the differential equation. Florin Diacu and Philip Holmes give a suggestive description of vector fields in [1]: "Thus, in a manner somewhat like that of a river current that carries flecks of foam and driftwood on its surface, leaving local evidence of its passage, the vector field drives individual solutions of the differential equation, starting at each point in phase space, to form a global phase portrait." An important property of solutions of differential equations is *continuity with respect to initial data*, which implies that solutions beginning close together will remain together at least for a while.

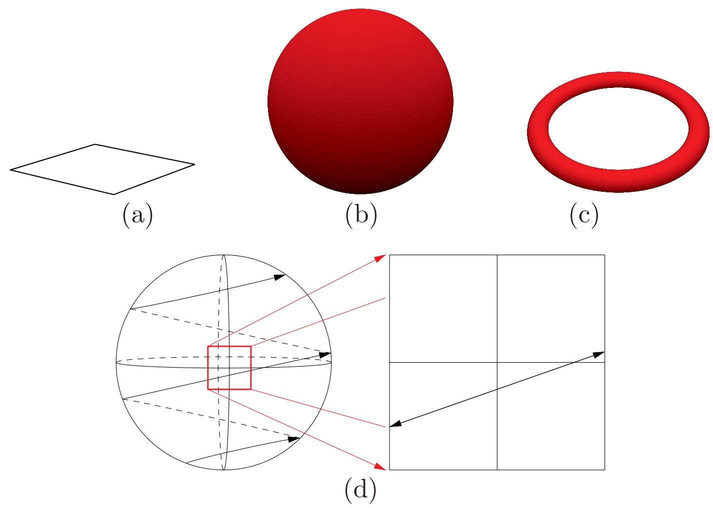

Figure 3: Manifolds: a) plane, b) sphere, c) torus, d) planar approximation of a sphere.

The phase space of our example is a plane. Phase spaces, however, can be of any shape and dimension, for example a sphere or a torus (see Figure 3). They are called *manifolds* and can have coordinates similar to the Earth's meridians and parallels. A solution to a differential equation is a curve, but it can also be a point. This is an *equilibrium solution*, a place in phase space where the variables remain constant, for example a stationary ball.

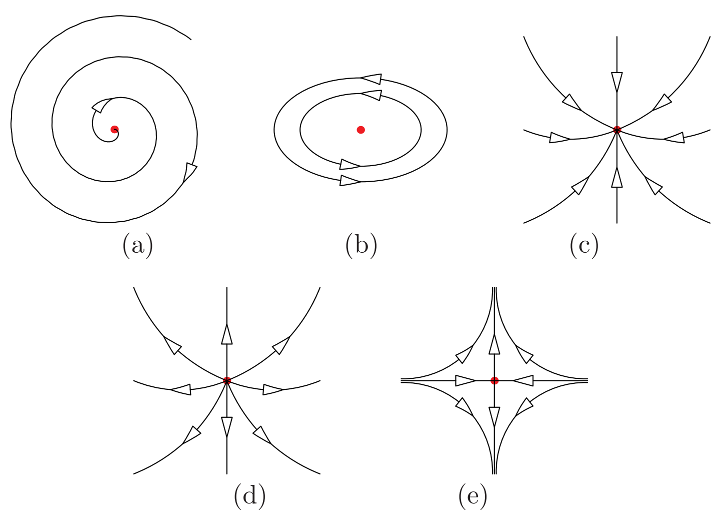

Figure 4: Common flows around equilibria: a) spiral source, b) periodic orbit, c) sink, d) source, e) saddle point.

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Vector fields around equilibria can have various shapes (see Figure 4). Some examples are the *spiral source*, where solutions spiral away from the equilibrium, the *periodic orbit*, where solutions move around the equilibrium neither approaching nor leaving it, the *sink*, where all orbits move towards the equilibrium. There is also the *saddle point*, where two orbits approach the equilibrium, two leave it, and all others skid by. This is a *separatrix* since it separates orbits with different types of behaviour. A last remark is that solutions tending towards an equilibrium can only attain it in infinite time.

Newton's contributions provided a new way of studying the motions of and interactions between celestial bodies. This led to the development of the field of *celestial mechanics*. The n-body problem, one of its central problems, still remains incompletely solved. It is a rich scientific mine in which the possibility of mathematical chaos was first observed.

Newtonian mechanics state that gravitation acts between bodies and thus produces motion. The force is proportional to the product of the masses, and inversely proportional to the square of the distance. Newton proved that celestial bodies can be modelled as *point masses*, that is, bodies reduced to points of a finite mass (see Figure 5).

Figure 5: A Newtonian model of 2 bodies (m_1 and m_2).

The n-body problem is formulated as such: consider n point masses $m_1, m_2, \ldots, m_n$ in three-dimensional space. Suppose they exhibit Newtonian forces of attraction. If the initial positions and velocities of the masses are given for some present instant t_0, determine the system's state for every past and future instant. In other words, a general solution is being asked for.

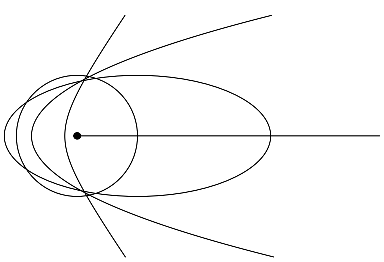

Figure 6: Possible motions of two bodies.

