# π in the Sky


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*Pi in the Sky* magazine is aimed primarily at high school students and teachers, with the main goal of providing a cultural context/landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

## Contributions Welcome

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## Cover Page

A multitude of mathematical games and puzzles, by Breeonne Baxter.

## Table of Contents

# The Draw of Mathematical Games and Puzzles

by Mark MacLean, Guest Editor, University of British Columbia

Mark MacLean and a trio of children in Colombia.

I was waiting in an airport recently and was surprised at the number of my fellow passengers who were working away at solving Sudoku number puzzles while they waited for our flight to board. Although these logic number puzzles have been around since the late 1970s, Sudoku became an international craze last year and now appears in most major newspapers on a daily basis. People who otherwise deal with almost nothing mathematical in their lives are attracted to the relatively simple rules and the challenges these Sudoku puzzles present. (The last mathematical puzzle that captured public attention was Rubik's Cube, which had a recent brief return to popularity.)

While many non-mathematicians enjoy Sudoku, three mathematicians at the University of Queensland, Australia, spent some time examining the mathematical basis of these popular puzzles. Their lovely article in this issue, "Seeking Solutions to Sudoku Squares," explores not only methods for solving these puzzles, but includes the mathematics behind the puzzle if you want to construct one of your own. Scattered throughout this issue of *Pi in the Sky* are Sudoku puzzles to test out what you learn from the article. (If you want a real challenge, try visiting http://www.samurai-sudoku.com.)

If you add strategy to logic, you get an interesting family of games known as Nim. These games are traditionally played with piles or rows of matches. In one version, players alternately remove some number of matches from one of these piles or rows, with the object to be the player who removes the last match on the table. (Other versions make the person who removes the last match the loser.) Anthony Quas's article, "Nim and Friends" talks about two versions, Nim and Nim Lite. In it, he explores the key mathematical question: "Is there a winning strategy?" I suggest that you get a box of matches or toothpicks to help you as you read through this article. It might be helpful to play a few games with a friend as you try to understand the strategy.

Games become more complicated (and interesting for some) if there is an element of risk attached to the play. The television game show *Deal or No Deal* is highly popular. Prize money is distributed unevenly amongst 26 briefcases and the contestant chooses one of these cases. He or she then gets to open, one at a time, six of the remaining cases to reveal the amounts in each case. At this point, a mysterious "Banker" tries to buy the contestant's original case for some amount of money. The contestant chooses to take the deal, or go on playing. If the contestant elects to refuse the deal, he or she chooses five cases and the process is repeated, and so on, with each round seeing four, three, two and finally one case opened until either the contestant takes the deal, or until there are two cases left and the contestant must decide whether or not to switch cases before opening one to see the prize. *Deal or No Deal* is a sophisticated version of an old game show, Monty Hall's *Let's Make a Deal,* and Ivar Ekeland presents a clear discussion of the probability involved in this game in his article, "Monty Hall and Probability."

We have many other wonderful articles in this issue on Mathematical Puzzles and Games. I hope you enjoy them. I encourage you to work through the challenge problems and puzzles we present in these pages, for surely the best part of mathematics is doing it.

## Sudoku puzzles

Sudoku is a logic-based placement puzzle. The object is to fill the grid so that every column, every row and every 3×3 box contains the digits 1 to 9.

**Easy**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 7 | | 1 | | | | 3 | 6 |
| 1 | 8 | | 3 | | 5 | 4 | | |
| | | 5 | | | 4 | | | 2 |
| 7 | | | | | | 9 | 4 | |
| 9 | | | 8 | | 7 | | | 3 |
| | 3 | 2 | | | | | | 8 |
| 8 | | | 9 | | | 1 | | |
| | | 7 | 5 | | 8 | | 9 | 4 |
| 5 | 1 | | | | 6 | | 8 | 7 |

**Hard**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 7 | | 8 | 1 | 6 | | 5 | |
| 1 | | | | | | | | 2 |
| | | 3 | | | | 8 | | |
| 7 | 4 | | 9 | | 5 | | 2 | 8 |
| | | | | 8 | | | | |
| 3 | 6 | | 1 | | 2 | | 9 | 7 |
| | | 6 | | | | 9 | | |
| 4 | | | | | | | | 5 |
| | 1 | | 2 | 9 | 4 | | 3 | |

To learn more on Sudoku puzzles, including a focus on the mathematical ideas that create Sudoku and similar puzzles, turn to page 7.

Solutions on page 22.

# Games, Puzzles and Problems

by Gordon Hamilton, Masters Academy and College

Games can be broken down into a sequence of puzzles alternating between players.

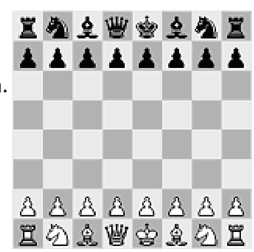
In 1898 Emanuel Lasker (chess world champion from 1894 to 1921) played Victor Wahltuch in England.

**Puzzle**: Below, find the decisive move for Lasker (white).

Usually, these game-based puzzles/problems become progressively more difficult to solve, as one moves backwards from the last move to the beginning of the game.

Earlier in the same game between Emanuel Lasker and Victor Wahltuch.

**Problem**: Find a decisive move for Lasker (white).

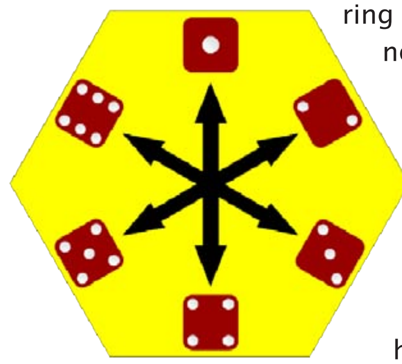


Let's call these **game-puzzles** if they are solved and **game-problems** if they are unsolved. This makes the challenge to find a decisive first move for chess a **game-problem**, because nobody has determined which first move guarantees a win. On the other hand, we refer to Nim (see related article on page 13), tic-tac-toe and rock-paper-scissors as a **game-puzzle,** because you can determine which first move will guarantee a win.

Beyond asking "what is a winning next move," games like Nano-Hopscotch (below) naturally provide a rich source of inspiration for **game-puzzles** and **game-problems**. Reciprocally, puzzles like the Polyanimal Puzzles later in this article are a rich source of inspiration for game designers.

## Nano-Hopscotch

Snap together a bunch of carbon rings to form a platform and then move from one carbon ring to another by rolling a dice. If a dice roll directs you to a carbon ring previously visited - ignore it and roll again. Also ignore any rolls that direct you off the platform. You win the game if you can visit all the carbon rings.



For example, starting at the dot on the heart-shaped platform below (Fig. a) - you couldn't move until you rolled a 3 or 6. You rolled a 3 and then couldn't move until you rolled a 2 or 4 (Fig. b). Finally you rolled a 4 and were stuck (Fig. c).

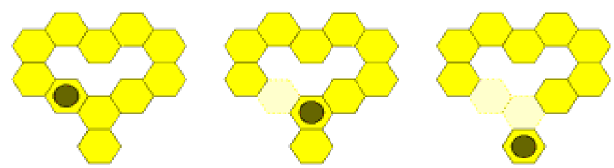


Fig. a Fig. b Fig. c

1) Where could you start on the heart-shaped platform in order to guarantee that you win? Where could you start so that you will win some of the time, but not all of the time?

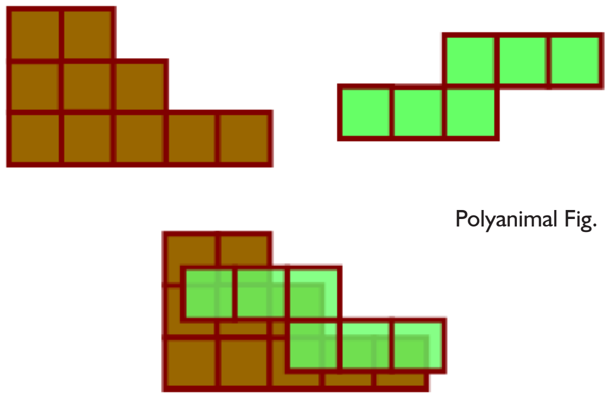
2) If possible, snap together a platform so that it is:

- impossible for you to win.
- impossible for you to lose.
- impossible for you to win from every carbon ring except one.
- possible, but not guaranteed, for you to win from every carbon ring.
- possible for you to win from every carbon ring with the same probability, $p$, where $0<p<1$. (I do not know the answer to this problem.)

3) Find a carbon ring in some platform where the chance of winning is exactly 3/4 if you start on it. (I do not know if 3/4 is the largest possible fraction less than 1.)

## Polyanimal Puzzles

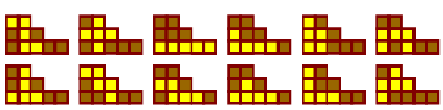
A Polyanimal can eat another Polyanimal if it fits inside. For example, in Polyanimal Fig. 1, the ugly brown giant slug (size 10) can eat the cute green garter snake because the garter snake (size 6) can be flipped over so that it fits inside the giant slug.



Polyanimal Fig. 1

1) Find three Polyanimals of sizes 3, 4 and 5 such that all three can all live happily together (without danger of one being eaten).

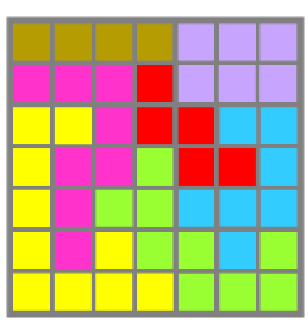
2) A farmer wishes to exterminate the 12 Pentomino pests. He considered renting an ugly brown giant slug (size 10) which can eat all of them (see Polyanimal Fig. 2, below), but wondered whether there was a smaller Polyanimal that would do the job.



Polyanimal Fig. 2

3) Another farmer doesn't have too much money and can only rent a Polyanimal of size 6. Which Polyanimal should he rent to eradicate the most Pentomino pests? If he gets a two-for-one deal, which two creatures of size 6 should the farmer get?

4) Find seven Polyanimals of sizes 4, 5, 6, 7, 8, 9 and 10 that can be happy together in a 7-by-7 cage (Polyanimal Fig. 3). For instance, the full cage contains Polyanimals of the right sizes, but the red fly (size 5) is unhappy because the green Venus flytrap (size 9) can eat her.



Polyanimal Fig. 3

5) Noah was told to build a refrigerator to rescue an infinite number of Polyanimals from global warming. What is the maximum number of Polyanimals of size 3 that he could safely rescue? After Noah rescued these, what is the maximum number of Polyanimals of size 4 that he can safely rescue? After Noah rescued these, what is the maximum number of Polyanimals of size 5 that he can safely rescue?

6) Design a Polyanimal Game.

## Twinkle Twinkle

*Games, puzzles and problems feed off each other. The idea for this problem came when playing connect-the-dots with a kindergarten student:*

Randomly scatter $S$ stars in a unit square universe. For each star – connect it to its two closest neighbours. Constellations result. On the right are the constellations created when 100 stars were scattered randomly.

Twinkle Twinkle - a starfield

The smallest constellation is a triangle. Describe how the largest constellation scales as $S \Longrightarrow$ infinity. (I do not know the answer to this problem.)

## About the author

Dr. Gordon Hamilton (gamesbygord@gmail.com) is a consultant for K-12 math/science with Master's Academy & College, Calgary. The puzzles, games and problems above were mostly created when Gordon worked with the Galileo Educational Network (http://www.galileo.org). The picture of Lasker is from http://en.wikipedia.org/wiki/Emanuel_Lasker.

*Solutions to these problems, as well as to other facinating mathematical challenges, please visit our website: http://www.pims.math.ca/pi.*

# *Pi in the Sky* Mathematical Haiku Contest Winners

We were pleased to receive many entries, 31 in all, to our Mathematical Haiku Contest from the 2005 issue of *Pi in the Sky*. Many thanks to everyone who submitted an entry. Thanks also to teacher Tiffany Godin of École Sainte-Marguerite-Bourgeoys in Calgary for encouraging her Math 31 students to submit a haiku to our contest. Thanks also go to Henry M. Knitter and his Math 20 class at John G. Diefenbaker High School in Calgary for their submission.

Our distinguished panel of judges selected Lisa Walpole's haiku as the contest winner. Lisa teaches senior high math at St. Martin de Porres High School in Airdrie, Alberta. Congratulations to Lisa, who won the $100 prize for her poem.

*A möbius strip*
*Continuously circling*
*Is there no escape?*

Our judges also selected two haikus to receive honourable mention: Don E. Kaplecki, who works at Discovering Choices Outreach High School in Calgary, receives honourable mention for his entry titled *Nature's Way*.

*One, one, two, three, five*
*Eight, thirteen, then twenty-one*
*Creates the pattern*

The Math 31 class at École Sainte-Marguerite-Bourgeoys also receives honourable mention for their entry.

*How do I love thee*
*Let me count the ways*
*T-I 83*

Congratulations to all of our participants and winners, for your creativity and enthusiasm!

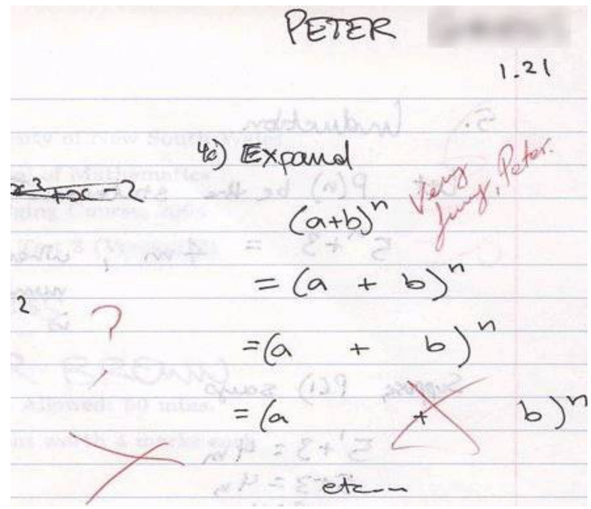
**David Leeming,**
*Managing Editor, Pi in the Sky*

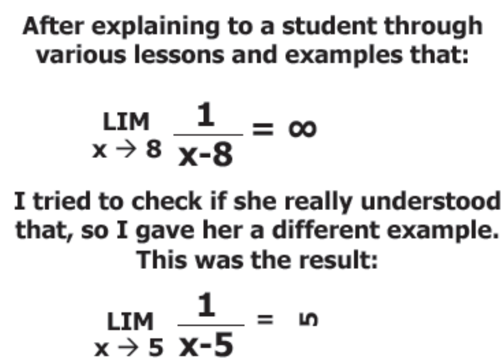
On behalf of *Pi in the Sky*, we would like to thank Alexander Melnikov (University of Alberta) and Len Berggren (University of Victoria) for their support of *Pi in the Sky*. Dr. Melnikov and Dr. Berggren's terms on the Editorial Board have ended.

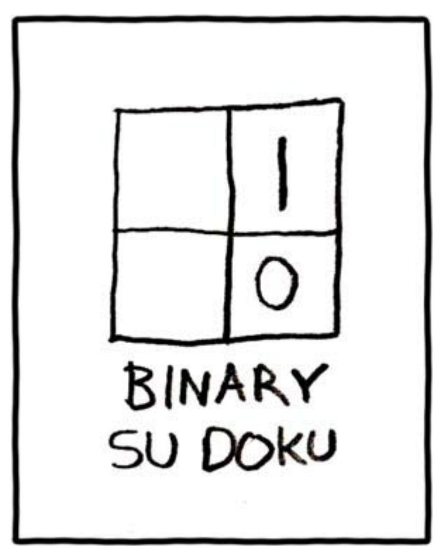
*Pi* welcomes Anthony Quas (University of Victoria) and Gordon Hamilton (Masters Academy and College) to the Editorial Board.

**Ivar Ekeland,**
*Pi in the Sky Editor-in-Chief*

# Math Jokes




*courtesy of xkcd.com*

Send your math jokes to us at pi@pims.math.ca!

# Seeking Solutions to Sudoku Squares

by Diane Donovan, Carlo Hämäläinen and Anne Penfold Street,
Centre for Discrete Mathematics and Computing, University of Queensland, Australia

In this brief article we focus on the mathematical ideas which underpin Sudoku and similar puzzles. We restrict ourselves to studying squares based on 4×4 arrays. However, everything discussed here can be extended to similar puzzles based on $n \times n$ arrays, where $n$ is any square number. Most of us have seen, by now, the 9×9 squares published in the puzzle section of newspapers.

Let us begin by carefully defining these objects. First we will define a well-known mathematical structure called a latin square and then, by placing extra conditions on the latin squares, we obtain Sudoku squares.