It's a difficult problem, the only solved case being $n = 2$. Here, the motion of one body with respect to the other is always a circle, an ellipse, a line, a parabola, or a hyperbola (see Figure 6). At a first look, the problem has three position and three velocity variables for each body, so a total of twelve, but this can be simplified. First, since gravity acts along a line, the bodies can only move in two dimensions. Next, since the centre of mass never changes, the position of one body with respect to the other will suffice. Since everything is relative, the problem can be further simplified to just four variables. As soon as $n = 3$ is considered, complications arise, leading to the possibility of chaos.

The man who first glimpsed at this phenomenon was the French mathematician Henri Poincaré. In 1885, he stumbled upon a contest printed in the journal *Acta Mathematica* and commissioned by King Oscar II of Sweden and Norway. The contest asked for a general solution to the n-body problem:

"Given a system of arbitrarily many mutually attracting material points that obey Newton's laws, try to find, under the assumption that no two points ever collide, a representation of the coordinates of each point as a series in a variable that is some known function of time, and converges uniformly for all real values of that variable. [The solution to] this problem. . . will considerably extend our understanding of the solar system. . ." [6].

Poincaré began with the three-body problem. Soon he reached the limits of quantitative mathematics, realizing that the problem would have to be considered *qualitatively*, the question being, "What do solutions to differential equations look like?" rather than "What is their formula?" His brilliant insights sent mathematics in a new direction. He had not solved the n-body problem, but on January 21, 1889, he was awarded King Oscar's prize for his discoveries.

It is somewhat ironical that chaos actually emerged from the search for its opposite: stability. The general question of stability concerning dynamical systems asks whether a system will endure small changes relatively unaltered. Mathematicians sought to confirm our solar system's stability, and for good enough reasons. They defined stability if no planets collide or escape from their orbits around the sun. It is a ten-body problem that must remain forever close to how it began. The general solution to the n-body problem incorporates the stability question. Poincaré was, in fact, searching for stability in a three-body problem when he encountered chaos.

The solution to a planetary problem is usually found through *perturbation*, which requires one to take the solution of a simple, but similar problem and progressively modify it to approximate the true solution of the differential equations. This is a natural way to approach planetary problems; one may start with a perfect ellipse, then alter it to match the true orbit. When applying a perturbation method, a *series approximation* is obtained. It is the attempt to obtain the solution through the sum of an infinite number of terms. Each term slightly modifies the solution, and the more terms taken, the better the resulting approximation. An example is the function $\sin x$, which can be represented as a series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots \tag{3}$$

The stability question occupied many mathematicians and went through much development. The first significant contribution to it came from Pierre Simon Laplace. In 1773, he proved that: "In the first power series approximation of the eccentricities, the major axes of the planets have no secular terms," [5] (see Figure 7).

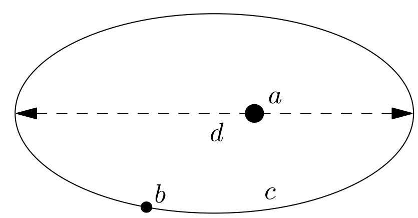

Figure 7: Terms in an orbit. a is a large body, b is a smaller body, c is the orbit of b relative to a and d is the major axis.

Eccentricity describes the "flatness" of an ellipse. Large eccentricity means long, thin ellipses, and small eccentricity means short, fat ellipses. The major axis is the longest line connecting two points on the ellipse. Secular terms relate to the increase of a time variable in the differential equations describing the bodies' motions. If there are no secular terms, time is not a changing factor, and thus the major axes don't "change" with time, implying that the ellipses are stable. If secular terms did appear, however, they could change things. Either they would cancel themselves out, thus yielding stability, or instability could occur. Laplace had only obtained this result for a first power series approximation, a first estimate of the planetary problem, so stability was not yet proven, but it was a step in that direction.

Joseph Louis Lagrange built off of Laplace's findings between 1774 and 1776, proving that: "For all order approximations of the eccentricities of the ellipses (given as orbits of the planets), for all order approximations of the sine of the angle of mutual inclinations, and for perturbations of the first order with respect to the masses, the solar system is stable in the sense that secular terms do not occur," [4]. This showed that in all order approximations of the eccentricities with respect to a first power series approximation relative to the bodies' masses, secular terms did not appear, and thus that case was stable.

Siméon Denis Poisson improved on Lagrange's result in 1808, showing that no secular terms appeared in the major axes respective to a second power series approximation relative to the bodies' masses. Poisson also offered a new definition of stability. He suggested a system was stable if the bodies repeatedly returned near their initial positions. This is a loose definition of stability, especially when considering Poincaré's recurrence theorem, which states that in a three-body problem, if the motion remains bounded and the bodies do not collide, they will return close to their initial positions. Once near the starting point, the argument can be repeated ad infinitum to obtain Poisson stability. This is similar to having a volume of gas V_0 at some arbitrary instant t_0 as in Figure 8. At t_1, it will have moved to V_1, which may be shaped differently, but is equal in volume. At some finite instant t_m, V_m must intersect V_0, because if not, all the volumes are disjoint, so the overall volume of the water would eventually grow. (The cases for which the recurrence theorem does not hold are negligible in amount. An example is when V_0 has zero volume.)