A *latin square*, of order $n$, is an $n \times n$ array in which each of the symbols $1,\dots,n$ occurs once in every row and once in every column. So, for example, the following 4×4 arrays are examples of latin squares of order 4.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 3 |
| 3 | 4 | 2 | 1 |
| 4 | 3 | 1 | 2 |

$L_1$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 3 |
| 3 | 4 | 1 | 2 |
| 4 | 3 | 2 | 1 |

$L_2$

A cell $(r,c)$ in a latin square is the intersection of row $r$ with column $c$. For example the intersection of row 3 and column 3, which is cell (3,3), contains the symbol 2 in $L_1$ and the symbol 1 in $L_2$, (both shown above). The cells of a latin square of order 4 can be divided up into four subsquares (or boxes):

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 1 | 4 | 3 |
| 3 | 4 | 2 | 1 |
| 4 | 3 | 1 | 2 |

=

| | |
|---|---|
| *Q*1 | *Q*2 |
| *Q*3 | *Q*4 |

For example, the box *Q*1 has the cells (1,1), (1,2), (2,1), (2,2). A *Sudoku square* of order 4 is a 4×4 latin square in which each of the four boxes contains each of the symbols 1,2,3,4 precisely once. In newspapers the usual size of a Sudoku square is 9×9, that is, a 9×9 latin square in which each of the 9 boxes contains each of the symbols 1,2,...,9 precisely once. If we take a 9×9 Sudoku square we will have 9 boxes:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 9 | 7 | 6 | 1 | 5 | 8 | 4 | 2 |
| 2 | 1 | 6 | 8 | 4 | 9 | 3 | 7 | 5 |
| 4 | 8 | 5 | 7 | 3 | 2 | 6 | 9 | 1 |
| 5 | 4 | 2 | 1 | 7 | 8 | 9 | 3 | 6 |
| 1 | 6 | 9 | 3 | 5 | 4 | 7 | 2 | 8 |
| 8 | 7 | 3 | 9 | 2 | 6 | 1 | 5 | 4 |
| 9 | 2 | 1 | 4 | 6 | 7 | 5 | 8 | 3 |
| 6 | 5 | 8 | 2 | 9 | 3 | 4 | 1 | 7 |
| 7 | 3 | 4 | 5 | 8 | 1 | 2 | 6 | 9 |

Neither of the latin squares $L_1$ and $L_2$ above is a Sudoku square, but if, in each case, we swap rows 2 and 3 we obtain the following two Sudoku squares, of order 4.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 2 | 1 |
| 2 | 1 | 4 | 3 |
| 4 | 3 | 1 | 2 |

$S_1$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 |
| 2 | 1 | 4 | 3 |
| 4 | 3 | 2 | 1 |

$S_2$

Mathematically, we can show that the two latin squares $L_1$ and $L_2$, of order 4, are not equivalent. By this we mean that it is not possible to turn one of these squares into the other by any combination of one or more of the following operations: changing the order of the rows; changing the order of the columns; or renaming the elements (for example, changing 1, 2, 3, 4 to 3, 2, 4, 1). However, we can also show that there are at most two non-equivalent latin squares of order 4, and so there are precisely two non-equivalent latin squares of order 4 (see the biographical note at the end of the article).

This information tells us that there are essentially two Sudoku squares $S_1$ and $S_2$, of order 4. To see this, assume that there are three "different" Sudoku squares of order 4. This means that there are three "different"

continued on page 8

latin squares of order 4, which we have just seen there are not. Hence the result follows.

We define a *Sudoku Puzzle* of order 4 to be a 4×4 Sudoku square with some cells containing the symbols 1,2,3,4 and other cells empty. This means that we require that the empty cells can be filled in such a way as to obtain a Sudoku square of order 4. We call a partial Sudoku square a Sudoku Puzzle only if it is completable, as specified by the following rule:

**Rule:** The symbols 1, 2, 3, 4 are to be placed in the empty cells of the 4×4 partial Sudoku square, in such a way that the completed array has each of the four symbols occurring once in every row, once in every column and once in every box *Q*1, *Q*2, *Q*3 and *Q*4.

A Sudoku puzzle is said to be *valid* if it has precisely one completion. Here is a valid Sudoku puzzle of order 4×4.

| | | | |
|---|---|---|---|
| 1 | 2 | | |
| | | | |
| 2 | | | |
| | | | 3 |

*P*

## Methods for solving Sudoku puzzles

The puzzle *P* is fairly easy to solve by hand, but is there a general procedure, an *algorithm*, that we can use on any puzzle? The algorithm should be made up of simple steps that we could code on a computer. For Sudoku puzzles, the algorithm is based on the array of alternatives. For each empty cell, we list the plausible symbols. We can do this by using the fact that each symbol can occur once in every row, once in every column and once in each of the four boxes. Once we have these lists we try to fill in some cell of the puzzle, and then keep repeating the process.

The steps involved in obtaining a solution to the example above are:

**Step 1** Construct the array of alternatives *A*(*P*) for the Sudoku puzzle *P*.

| | | | |
|---|---|---|---|
| 1 | 2 | | |
| | | | |
| 2 | | | |
| | | | 3 |

*P*

| | | | |
|---|---|---|---|
| | | {3,4} | {4} |
| {3,4} | {3,4} | {1,2,3,4} | {1,2,4} |
| | {1,3,4} | {1,4} | {1,4} |
| {4} | {1,4} | {1,2,4} | |

*A*(*P*)

Two corner cells have just one symbol each in their lists so we can fill in these cells in *P*.

**Step 2** Update the square *P* and then construct a new array of alternatives for the updated square.

| | | | |
|---|---|---|---|
| 1 | 2 | | 4 |
| | | | |
| 2 | | | |
| 4 | | | 3 |

| | | | |
|---|---|---|---|
| | | {3} | |
| {3} | {3,4} | {1,2,3} | {1,2} |
| | {1,3} | {1,4} | {1} |
| | {1} | {1,2} | |

We can now see that two more 1s are forced and so are two more 3s.

**Step 3** Update the square shown on the left above and construct the array of alternatives for this square.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | | | |
| 2 | | | 1 |
| 4 | 1 | | 3 |

| | | | |
|---|---|---|---|
| | | | |
| | {4} | {1,2} | {2} |
| | {3} | {4} | |
| | | {2} | |

At this point we could easily complete the square by hand, but for the sake of demonstrating our algorithm we will continue the process to the end.

**Step 4** Update the square shown above on the left and construct the array of alternatives for this square.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | | 2 |
| 2 | 3 | 4 | 1 |
| 4 | 1 | 2 | 3 |

| | | | |
|---|---|---|---|
| | | | |
| | | {1} | |
| | | | |
| | | | |

***Solution***: Now each of the remaining empty cells has only one possible entry and so we obtain our solution.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 |
| 2 | 3 | 4 | 1 |
| 4 | 1 | 2 | 3 |

More generally, the square *A*(*P*) is known as the *array of alternatives* for the puzzle *P*. It has the properties that for each empty cell (*r*,*c*) of *P*, the corresponding cell (*r*,*c*) of *A*(*P*) contains the list of all symbols not occurring in row *r* or column *c* of *P*. All other cells of

# Sudoku Squares

continued

$A(P)$ are empty. If there exists a cell, say $(i,j)$ of $A(P)$ which contains only one alternative, say symbol $x$, then we say that the symbol $x$ is forced to occur in cell $(i,j)$ of $P$.

For instance, at Step 1 the only alternative for cells (1,4) and (4,1) was the symbol 4. Hence the symbol 4 was forced to occur in these cells of $P$. Indeed, at each step in the solution process we obtained forced entries.

## The construction of Sudoku puzzles

For the moment let's forget about solving these puzzles, and instead think about trying to construct them. What sort of properties should our partial array have? It's easy to find partial squares that complete to a Sudoku square. The hard part is trying to find a partial square that has just one solution. To explore this idea consider the partial Sudoku square $Q$, (below on the left) and its array of alternatives $A(Q)$ (on the right).

| | | | |
|---|---|---|---|
| | 2 | | |
| 3 | | | |
| 2 | | | |
| | | | 3 |

$Q$

| | | | |
|---|---|---|---|
| {1,4} | | {1,3,4} | {1,4} |
| | {1,4} | {1,2,4} | {1,2,4} |
| | {1,3,4} | {1,4} | {1,4} |
| {1,4} | {1,4} | {1,2,4} | |

$A(Q)$

With a little work we can see that there are at least two solutions (shown below). One reason for this is that both the symbols 1 and 4 are missing from the partial Sudoku square $Q$ so that, if we find either of the two solutions below, we can obtain the other by swapping 1 and 4.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 |
| 2 | 3 | 4 | 1 |
| 4 | 1 | 2 | 3 |

| | | | |
|---|---|---|---|
| 4 | 2 | 3 | 1 |
| 3 | 1 | 4 | 2 |
| 2 | 3 | 1 | 4 |
| 1 | 4 | 2 | 3 |

Thus we can state the following property.