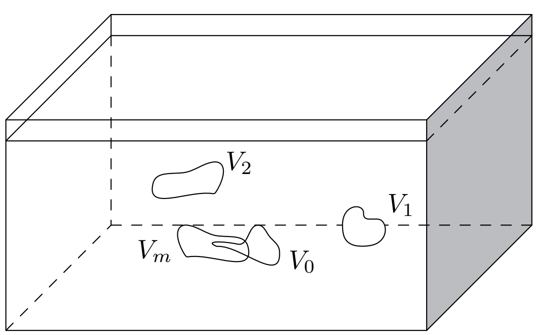

Figure 8: Tank of water analogy for the proof of the recurrence theorem.

The trend of advances pointing towards stability ended in 1878 with the Romanian mathematician Spiru Haretu, who proved that secular terms appeared in the major axes when a third power series approximation is taken with respect to the bodies' masses, thus implying that instability might occur in the solar system.

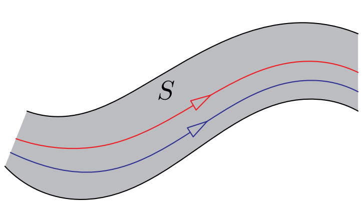

Figure 9: Curve S is stable if surrounding solutions remain nearby forever.

In 1892, the Russian mathematician Aleksandr Mikhailovich Liapunov proposed a definition of stability that addresses all differential equations. For a solution to be stable, any other solutions starting near it must remain close forever. The curve S in Figure 9 is stable if all other solution curves starting close by follow it forever (e.g. curve T). Stability should not be confused with continuity with respect to initial data, which demands only local nearness. As soon as a nearby solution strays away, the solution in question is unstable.

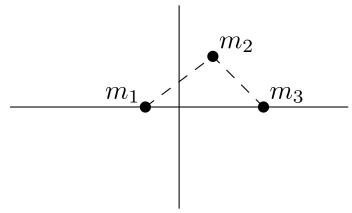

Figure 10: The planar restricted three-body problem.

Stability is more obvious around equilibria, which are deemed stable if no surrounding solutions leave their vicinity, and unstable if any do. Recalling the possible shapes of the vector fields around equilibria, examples of stable equilibria are the periodic orbit and the spiral sink, and examples of unstable equilibria are the source and the saddle point. Asymptotic stability occurs when nearby solutions also tend to an equilibrium.

Poincaré looked for stability in a restricted three-body problem, in which two large bodies, m_1 and m_2, follow elliptical orbits similar to a two-body problem, while a third small body does not influence them (see Figure 10). Poincaré found that the third body would asymptotically approach an equilibrium solution, only to leave and then return in infinite time. He named this a *homoclinic* orbit (see Figure 11). Despite appearing similar in phase space, homoclinic and periodic orbits are very different. In infinite time, one would travel around a periodic orbit infinitely many times, but only once around a homoclinic orbit.

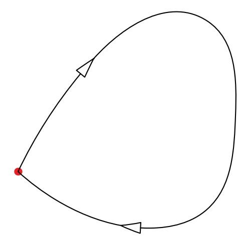

Figure 11: A homoclinic orbit.

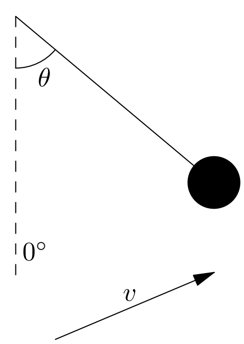

Figure 12: A simple pendulum.

Since a restricted three-body problem is still difficult to examine, we consider a simpler analogous example: the pendulum. A bob fixed to a rigid bar that swings back and forth because of a pivot conserving its energy (see Figure 12). The variables describing its state are the angle formed by the rod and its rest position θ and its angular velocity v. Consequently, this system's phase space is a plane axes θ and v. The pendulum has two equilibria. The first is the lowest position ($\theta = 0^o$), and the second the highest position ($\theta = \pm 180^o$). In both positions, the variables remain constant; however, the first equilibrium is stable whereas the second is not.

Solutions near the stable equilibrium do not leave its neighbourhood, but those near the unstable equilibrium do (see Figure 13). The equilibria appear as points (there are actually three points because of the unstable equilibria at 180^o and -180^o). Periodic orbits (corresponding to the pendulum swinging) begin around the stable equilibrium, and the unstable equilibria lie at the flow's extremities. The orbit joining them is *heteroclinic*, which means that it leads to an

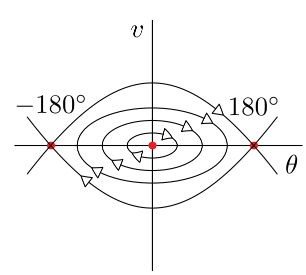

Figure 13: The flow of a simple perfect pendulum.

equilibrium different from where it started.

In his work, Poincaré examined periodic orbits. Continuity with respect to initial data led him to conclude that solutions near a periodic orbit could have one of three shapes: periodic motion around the orbit, inward spiral motion, or outward spiral motion. He found a cross section allowed effective study of such solutions (see Figure 14). The transversal line D cuts through all the orbits, thus transforming them into a series of points on a line. Iteration of the periodic orbit produces a single point p. Starting at o_0 on the other orbit, first point o_1 is met, then $o_2, o_3, \ldots$. This generates a *first return map*, or *Poincaré map*, assigning to each point its next iterate on the cross section. Denoting the Poincaré map by P, $P(o_0) = o_1$, $P(o_1) = P(P(o_0)) = o_2, \ldots$, we see that all of the images of o_0 generated by the map lie on the orbit through o_1. The Poincaré map reduces dimension by one and changes the orbit's setting from continuous time to discrete time.