**Property 1** A valid Sudoku puzzle of order 4 must contain at least 4-1=3 distinct symbols. (Obviously, this can be extended to 9-1=8 symbols for a valid Sudoku puzzle of order 9, etc.)

We can also see that we must have at least one symbol in each of the pairs of rows 1 and 2, and rows 3 and 4, and likewise for the columns. This is because rows 1 and 2, for instance, are contained in the boxes $Q_1$ and $Q_2$, and similarly for the other pair of rows and the pairs of columns. This leads to Property 2.

**Property 2** A valid Sudoku puzzle of order 4 must contain at least one symbol in rows 1 and 2, and at least one symbol in rows 3 and 4. A valid Sudoku puzzle of order 4 must contain at least one symbol in columns 1 and 2, and at least one symbol in columns 3 and 4.

Another property that is not so obvious is described below. Consider cells (1,1), (1,2), (3,1), (3,2) of the Sudoku squares $S_1$ and $S_2$.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 2 | 1 |
| 2 | 1 | 4 | 3 |
| 4 | 3 | 1 | 2 |

$S_1$

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 3 | 4 | 1 | 2 |
| 2 | 1 | 4 | 3 |
| 4 | 3 | 2 | 1 |

$S_2$

| | | | |
|---|---|---|---|
| 1 | 2 | | |
| | | | |
| 2 | 1 | | |
| | | | |

$I$

| | | | |
|---|---|---|---|
| 2 | 1 | | |
| | | | |
| 1 | 2 | | |
| | | | |

$I'$

Any valid Sudoku puzzle, say $P$, which is a subset of $S_1$ or $S_2$ must contain an entry in one of the cells (1,1), (1,2), (3,1), (3,2), as shown in $I$. The reason for this is that, if it doesn't, then there is nothing to stop us swapping these symbols around, as shown in the square $I'$.

Similarly we can show that $P$ must contain at least one entry from the cells (2,1), (2,2), (4,1), (4,2), one from (1,3), (1,4), (3,3), (3,4) and one from (2,3), (2,4), (4,3), (4,4). Thus we may deduce Property 3.

**Property 3** Any valid Sudoku puzzle of order 4 contains at least four entries.

Since the 16 cells we have just considered are all in different positions, we can deduce Property 3. Don't worry if you can't see this particular property immediately, but it is true.

Insights such as these give us some indication on how to construct valid Sudoku puzzles.

## Harder Sudoku Puzzles

Newspaper Sudoku puzzles are usually graded as being easy, medium, or difficult to solve. What about 4×4 Sudoku puzzles? It turns out that they are all "easy" in the sense that, at each stage of the solution, there is at least one cell in the array of alternatives which contains only one symbol.

We will attempt to build a "hard" 4×4 valid Sudoku puzzle, $P$. By hard we mean, that in addition to being a valid puzzle, its array of alternatives exhibits the following properties:

**H1:** any empty cell of $P$, and thus any filled cell of $A(P)$, will contain at least two distinct possible alternatives, making our puzzle harder to solve;

**H2:** no row or column of $P$ is completely filled and no symbol occurs four times in $P$, avoiding redundant information.

continued on page 11

# Monty Hall and Probability

by Ivar Ekeland, University of British Columbia

Many of us have trouble enough dealing with numbers. However, when it comes to probabilities, the situation is much worse. It is easy to devise simple problems that will baffle even sophisticated mathematicians.

This may be because people have been counting and measuring for millennia, while probability is a mere four hundred years old. The ancient Greeks, for instance, knew that there were infinitely many prime numbers and were familiar with Pythagoras' theorem. But they would not have understood what we mean when we say that flipping a coin gives heads with a probability 1/2. The idea appears for the first time in 17th century France, and it seems that there has not been time enough for the next generations to get used to it.

A classical example of a baffling problem is the so-called the Monty Hall paradox. According to Wikipedia: *Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what lies behind the doors, opens another door, No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?*


You chose door number 3. The goat is not behind door number 1. Do you change your mind and pick door number 2?

Most people say no: they prefer to stick to their original choice. When they first choose, all three doors were closed, and they picked one at random, so the probability that it was the right one was 1/3. Now only two doors are closed, and they figure that since they still don't know where the car is, the two doors must have equal probability, namely 1/2. There is no reason to change, and indeed, when given the opportunity, most people do not.

In fact, it is a mistake: door No. 1 still has a probability 1/3 of winning, but door No. 2 now has a probability of 2/3, so switching doubles your chances of winning. The preceding argument would be correct if door No. 3 had been open to begin with, but it was not. What happened is that all doors were closed, so that there was a probability 1/3 that the car was behind door No, 1, and a probability 2/3 that it was behind one of the others. If it actually is behind door No. 2 or door No. 3, the host will open the other one, but that will certainly not affect the fact that the car is in the block {2,3}. So there is still a probability 2/3 that it is in the block {2,3}, and if it is not in 3, which the host showed by opening that door, then there must be a probability 2/3 that it is in 2.

You are not convinced? There are more sophisticated arguments, which you will find in the Wikipedia article. You have read them and you are still not convinced? Welcome to a large club of confused contestants! The Monty Hall paradox has a long history, and many people adamantly refuse to be convinced that one should switch. The last resort is to try it. On http://www.shodor.org/interactivate/activities/SimpleMontyHall, you will be able to try the two strategies, of switching and not switching, and finding out for yourself, which is the right one.

The Monty Hall paradox is a misnomer. A paradox is a statement that contradicts itself, but there is no contradiction in this example, only a wrong answer to a reasonable question. Now here comes a true paradox in probability. I found it in "A Mathematician's Miscellany," a collection of thoughts and remarks by the Cambridge mathematician John Littlewood. It goes as follows: Imagine a pack of cards, each of which has one number on one side and the number directly above on the other. There is one card with 1 on one side and 2 on the other, two cards with 2 on one side and 3 on the other, four cards with 3 and 4, eight cards with 4 and 5, and so on, ad infinitum, so that there are $2^{n+1}$ cards with $n$ on one side and $n$+1 on the other.

Let us now use that pack to play a game of chance.

# Monty Hall and Probability
continued

Two people, Ann and Brian, stand facing each other, while the host draws one card from the pack and puts it between them, so that each can see the side facing him or her, but not the other side. The winner is the one who sees the lowest number.

What is Ann's probability of winning? If she sees a 1, the other side must be a 2, and she has won. If she sees a number $n>1$, the hidden number, on the other side of the card, is either $n$-1 or $n$+1. In the first case she loses, in the other case she wins. Since there are twice as many cards of the second type as there are cards of the first, she has a probability 2/3 of winning.

Unfortunately, the same argument holds for Brian: he also has a probability 2/3 of winning. Since one must win, but not both, the two probabilities should sum to 1, so that 2/3 + 2/3 =1, a remarkable equality, which should never have been published in *Pi in the Sky*.

This is a true paradox. The argument is perfectly correct. The only problem with it is that there is no such pack of cards. One cannot physically construct a pack with infinitely many cards. Even if one could, what this argument shows is that one could not draw at random a card from it, otherwise one would end up with a contradiction. In other words, even in mathematics, there is no such thing as drawing a card at random from an infinite pack.

# Sudoku Squares
continued

We will start with an empty array $P_0$ and a full array of alternatives $A(P_0)$. Then iteratively we will try to construct $P_1$, $P_2$,... and simultaneously $A(P_1)$, $A(P_2)$,... so that eventually $A(P_i)$ has the desired Properties H1 and H2, mentioned above.

| | | | |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |

$P_0$

| | | | |
|---|---|---|---|
| {1,2,3,4} | {1,2,3,4} | {1,2,3,4} | {1,2,3,4} |
| {1,2,3,4} | {1,2,3,4} | {1,2,3,4} | {1,2,3,4} |
| {1,2,3,4} | {1,2,3,4} | {1,2,3,4} | {1,2,3,4} |
| {1,2,3,4} | {1,2,3,4} | {1,2,3,4} | {1,2,3,4} |

$A(P_0)$

We will look at various cases, and we may as well start by assuming that cell (1,1) of $P_1$ contains the symbol 1 (can you explain why we can do this?). To explore possibilities, we will also assume that symbol 2 occurs in cell (2,3) of $P_1$. This situation is summarized as:

| | | | |
|---|---|---|---|
| 1 | | | |
| | | 2 | |
| | | | |
| | | | |

$P_1$

| | | | |
|---|---|---|---|
| | {2,3,4} | {3,4} | {3,4} |
| {3,4} | {3,4} | | {1,3,4} |
| {2,3,4} | {1,2,3,4} | {1,3,4} | {1,2,3,4} |
| {2,3,4} | {1,2,3,4} | {1,3,4} | {1,2,3,4} |