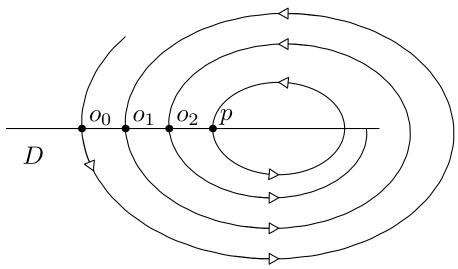

Figure 14: Taking a cross section of an inward spiral orbit and a periodic orbit.

Returning to the pendulum, consider an additional effect. Suppose it is periodically shaken every T seconds with varying amounts of force. The system is now be described by three variables θ, v, and t (time). Ignoring the new effect, the simple pendulum's motions would appear as in Figure 15, in which the equilibria appear as straight lines.

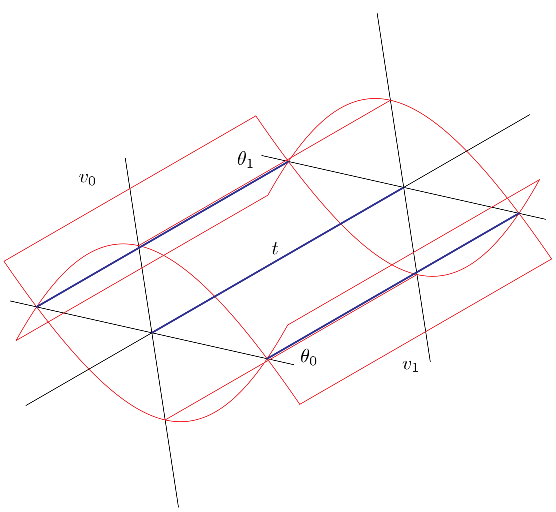

Figure 15: The extended phase space of a simple perfect pendulum.

The periodic perturbing force deforms the phase space and the manifolds become warped—with past and future motions depending on when the periodic motion occurs. They will only meet when the system's energy balances out, allowing the solution to return towards an equilibrium. A simpler way to examine this system is through a Poincaré map, for which the time setting changes from continuous to discrete. In our pendulum, time is important because of the periodic force. Let it repeat every T seconds, thus creating natural discrete "blocks" of T seconds.

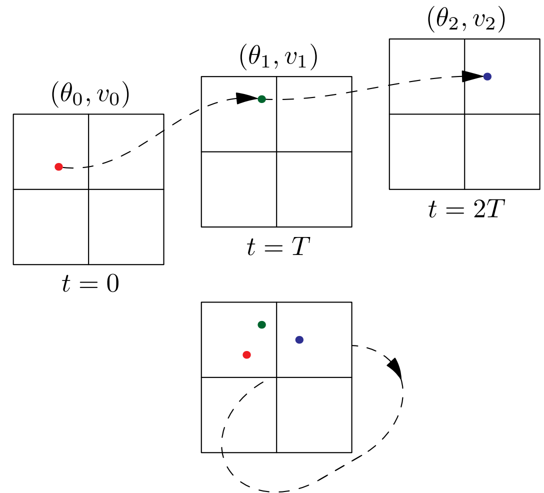

Figure 16: A Poincaré Map is obtained by taking a cross section every T seconds.

Begin by taking a cross section at $t = 0$, thus obtaining a certain point (θ_0, v_0). By following the orbit to $t = T$, another point (θ_1, v_1) is obtained. Then t is reset to 0, and the orbit is followed to (θ_2, v_2). By repeating this, a Poincaré map is obtained. It is as if the pendulum's phase space is sliced every T seconds (see Figure 16). Taking a Poincaré map of the phase space reduces its dimension to two and transforms the stable and unstable manifolds into curves. Since the unstable motion repeats every T seconds, it appears as a fixed point. Figure 17 shows the unperturbed pendulum's Poincaré map. The two manifolds join smoothly, moving away from p and back to it. In the perturbed pendulum's Poincaré map, the manifolds have been disturbed, so they appear as two separate curves (see Figure 18). Here U denotes the unstable manifold and S denotes the stable manifold. While S moves towards p with forward iterations of the map, U moves towards p with inverse iterations.

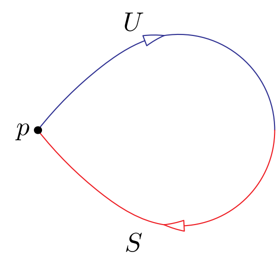

Figure 17: The stable and unstable manifolds join to form a smooth line in the unperturbed pendulum's Poincaré map.

Sometimes the two manifolds intersect, say at q_0. Forward iteration of the map generates $P(q_0) = q_1$, $P(q_1) = q_2, \ldots$, which lie on the stable manifold. Inverse iteration, generates $P^{-1}(q_0) = q_{-1}$, $P^{-1}(q_{-1}) = q_{-2}, \ldots$, which lie on the unstable manifold. Thus q_n approaches p as n goes to infinity, as does q_{-n}. The point q_0, however, belongs to both manifolds. It is a transversal homoclinic intersection, and also part of the sequence going from q_{-n} to

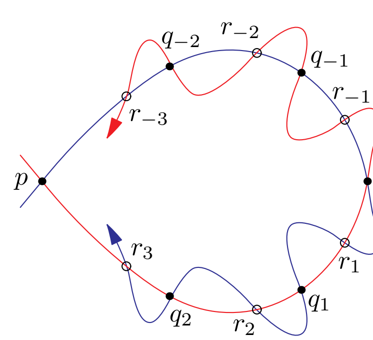

Figure 18: The perturbed pendulum's Poincaré map.

q_n. Consequently, all points q_{-n} to q_n must belong to both manifolds, causing them to weave about so as to land on every point in the sequence, thus producing infinitely many more transverse homoclinic intersections (see Figure 18). Furthermore, a manifold has to cut across the other in the same direction every time, meaning there is at least one more intersection series r in between the points in the q series.

Imagine each manifold as a line going in one direction. Upon intersection, continuity of solutions with respect to initial data causes each line to move toward the other's path; however, each must also continue in its direction. This produces further intersections, and the pattern continues. The manifolds are like lines trapped and pulled between two "directions."

As the manifolds approach the saddle point p, they get stretched out. This causes further intersections between them, producing *secondary homoclinic points* (see Figure 19). These feel the same effect, leading to *tertiary homoclinic points*, and so on ad infinitum. The resulting picture is so complex that Poincaré dared not draw it. He called it a *homoclinic tangle*. This case resembles the restricted three-body problem, and led Poincaré to the conclusion that accurate long-term predictions are impossible in such systems.

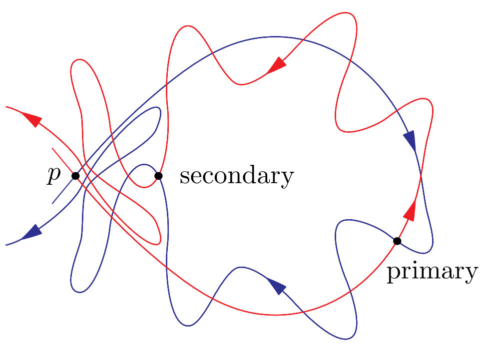

Figure 19: The saddle point's stretching effect produces infinitely many more transverse homoclinic intersections.

And this is how chaos was born from stability. Since then, Poincaré's ideas grew into a coherent mathematical theory called nonlinear dynamics, which is now further developed by thousands of experts from all over the world.

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convergence

An Invitation and Call for Papers

Victor Katz†

Convergence: Where Mathematics, History and Teaching Interact, is the Mathematical Association of America's new online magazine on the history of mathematics and its use in teaching. Part of MathDL, the mathematics digital library, Convergence is aimed at teachers of grades 9–14 mathematics, be they secondary teachers, two- or four-year college teachers, or college teachers preparing secondary teachers. ("Grade 9–14 mathematics" encompasses algebra, synthetic and analytic geometry, trigonometry, probability and statistics, elementary functions, calculus, linear algebra, and differential equations.) The editors, Victor J. Katz, from the University of the District of Columbia, and Frank Swetz, from Penn State University, Harrisburg, welcome all members to log on to the Convergence website (http://convergence.mathdl.org) and see what the magazine has to offer.

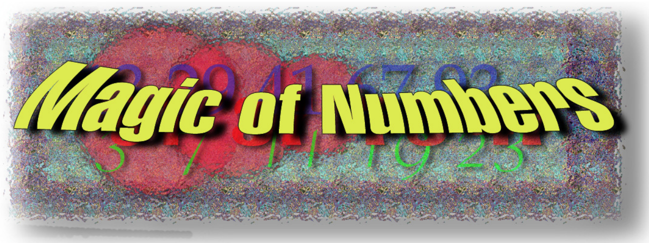
Among the types of material appearing in Convergence are:

- Expository articles dealing with the history of various topics in mathematics curricula. These may contain interactive components and colour graphics.
- Translations of original sources, generally accompanied by commentary showing the context of the works.
- Reviews of current and past books, articles, and teaching aids on the history of mathematics of use to teachers, as well as reviews of websites providing information on the history of mathematics.
- Classroom suggestions. These may be self-contained articles showing how to use history in the teaching of a particular topic or they may be materials closely related to a main article, showing in some detail how to use the article in a classroom setting.
- Historical problems. These problems will appear in a section entitled "Problems from another time".
- What Happened Today in History? Each day, there will be a listing of 2–3 historical "mathematical events" that happened on that date.
- Quotation of the day. A new and interesting quotation about mathematics from a historical figure will appear in this section each day.
- An up-to-date guide to what is happening around the world in the history of mathematics and its use in teaching. The magazine will report on past and future meetings.

The magazine is currently free to all, due to the support of the National Science Foundation, but registration is required to access the site. A small subscription fee will be charged beginning in 2006.

Currently, we have a limited supply of articles in our pipeline. Because our goal is to bring out new material on a regular basis, we need a continual flow of articles and classroom suggestions. We therefore welcome your ideas for articles as well as your completed manuscripts. Please contact Victor Katz at vkatz@udc.edu for more information.

† **Victor Katz** is a Professor of Mathematics at the University of the District of Columbia. His e-mail address is vkatz@udc.edu.



Divisibility by Seven

Jeremy Tatum[†]

Most people know how to test whether a very large number is divisible by 2, 3, or 5, and perhaps even by 11. However, testing to see whether a large number is divisible by 7 is not as easy. In the March 2003 issue of π *in the Sky* Edwin Charles and I published an article showing how to test whether a given large number is divisible by 7, 13, or 17, or indeed any prime number up to 97, and we did our best to explain why the method worked.