$A(P_1)$

However, when we apply the ideas we have just discussed to Box Q1 we see that symbol 2 is forced to be in cell (1,2) of $P_2$, and that in Box Q2 symbol 1 is forced to be in cell (2,4) of $P_2$, giving:

| | | | |
|---|---|---|---|
| 1 | 2 | | |
| | | 2 | 1 |
| | | | |
| | | | |

$P_2$

| | | | |
|---|---|---|---|
| | | {3,4} | {3,4} |
| {3,4} | {3,4} | | |
| {2,3,4} | {1,3,4} | {1,3,4} | {2,3,4} |
| {2,3,4} | {1,3,4} | {1,3,4} | {2,3,4} |

$A(P_2)$

By Property 1 we know that one of symbols 3 and 4 must occur in $P_3$, and by the constraints, Properties H1 and H2, that we have placed on $A(P_3)$ we know that neither of them can occur in rows 1 or 2. So we can assume that cell (3,1) of $P_3$ contains symbol 3. But this is not possible, as it leads to only one alternative for cell (2,1) of $A(P_3)$ (namely {4}). If we choose to place symbol 4 in cell (2,1) of $P_3$, we violate Properties H1 and H2.

Hence we may assume that this case is not possible and so, if cell (1,1) of $P_3$ contains symbol 1, then we can't place any of the symbols 2,3 or 4 in cells (2,3),(2,4),(3,2),(4,2) of $P_3$. Note also that the case with symbol 1 in cell (1,1) of $P_3$ and symbol 2 in cell (1,2) is also equivalent to this case, and hence not possible.

Now consider the case with symbol 1 in cell (1,1) of $P_3$ and symbol 2 in cell (1,3).

| | | | |
|---|---|---|---|
| 1 | | 2 | |
| | | | |
| | | | |
| | | | |

$P_3$

| | | | |
|---|---|---|---|
| | {3,4} | | {3,4} |
| {2,3,4} | {2,3,4} | {1,3,4} | {1,3,4} |
| {2,3,4} | {1,2,3,4} | {1,3,4} | {1,2,3,4} |
| {2,3,4} | {1,2,3,4} | {1,3,4} | {1,2,3,4} |

$A(P_3)$

continued on page 12

# Sudoku Squares
continued

By Property 1 we may assume symbol 3 occurs in at least one cell of $P_3$. It can't occur in columns 2 or 4. So we may assume symbol 3 occurs in cell (3,1) (all other possibilities are equivalent or reduce to an earlier case). Thus we have:

| | | | |
|---|---|---|---|
| 1 | | 2 | |
| | | | |
| 3 | | | |
| | | | |

$P_3$

| | | | |
|---|---|---|---|
| | {3,4} | | {3,4} |
| {2,4} | {2,3,4} | {1,3,4} | {1,3,4} |
| | {1,2,4} | {1,4} | {1,2,4} |
| {2,4} | {1,2,4} | {1,3,4} | {1,2,3,4} |

$A(P_3)$

Now this square still has at least two completions, so we must add another entry. If we add an entry to any cell other than (3,3) we are reduced to an earlier argument. Thus we may assume we are adding either symbol 1 or symbol 4 to cell (3,3). In the former case we have

| | | | |
|---|---|---|---|
| 1 | | 2 | |
| | | | |
| 3 | | 1 | |
| | | | |

$P_4$

| | | | |
|---|---|---|---|
| | {3,4} | | {3,4} |
| {2,4} | {2,3,4} | {3,4} | {1,3,4} |
| | {2,4} | | {2,4} |
| {2,4} | {2,4} | {3,4} | {2,3,4} |

$A(P_4)$

but there are still two completions and adding any entry gives an earlier case. So alternatively we take

| | | | |
|---|---|---|---|
| 1 | | 2 | |
| | | | |
| 3 | | 4 | |
| | | | |

$P_4$

| | | | |
|---|---|---|---|
| | {3,4} | | {3,4} |
| {2,4} | {2,3,4} | {1,3} | {1,3,4} |
| | {1,2} | | {1,2} |
| {2,4} | {1,2,4} | {1,3} | {1,2,3} |

$A(P_4)$

Once again we have two completions and the argument reduces to an earlier case.

There are still a few more cases to argue, for instance symbol 1 in cell (1,1) and symbol 2 in cell (2,2), but the reasoning so far seems to suggest that it is not possible to construct a "harder" Sudoku puzzle of order 4. We have checked this result by generating all possible valid Sudoku puzzles of order 4 and the corresponding arrays of alternatives, and in all cases there is always at least one empty cell of the Sudoku puzzles for which there is only one possible entry. You might like to see if you can complete the theoretical proof by checking the remaining cases.

## Squares of order greater than 4

If we want harder puzzles we need to look at orders greater than 4. We can look at Sudoku squares of order 9, but they are pretty big. Alternatively we could define a new puzzle! Even a 6×6 array is pretty big so, although it would be nice to have subsquares or boxes sitting inside our array, we are going to choose a smaller case.

We define a *latin square Puzzle*, of order 5, to be a 5×5 array with some cells containing the symbols 1, 2, 3, 4, 5 and other cells empty. However, we require that these empty cells can be filled in such a way as to obtain a latin square of order 5. That is, we take a partial latin square of order 5 and fill in the empty cells using the following rule.

**RULE:** The symbols 1, 2, 3, 4, 5 are to be placed in the empty cells of the 5×5 partial latin square, in such a way that the completed array has each of the five symbols occurring once in every row and once in every column.

This latin square puzzle is said to be *valid* if it has only one completion. The following example is a valid 5×5 latin square puzzle, but in terms of the discussion given above, the array of alternatives can't be completed by considering only forced entries. So in this sense it can be termed a "hard" latin square puzzle.

| | | | | |
|---|---|---|---|---|
| | | | | 5 |
| | 1 | 4 | | |
| 3 | | 5 | | |
| 4 | | 2 | | |
| | | | 2 | 4 |

*P*

| | | | | |
|---|---|---|---|---|
| {1,2} | {2,3,4} | {1,3} | {1,3,4} | |
| {2,5} | | | {3,5} | {2,3} |
| | {2,4} | | {1,4} | {1,2} |
| | {3,5} | | {1,3,5} | {1,3} |
| {1,5} | {3,5} | {1,3} | | |

*A(P)*

Finally, to find out more about Sudoku puzzles you might like to look at the Wikipedia website, http://en.wikipedia.org/wiki/Sudoku.

We thank Ken Gray for helpful discussions.

***Bibliographic note***: J Denes and A D Keedwell, Latin Squares and Their Applications, The English University Press Ltd., London 1974.

**To play Sudoku online:**

http://www.websudoku.com
http://www.dailysudoku.com
http://www.sudokupuzz.com

# Nim and Friends

by Anthony Quas, University of Victoria

Nim is a family of games played with matchsticks, where each player takes it in turn to remove matches according to various rules, where the winner is the player who takes the last match (or in some versions the player forcing the other player to take the last match).

We will talk here about two versions of the game that I will call Nim and Nim Lite.

In Nim Lite, there is just one pile of any number of matches (games often start from 15), and players take turns to remove 1, 2 or 3 matches. The player removing the last match is the loser.

In Nim, there are several piles of matches. Players take turns to remove any (positive) number of matches from any single pile. In this version, the player removing the last match is the winner. A game might start for instance with 5 piles of 1, 2, 3, 4 and 5 matches. In this case a possible move would be to take two matches from the 4 pile leaving piles of 1, 2, 3, 2 and 5 matches. Another possible move would be to take all of the 3 pile, leaving four piles of 1, 2, 4 and 5 matches.

*photograph by Katia Grimmer-Laversanne*

In Nim Lite, players take turns removing 1, 2 or 3 matches from a single pile. The player to remove the last match is the loser.

Of course, as mathematicians, when faced with a game like this, it is natural to look for a *winning strategy*. What would this mean? It cannot mean a way of winning from every position, because if both players knew the winning strategy, then the first player would expect to be able to win from where he or she started, yet whatever the first player did, the second player would expect to be able to win from there as well.

Having said this, there are some obvious winning positions. In Nim Lite, for instance, a position where there are exactly 2, 3 or 4 matches remaining is a winning position for whoever is about to play. The player should simply take all of the matches but one. The other player is then forced to take the last match.