Since then, I have discovered a quite different method of testing for divisibility by 7—though I have to confess that I have really no idea why it works! In this short article I describe the method—in the hope, perhaps, that some bright reader might be able to provide an explanation.

The key to the test is the number 546231. This number is itself divisible by 7, though I don't know whether that has any relevance. Its factors are $3 \times 7 \times 19 \times 37 \times 37$, but again I don't know whether that is relevant.

Regardless, here is the method. Let us suppose that we want to test whether the number 6065534139 is divisible by 7. Under this number we write the number 546231 repeatedly, *starting from the right*

6	0	6	5	5	3	4	1	3	9
6	2	3	1	5	4	6	2	3	1

Under this we write the products of the two numbers in each column:

36	0	18	5	25	12	24	2	9	9

Add these products together—it comes to 140. If this sum, 140, is divisible by 7, the original number is divisible by 7. If you are not sure whether 140 is divisible by 7, you can repeat the process:

1	4	0
2	3	1
2	12	0

The sum is 14. And if you are still unsure whether 14 is divisible by 7, you can do it once more:

1	4
3	1
3	4

The sum is 7, and so the original number (as well as 140 and 14, as if you didn't know!) is divisible by 7.

You might like to test the number that was used as an example in the article mentioned, namely

698664808849557661972934437230757991 1.

For the record, this number is divisible by 7—but see if the method works on it.

As I said, I don't know *why* it works, and I'd be happy if someone could come up with an explanation. Also, I wonder, are there other numbers analogous to 546231, which can be used to test for divisibility by 13, by 17, by 19?

[†] **Jeremy Tatum** is a former professor in the Department of Physics and Astronomy of the University of Victoria. His e-mail address is universe@uvvm.uvic.ca.

Galileo's Punishment

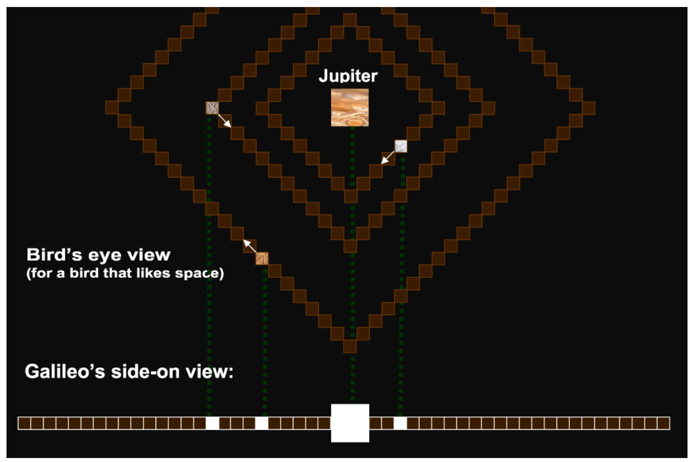
A math challenge problem by Gord Hamilton[†]

Upon Galileo's death, Jupiter (king of the Roman gods) ordered Pluto (lord of the underworld) to punish Galileo for failing to respect his planet's privacy.

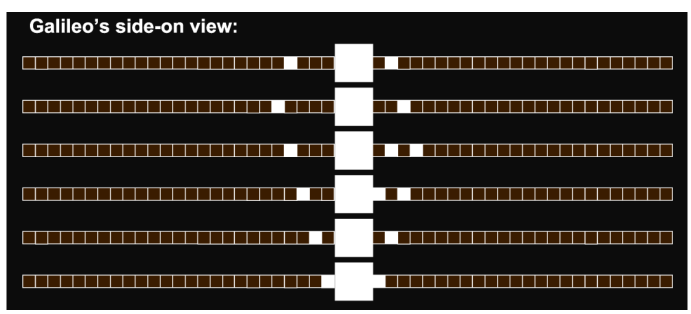
"Galileo is a telescope wielding rogue," said Jupiter. "Just as he wandered from one town to another in life, let him wander from one universe to another in death."

And thus it was that Galileo awoke from death in a universe with alien physical laws:

- Every heavenly body is a cube.
- Moons move in diamond orbits centred on their planet.
- No two moons occupy the same orbit.
- A moon can touch, but cannot pass through its planet.
- Every 24 hours all moons jump one space around their orbit.



1. The picture below shows 6 of Galileo's sightings of a planet and her 3 moons. On graph paper, with the planet occupying 9 basic squares, and each moon only 1 such, draw a possible "bird's eye" view of the 3 moons with time labels 1 to 6 inscribed in each moon-square.



2. Is it possible to draw a similar arrangement of 6 sightings of n moons such that Galileo sees only $n-1$ or fewer? If so do it, if not explain why.
3. Repeat the last problem but allow any number of sightings.

[†] **Gord Hamilton** can be reached via his web site www.galileo.org/math/puzzles.html.

Math Challenges

Problem 1. Find the largest value of $x_{m,n} = \dfrac{1}{m+n+1} - \dfrac{1}{(m+1)(n+1)}$ where m, n are positive integers.

Problem 2. For any positive integer n, let $S(n)$ denote the sum of its digits in decimal notation. If $S(n) = 50$ and $S(15n) = 300$, find $S(4n)$.

Problem 3. Let A be a nonempty set of positive integers such that if $a \in A$ then $4a$ and $\lfloor \sqrt{a} \rfloor \in A$ (here $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a). Prove that A is the set of all positive integers.

Problem 4. Let $\mathbb{N}$ denote the set of positive integers and $f : \mathbb{N} \to \mathbb{N}$ a function such that $f(f(n)) + 2f(n) = 3n + 4$ for every $n \in \mathbb{N}$. Find $f(2006)$.

Problem 5. Let AA', BB', CC' be the angle bisectors of ΔABC. If $\widehat{B'A'C'} = 90^o$, find $\widehat{BAC}$.

Send your solutions to π *in the Sky*: Math Challenges.

Solutions to the Problems Published in the December, 2004 Issue of π *in the Sky*:

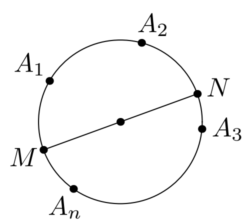
Problem 1. (Solution given by Jerry G. Ianni from Leonia, New Jersey.) Yes, it is possible! Even though the combined total number of patients treated by each hospital is the same, the number of patients treated for each specific disease is allowed to be different. Suppose that 90% of the patients treated for disease D_1 at hospital H_1 are cured and that 100% of the patients treated for disease D_1 at hospital H_2 are cured. Suppose also that 40% of the patients treated for disease D_2 at hospital H_2 are cured. It seems that hospital H_2 would generally have a better performance record. However, suppose both hospitals treat 5000 patients over the course of one year for these two diseases. If hospital H_1 treats 4500 patients for D_1 and 500 patients for D_2, the total number of patients cured will be $0.9(4500) + 0.4(500) = 4250$. This figure is 85% of all the patients. On the other hand, if hospital H_2 treats 3000 patients for D_1 and 2000 patients for D_2, the total number of patients cured will be $1(3000) + 0.5(2000) = 4000$. This figure is 80% of all the patients. Thus, hospital H_1 cures a greater percentage of all the patients during the year. The practical reason that hospital H_1 did better is because it treated a much greater percentage of patients for a disease (D_1) that it treats well whereas hospital H_2 treated a large percentage of patients for a disease (D_2) that it does not treat so well.

Problem 2. Let $(i_1, i_2, \ldots, i_n)$ be a permutation of $1, 2, \ldots, n)$. Provided that $k < n$, the pair (i_k, i_n) is called an inversion if $i_k > i_n$. The interchange of two neighbouring numbers changes the parity of the number of inversions of a permutation. The permutation $(1, 2, \ldots, n)$ has zero inversions, and if 2005 operations of interchanging of two neighbouring numbers are performed, the number of inversions will be odd and, thus, the original arrangement could not be obtained.

Problem 3. Let $x_1, x_2, \ldots, x_{100}$ be the given numbers. None of the 300 numbers $x_1, x_2, \ldots, x_{100}, x_1 + 2, x_2 + 2 \ldots, x_{100} + 2, x_1 + 5, x_2 + 5, \ldots, x_{100} + 5$ is greater than 299. Therefore by the box principle (see π *in the Sky*, June, 2000), two of them, say $x_i + m$ and $x_j + n$ are equal, where $m, n \in \{0, 2, 5\}$ and $m \neq n$. Therefore $|x_i - x_j| = |m - n| \in \{2, 3, 5\}$ as required.
Jerry G. Ianni from Leonia, New Jersey gave a different solution that works only if the integers are ≤ 296.

Problem 4. We must have $x = y = z$. Indeed, if for example we assume that $x < y$, then from the last two equations we get $(y+z)^5 < (z+x)^5$; hence $y < x$, which is a contradiction. Similarly, assuming any of the other possibilities results in contradictions.
Taking $x = y = z$ in the first equation, we get $(2x)^5 = x$; hence $x = 0$ or $x = \pm\frac{1}{2\sqrt{2}}$. Therefore the solutions of the system are $(0,0,0), (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}), (\frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}})$.
A similar solution was given by Jerry G. Ianni from Leonia, New Jersey.

Problem 5.



Let MN be a diameter of the circle such that $M, N \neq A_i$, $i = 1, 2, \ldots, n$. Writing the triangle inequality for $\Delta A_i MN$ we get

$$A_iM + A_iN > MN = 2, i = 1, \ldots, n$$

and thus

$$\sum_{i=1}^{n} A_iM + \sum_{i=1}^{n} A_iN > 2n.$$

Therefore, either $\sum_{i=1}^{n} A_iM$ or $\sum_{i=1}^{n} A_iN$ is greater than n.

Editor's Note. We received interesting solutions for some problems from the September 2003 issue of the magazine. Here are two of them:

Problem 2 (Math Challenges π *in the Sky*, September 2003): Find all distinct points (x, y) of integers that are solutions of the equation $x^2 - xy + y^2 = x + y$.
Solution by Yuming Chen and Edward T. H. Wang, Wilfrid Laurier University, Waterloo.
Rewrite the given equation as $(x-1)^2 + (y-1)^2 + (x-y)^2 = 2$. Hence, $((x-1)^2, (y-1)^2, (x-y)^2) = (1,1,0)$ or $(1,0,1)$ or $(0,1,1)$. By simple computations, we easily find that $(x, y) = (2,2)$ or $(0,0)$ in the first case, $(x,y) = (2,1)$ or $(0,1)$ in the second case, and $(x,y) = (1,2)$ or $(1,0)$ in the third case, yielding a total of six distinct pairs of solutions.