This suggests that we should classify positions in the game into winning and losing positions. We will call them *W* and *L*. The idea will be that if you start to play in a position in *W*, then if you play perfectly, whatever your opponent does, you should be able to win the game; whereas if you play starting from a position in *L*, then if your opponent plays perfectly, he should be able to win.

Let us study Nim Lite some more. We will label positions by the number of matches. Clearly 1 is a losing position (we will write $1 \in L$) as you are then forced to take the last match. As remarked above 2, 3 and 4 are winning positions $2,3,4 \in W$). Starting from 5, you can take 1, 2 or 3 matches leaving your opponent in 4, 3 or 2 from which they can win.

It follows that $5 \in L$. Continuing to argue this way, you can eventually classify all positions as winning or losing.

Notice that the reason that 5 was in *L* was that any move you made left your opponent in *W*. This gives us a way to build up *L* and *W* more systematically:

**Step 0** (Initializing) Start off by putting into *L* all positions where every move causes you to lose in 1 turn; similarly put into *W* all positions where you can win in a single turn. (For Nim Lite, this means putting 1 into *L*, whereas for Nim, this means putting all combinations with only one non-empty pile into *W*.)

**Step 1:** (Expanding *W*) Find all positions not already classified where there is a move that leaves the game in *L*, and add these positions to *W*.

**Step 2:** (Expanding *L*) Find all positions not already classified where **every** move leaves the game in a position in *W*; Add these positions to *L*.

**Repeat:** Repeat steps 1 and 2 until all desired states are classified.

For Nim, we will encode positions by recording the number of matches in each pile (e.g. the position (3,2,1) means that there are 3 piles with 3, 2 and 1 matches). Since empty piles are not relevant to the game, we will write (4,3) instead of (4,0,3). Also the position (5,3,1) should be counted as the same as the position (3,1,5) as the order of the piles doesn't affect the state of the game. For clarity, we will list the piles in decreasing order of size.

By following the procedure described above, we no-

continued on page 14

tice that any position with just one pile (i.e. any position encoded as ($n$) ) belongs to $W$ (the player should just take all the matches).

If we try to expand $W$, there are no positions where

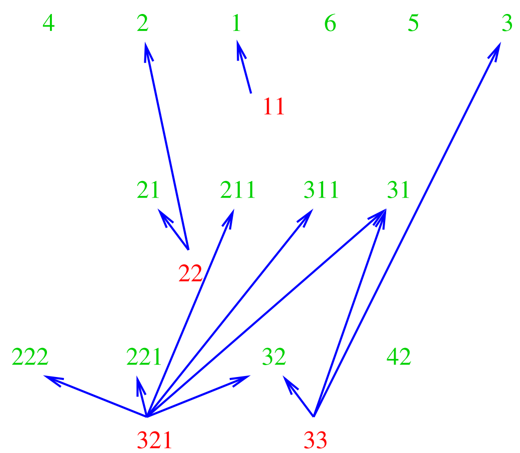



**Figure A**: Any move from a losing (red) position takes you to a winning (green) position.

every move takes you into $L$ (because $L$ doesn't have anything in yet).

Let's try to expand $L$. We're looking for positions where every possible move takes you into $W$ (the positions with just a single pile). We see that we any position like this must have 2 piles; and since we require that every move leaves you in $W$ the only position to add to $L$ is (1,1). (Whichever match you take, you're left with the position (1), which is a winning position for your opponent). Anything else doesn't have the right property (e.g. (2,1) doesn't work because there are some moves that leave the opponent in $W$, i.e. leaving (1) or (2) but also some moves that don't leave the opponent in $W$, i.e. leaving (1,1).)

So far $W = \{(n) : n \geq 1\}$; $L = \{(1,1)\}$. We try again to expand $W$. Anything where you can leave the opponent in $L$ will do. We can check that $(n,1)$ for $n \geq 2$ should be added (as you can take off all but 1 match from the first pile leaving a position in $L$). Similarly, $(n,1,1)$ for $n \geq 1$ should be added to $W$ as you can take the entire first pile leaving your opponent a position in $L$.

Expanding $L$ is a bit more tricky now, but starting from (2,2), we either get to (2,1) or (2,0). Since both are in $W$, this must be a losing position. (1,1,1,1) is another losing position as from here you have to go to (1,1,1), which is a winning position.

Working in this way, we can get a complete classification of positions with no more than a fixed number of matches. This classification is shown for all positions involving no more than 6 matches with no more than 3 piles in Figures A and B. The first shows that whatever you do from a losing position, you hand your opponent a winning position, whereas the second shows for each winning position how to hand your opponent a losing position. This second figure therefore constitutes our winning strategy: whatever position you are handed, if it is a winning position, hand your opponent a losing position (so that if you are handed 221, Figure B shows that you should take the single match and hand your opponent 22). If it is a losing position, there is nothing you can do to help yourself, so take a single match.

Notice that in each of the $W$ positions, there is *some*

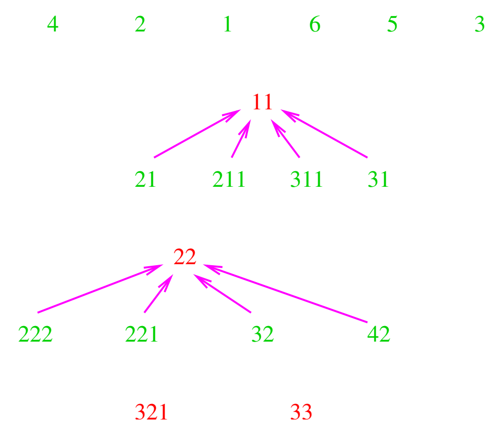



**Figure B**: From any winning position, there is a move to a losing position.

move that takes it to the $L$ on the next row up; whereas for each of the $L$ positions, *any* move leaves the game in a $W$ position on one of the higher rows.

One question that we haven't answered so far is "why do *all* positions get classified?" To answer this, suppose that at some stage, all positions involving 6 matches are classified. Then at the next stage, for any position involving 7 matches, **either** all positions that you can move to are $W$ positions (in which case the given position should be an $L$); **or** there is at least one position that you can move to which is an $L$ (in which case the given position should be a $W$). At the next stage, all positions involving 8 matches will be classified and so on. (Readers familiar with induction will recognize this as being an argument by mathematical induction).

In one sense, this solves the problem. Given a computer, it is now possible to enumerate all winning and losing positions up to 20 matches (or 200 matches or 2000 matches). As a human player though, this isn't very satisfactory. It would be nice to have a way of working out if a position is in $W$ or $L$ without going through all of the above computation; and also: if you're in $W$, telling you how to get into $L$.

This was in fact done by Charles Bouton in 1901. His rule was as follows: write the number of matches in each pile in binary and take the **exclusive-or** of these numbers. A losing position is one in which the answer is 0. All other positions are winning positions. For example if the piles contain 5, 4, 3, 2 and 1 matches, we first write these in binary as 101, 100, 11, 10 and 1.

Taking the exclusive-or is similar to addition without carry. We look at the 1s column and "add" to make 1 if there are an odd number of 1s; 0 if there are an even number of 1s. Similarly in all of the other columns. We'll write $\oplus$ for this operation. As an example, we calculate $101\oplus100\oplus11\oplus10\oplus1$. First, we'll make the numbers all have the same number of digits so we're calculating $101\oplus100\oplus011\oplus010\oplus001$. In the 4s column, there are two 1s so the "sum" is 0. In the 2s column, there are also two 1s so the sum is 0, whereas in the 1s column there are three 1s so the sum is 1. The "exclusive-or sum" is 001.

This operation has all the nice properties of regular addition (commutativity: $a\oplus b = b\oplus a$, associativity: $(a\oplus b)\oplus c = a\oplus(b\oplus c)$). In addition, each element is its own inverse so that $a\oplus b=0$ if and only if $a=b$. We will use these properties below.

In this notation, we have $L=\{(a_1,\ldots,a_k) : a_1\oplus\ldots\oplus a_k=0\}$. All other positions belong to $W$. Since in the case where there are 5, 4, 3, 2 and 1 matches, $101\oplus100\oplus11\oplus10\oplus1\neq 0$, this is a winning position. To play perfectly, the player then has to move to a position whose exclusive-or sum is 0. This can be done in various ways. In fact, moving to any of (4,4,3,2,1), (5,4,2,2,1) or (5,4,3,2) will do the trick (e.g. to see that $(5,4,2,2,1) \in L$, we calculate $101\oplus100\oplus010\oplus010\oplus001=000$).

Whenever the exclusive-or sum is 0, any move will move it away from zero. To see this, we argue as follows. Suppose that the piles are $a_1,a_2,\ldots,a_k$ and $a_1\oplus a_2\oplus\ldots\oplus a_k=0$ and that we are going to change the first pile. We exclusive-or both sides of the equation with $a_1$ giving $a_1\oplus a_1\oplus a_2\oplus\ldots a_k=a_1$. Now, since $a_1\oplus a_1=0$, we see that $a_2\oplus\ldots a_k=a_1$. If we replace $a_1$ by $b$, the exclusive-or sum is now $b\oplus a_2\oplus\ldots a_k=b\oplus a_1$ (using the previous equation), which by the properties listed above is non-zero. Exactly the same analysis works if we change a different pile. We have checked that any move starting from a position in $L$ moves it to a position in $W$.

We also need to show that if the exclusive-or sum is non-zero, then there's a move which will return it to 0. Suppose that $a_1\oplus\ldots\oplus a_k=b\neq0$. Then we have to replace one of the $a_i$ by $a_i\oplus b$ to make the exclusive-or sum back to 0. We also need to ensure that $a_i\oplus b$ is smaller than $a_i$ (so that it corresponds to removing matches from the $i^{th}$ pile). To see this, suppose that the leading digit of $b$ is a 1 in the $j^{th}$ digit (corresponding to the $2^j$ column). Then at least one of the $a$s must have a 1 in the $j^{th}$ digit, $a_i$ say (otherwise the exclusive or sum of the $a$s would have a 0 in the $j^{th}$ digit). Forming $a_i\oplus b$ gives a number which now has a 0 in the $j^{th}$ column but is equal to $a_i$ in all more significant digits. This means that $a_i\oplus b$ is smaller than $a_i$. This gives a strategy for returning the configuration to the losing set. This shows that for any position in $W$, there is a way to move to a position in $L$.

The only thing left to make sure that this is a complete strategy is to ensure that the situation Ø in which there are no matches remaining is in $L$. This corresponds to the fact that if your opponent took the last match, they won the game. This is the same as saying if you add up no numbers, you get 0. Similarly, taking the exclusive-or of the empty set gives 0.

This gives a complete solution to Nim. We give one more example to show how to use it in practice. Suppose a player is faced with the combination (9,8,5,2). What should she do? We compute $1001\oplus1000\oplus101\oplus10=110$. This means that we need to exclusive-or one of the pile sizes with 110 so as to reduce the number of matches. $1001\oplus110=1111$ representing the number 15 which is greater than 9 is not acceptable. Similarly $1000\oplus110=1110$ which is greater than 8 so this move is not acceptable either. On the other hand $101\oplus110=11$ representing 3. This tells us that a suitable move (in fact the only winning move) is to remove 2 matches from the 5 pile leaving 3.

We leave readers with a challenge: Find a complete solution to Nim Lite. This should be done by building up the set of winning and losing positions and then attempting to give a complete description of the sets $L$ and $W$. It should then be possible to prove that any move from a position in $L$ leaves a position in $W$, whereas any position in $W$ should have a move leaving the game in state $L$. Since the object was not to take the last match, it should also be the case that 0 belongs to $W$ (corresponding to the fact that if you are handed the 0 position, your opponent just took the last match so that you won).

# Das Mathematikum

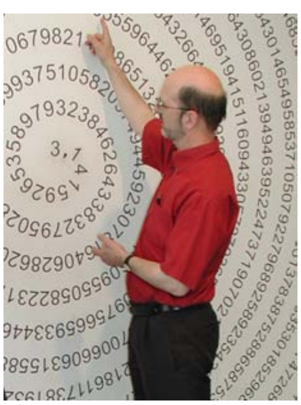
*What follows is a conversation with Albrecht Beutelspacher, Professor at the University of Giessen in Germany, and creator as well as director of the world's first large-scale mathematics museum. The interview took place in the summer of 2004, with Professor Günter Törner of Universität Duisburg-Essen and Klaus Hoechsmann of PIMS asking the questions for Pi.*

**PI: Herr Beutelspacher, how did all this come about? How did you even get the idea? Was it a sort of flash or did it dawn on you slowly? Then, how did you go about making it a reality?**

AB: It was no flash at all; rather, it was a slow and time-consuming development. About 10 years ago, I started to work with students on geometrical models. They were asked to build models and then give mathematical explanations about what was behind them. After some initial difficulties, this became so good, with such nice and beautiful models and such clear explanations that I suggested we create a small exhibition out of it — a post-semester exhibition. That took another half-year to prepare because we wanted to make it as attractive as possible. This was the first exhibition of *Mathematik zum Anfassen* (mathematics for touching), as we called it then. That was in 1994.

**PI: Exactly 10 years ago.**

AB: Yes — this turned out to be a great success. Since everybody was happy about it, I said: why not repeat this seminar? So we did, once more with new students and different models — and again it worked. Then the first enquiries came to us from other institutions, universities, schools and so on: "Could we have this exhibition?" In other words, it became slowly, but then with increasing demand, a travelling exhibition. I think the turning-point came at the 1998 World Congress Of Mathematicians in Berlin, a huge conference with mathematicians from all over the world. I was told that, for the first time in the entire history of this congress, the organizers had the idea of addressing the public as well.



Professor Beutelspacher with $\pi$

Therefore, they rented the *Urania*, a famous building for popularization of science, and we faced the challenge, but also the opportunity, of showing our *Mathematik zum Anfassen* in its basement, while on the upper floors there were mathematical talks by famous people. It was a huge success. It was acknowledged as a really good thing by the Fields Medalists on the one hand and the Berlin school children on the other, and I cannot imagine a greater difference in the approach to mathematics than of these two groups. So I knew we are on the right track, and since it was such a resounding success, it was natural to ask: why not make something permanent along these lines? This was the birth of the phrase "Mathematics Museum," but several years went by before the word became flesh. We had ideas, and everybody said "very good," but nobody knew how to proceed. Then, around the year 2000, we had an unexpected breakthrough: the City of Giessen gave us this building and hadn't even been asked.

**PI: Why Giessen of all cities?**

AB: I'm a Professor at the University of Giessen. You can do a project like this on such a small budget only if you are on location. There are several conditions, but this is one of the necessary ones. Also, I might say that Giessen has very few cultural, scientific attractions for people from outside town. So there was a need, there had always been a perceived need, to have something spectacular. I had many talks with the mayor, who happened to be a physicist, and therefore very sympathetic. Finally, the City donated this building and, simultaneously, the Ministry of the Land Hessen pitched in a substantial amount of money for restructuring it.

**PI: When we mentioned this place to others, we usually call it the world's first mathematics museum. Is this correct? Recently we heard that something similar was being created in France. Do you know anything about that?**

AB: If you take the word "museum" in the very strict sense — an exhibition of valuable objects like books and antique models — then there are many museums of mathematics, many collections of unique old things. I did not wake up one morning and said "I will create the world's first mathematics museum", it just

A great multitude of visitors

gradually happened. Experts in the United States also told me that there was, as yet, nothing similar. Some of the big science centres do have sections devoted to mathematics, but there is no institution where mathematics is at the heart of everything.

**PI: Since we are from *Pi in the Sky*, we would like to hear more details about the models and the kinds questions they prompt — to present them our readers, who would perhaps wish to make their own. But first we would like to get some of the more superficial things out of the way, if you don't mind. So, for example, how many visitors does the *Mathematikum* have?**

AB: This is not superficial at all, but very exciting. Of course we had to name numbers before the opening the place, and I mustered all my courage to say that we would have 60,000 visitors a year. Nobody believed me. But we more than doubled that number. We had over 130,000 in the first year, and we are registering an increase of 20% in the second. So we are heading toward 160,000 a year in this relatively small city. By now, more than 200,000 visitors come here to do mathematics.

How will these two wedges ever make a regular tetrahedron?

**PI: We leave that to our readers to figure out how many visitors per day that represents. Our question to you is: are these mostly teachers coming in with whole classes of students? Or are there also some "free" visitors?**

AB: There is a very clear partition of visitors. Monday to Friday mornings: classes, many classes. We try very hard to organize them so that not too many are in the museum at the same time. But in the afternoon and especially on weekends and holidays, we have many private visitors. And this is our *real* success: 60% of our visitors are private, or "free" as you called them. Adults, groups of adults, groups of children, everybody comes to visit. They have birthday parties here for kids, but also for adults, and so on.

**PI: Are they mostly from near Giessen, or do they come from farther away?**

AB: At least 50% come from farther away — which in Germany means more than 30 kilometres.