Problem 3 (Math Challenges π *in the Sky*, September 2003): Find the largest subset $A \subset \{1, 2, \ldots, 2003\}$ such that for all $a, b \in A$, $a + b$ is not divisible by $a - b$.
Solution by Yuming Chen and Edward T. H. Wang, Wilfrid Laurier University, Waterloo.
More generally, call a subset A of $\mathcal{U} = 1, 2, \ldots, n$ "good" if for all $a, b \in A$, $a + b$ is not divisible by $a - b$.
Let $f(n)$ denote the maximum cardinality of a "good" subset of $\mathcal{U}$. We claim that $f(n) = \lceil \frac{n}{3} \rceil$ where $\lceil\ \rceil$ denotes the ceiling function. Let A be a "good" subset of $\mathcal{U}$. Then clearly A can not contain two consecutive integers since their difference would divide their sum. Furthermore, A can not contain two integers that differ by 2 since if $a, b \in A$ such that $a - b = 2$, then $a + b$ must be even and thus divisible by 2. Hence, for all $k \in \mathcal{U}$,

$$|A \cap \{k, k+1, k+2\}| \leq 1.$$

It follows that $f(n) \le \lceil \frac{n}{3} \rceil$.

To see that the upper bound can actually be attained, let $G = \{k \in \mathcal{U} : k \equiv 1 (\mathrm{mod} 3)\}$. Then for all $a, b \in G$, $a - b \equiv 0 (\mathrm{mod} 3)$ and $a + b \equiv 2 (\mathrm{mod} 3)$. Hence, $a - b$ does not divide $a + 3$. Clearly, $|G| = \lceil \frac{n}{3} \rceil$. This completes our proof. In particular, for $n = 2003$, we have the given problem and the answer is $f(2003) = \lceil \frac{2003}{3} \rceil = 668$.

We also note that Problems 1 and 4 from *π in the Sky*, September 2003 were solved by Yuming Chen and Edward T. H. Wang and, respectively, Edward T. H. Wang and Kaiming Zhao.

From our Readers

Dear Pi:

Klaus Hoechsmann seems to ask the wrong question (Dec. '04) when he wonders about the "703" in the Body Mass Index. What puzzles me is, why the square of the height, instead of the seemingly obvious cube? Using the cube would give a measure of pudginess or build (ignoring a multiplicative constant, it's the proportion one occupies of a cube of the same height as oneself), and it's size-invariant: if you scale a person up proportionally from 1.5 to 2 meters in height, this measure stays the same. But doing that with the official BMI definition implies a 33% increase of BMI. So if BMI is not intended to be a size-invariant measure of build, what is it intended to measure, and why? Can anybody inform us?

Ed Hughes (Ottawa)

Dear Ed:

Thanks for taking the time to think about this formula—a rare and noble deed. At first I was just as surprised as you, but eventually I changed my mind. Pudginess itself is not the problem, it seems to be allowed in small people—but I have never heard a tall person, say, a Sumo wrestler, referred to as "pudgy."

The lanky Don Quixote and his pudgy side-kick Sancho Panza.

Take Don Quixote and his side-kick Sancho Panza as depicted here on the left (I had planned to do this with Goofey and Mickey Mouse—but they are somebody's intellectual property). The two valiant Spaniards seem to be close to what you had in mind: Sancho about 75% of his master's height, as a generous estimate. By your cube idea, this would allow their weights to be in the ratio 64 to 27. If Sancho weighed 108 pounds, a modest mass for such a hearty eater, the Don would be allowed a hefty 256—way too much for his slender mare.

Among people who like to pronounce foreign names, the BMI is known as the Quetelet-Index, after the Belgian mathematician Adolphe Quetelet (1796–1874) who invented it. He is sometimes referred to as the "patriarch of statistics" (not as its father since too many folks claim that honour), and is famous for fitting all kinds of human measurables to the normal curve, in a Quixotic quest for the "average man" (the term, "l'homme moyen" is also his). He began with a sizable data base of 5000 Scottish soldiers, and we can be sure that he laboured diligently to produce a measure of obesity that would still include our picaresque heroes as "normal" types, albeit near the fringe.

But quite apart from any faith in Adolphe, one can argue that a Señor Panza scaled to the height of the Man of La Mancha would likely need medical attention. Even a normal walk would have him panting as each square yard of lung surface must supply oxygen to 33% more flesh, and the arches of his feet would fall as the pressure on his soles increased by the same percentage—not to mention the pain in his knees.

Hoping to hear from you again sometime.

Sincerely,
Klaus

Mathematical Haiku Contest

A haiku is a poem or verse with a strict form of three (unrhymed) lines of five, seven, and five syllables.

For example,

A haiku with rhyme
Would be profoundly sublime
Pity it won't work
Anon

Some time ago, the Department of Mathematics and Statistics at the University of Victoria held a party for which the 'price' of admission was a mathematical haiku. Many fine haikus were forthcoming. Two examples are given below.

Mathematics is:
A flight from reality
Tell me, what is real?
Jacobus Swarts

Chains bounded above?
Want maximal element?
Well now you have it.
John Phillips

You get the idea. So, the editors of *π in the Sky* are asking for your best mathematical haiku. The rules are simple: it must be your own, you may submit more than one, and you must submit no later than May 31, 2006. Please send all entries to *π in the Sky Mathematical Haiku Contest* at pi@pims.math.ca. Entries will be judged by the editors. Our beloved Editor-in-Chief, Ivar Ekeland, will be awarding a prize of CAD$100 for the top mathematical haiku.

Good luck!

David Leeming
Managing Editor, *π in the Sky*

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