**PI: Of course there is an effect on the visitors themselves — but would you say of the general population? Is there a noticeable improvement in their attitude towards mathematics?**

AB: There is certainly no immediate improvement of mathematical skills in the proper sense, and this is not the aim of the *Mathematikum*. But, yes, I would say there is a more positive attitude. We have not done any statistical research on this but think, for instance, if ordinary people, or important people, people who have nothing to do with mathematics, come here to do a ceremony, they feel that mathematics for them is not a word they are afraid of. They love it.

**PI: Mathematics is an activity that is slow and concentrated. Right?**

AB: Fast and concentrated...

**PI: I'm trying to find a way of contrasting what you do as a mathematician, or what we do as mathematicians, with the kind of goings-on in the *Mathematikum*. As you know, we usually sit quietly and scratch our heads or look at the ceiling.**

AB: But then all of a sudden we have a flash of insight — hopefully.

**PI: Well that's what we're looking for, right. When we are sitting looking at the ceiling we are fishing for insight. For an idea to pop onto the screen.**

AB: Exactly, of course, with much more complex thoughts in scientific mathematics. But exactly the same thing happens here. We have these experiments. People do it by their hands or by their feet or whatever, so there is a more or less difficult problem to solve. Then suddenly they have a flash of insight (maybe they catch a hint from another visitor) and so they have become richer by these insights, or at least mathematically richer, than when they came in.

**PI: Can you give us some examples of this? We shall gladly give you time to think about it, to pick good examples or some outstanding examples where you can actually notice that a person has had an insight.**

AB: This can be observed very well in our puzzles. For instance, there is one puzzle is to make a three-sided pyramid of only two identical pieces. And you would think: well, either I can do this by chance or systematically, but most will notice that none of their approaches will work. You need the right idea. And so people try and it's amazing how they try and they come close to the solution. You see by how they stare at it, that they are so near and yet don't see it. And then, all of a sudden, they find the final turn they have to make, and you see — if you look at their eyes — they are happy. They won't forget this for a long time — how this pyramid is built of two pieces.

continued on page 18

**PI: Yes, we remember that one, we'll look at it again** **(see middle of page 17 for a picture)****. But, do you have another one, just to add some variety?**

AB: There is another very complex experiment dealing with randomness. There is a long queue of dice you have to make and then you have to do some kind of counting. If you would like I can explain it in more detail. And there is a very, very surprising result. You do this once, and then make a correction. And if you repeat this with other initial values, you always end at the same place. This is a great surprise, and people do this again and again: they really want to understand why this is so. This is an experiment you have to get into because it's a complex...

Roll a die to know where to start, then count your steps as marked.

**PI: This is the kind of thing we can use as an illustration for *Pi in the Sky* — let the people figure it out.**

AB: This is the experiment with the greatest, no, the longest time people spend on. So, they stay there for 10 minutes, 15 minutes, even longer. Whereas other experiments it is only a flash and then they go.

**PI: What about tying things together? Mathematics is not only a collection of disjoint puzzles — it is a kind of landscape, right? And how can you try to reflect that, or do you try?**

AB: First of all it's true, that's also my idea of mathematics. This is not yet done through the experiments. But, for instance, we have pyramid problems at several places. They also come up, for instance, when you stick a tetrahedron into a soap fluid, and pull it out, and you see this famous point in the middle. If you have done the solid tetrahedral experiment, this the point where the pyramid is cut into two pieces. So, there is a deep connection.

**PI: That is a really good example. And then, questions... reactions... do you remember any particularly striking question?**

AB: Very often in the *Mathematikum* but also on the floor. Many people come to me and simply seem to feel the need to say how nice, how beautiful, and so on, this is. That is one reaction, and the other one is that we notice how many people want to learn more about mathematics. Not everybody coming here — but a certain percentage and this remains a good number.

**PI: What kind of percentage, roughly? An old saw has it that about 5% of any population has extraordinary tastes.**

AB: I have no idea because we can satisfy only a small part of the demand. So we have lectures for eight to 12-year-old children, we systematically present our exhibits several times every week, and we have this regular event where we introduce a mathematician as a person. We shall also have other events and formats.

**PI: Can you give some examples of these mathematicians introduced as persons?**

AB: *Mathematik auf dem Sofa* — this talk at the *Mathematikum*. I invite a mathematician and we chat for one hour. I ask questions like: When did you first notice that mathematics could be something for you? Or, how are you doing it today? Are you studying before breakfast? Or, what are your dreams for the next 10 years? And so on. Our first guest was the president of the German Mathematical Society, then we had a lady from Duisburg, who does didactics of mathematics.

We also had a man from industry — the head of a company making gambling equipment. Last month we had a mathematician who became a journalist, and is now a Waldorf teacher, so a very interesting career. First science, then journalist, then the Waldorf teacher. It's very interesting and the only complaints the auditorium has are: "I would like to know more about this person."

Ancient technology: Loose but solid without glue or mortar.

**PI: What would you say if somebody asked you about a particularly memorable moment for you, in this context of the *Mathematikum*. Something where you personally feel that...**

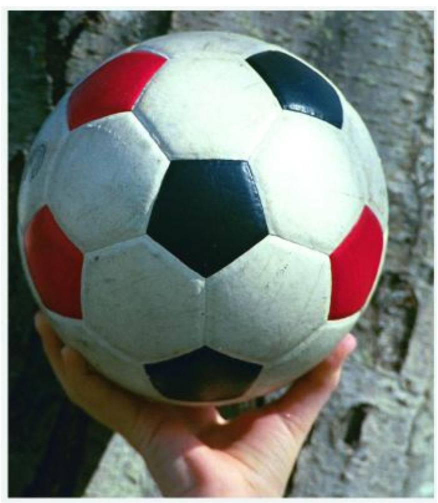
AB: So many moments.

**PI: Well, that's a good answer.**

AB: When we opened this, I didn't say I am proud of it: I said I feel very happy. Because in so many situations, we simply were lucky that it worked out. So, if you speak to politicians, it's unfortunate you have to convince not one, but two of them — belonging to different parties. In our case, the representatives of the City and of the Land did not like each other at all. I had to make sure that they did not clash, and therefore first talked to the one and then to the other. It was touch-and-go on several occasions. We were also lucky with the architect: we had a beautiful building, and he did a really professional job. His numbers were correct, the time and the amount of money — and he gave us excellent advice.

**PI: There must have been a few negative reactions.**

AB: I simply didn't notice it. If you do such a project, which is certainly the biggest project I will ever do in my professional career, it is clear that a necessary condition is that I was working 100% juggling ideas, time, connections, money and so on. Otherwise it wouldn't have worked. So you simply... your brain — my brain was made in such a way that, I just don't know. I can imagine that somebody said, "No, this is not good, and mathematics is quite different. You had better prove a theorem and not waste your time with these things." But I simply said... I even didn't say it, I simply did not listen, I did not hear.



How many hexagons all round?

**PI: Let us come back for a minute to the original projects. You started with some projects in geometry, can you give some examples?**

AB: The project was for second-year students and I simply gave them a list of objects of models. So it was the icosahedron, Möbius strip, tesselations, to make a round movement into a straight movement, linkages... things like that. They could choose. They could also make their own suggestions. It was very open. The only requirement: they have to really build something themselves and they have to explain some of the mathematics behind it.

**PI: Really explain it? Not just...**

AB: To speak frankly, in the first round it was difficult. Because, the difficulty was not building the models, everybody did this and then came to my office very proud. But they didn't find the mathematics in it. So they learned mathematics, they learned equations, they learned derivation, integration and things like that. They knew this, but the idea that mathematics is related to the things I can really touch or the space I inhabit, or whatever — this was an idea they were not familiar with. This was a shock for me. Then of course we had discussions about it.

**PI: Just one example, can you conjure up some example of some object that would...?**

AB: Yes, look at one of the Platonic solids, what we in Europe call a football... what are the mathematics of the soccerball? It is full of beautiful mathematics. You can first ask: What are the tiles it is made of? How many of them? How do they fit together? Is it mathematically true or is it just a rough approximation?

**PI: In 2000, we had a poster campaign in the buses in Vancouver, and one of the questions was about a soccerball. The pentagons on it were half black and half red. You're cueing me to repeat this in *Pi in the Sky* as an example.**

AB: My first question usually when I do the soccerball with kids is: How many pentagons are there? If it's not working with hexagons, how many pentagons? You need some clever counting, and this clever counting means you intuitively use the symmetry of the whole thing and so already this is lots of mathematics.

**PI: If we wanted to imitate *Mathematikum*, you started gradually, this didn't come out in one piece in one day. What kind of advice would you give? Supposing we wanted to do something like that in Vancouver.**

AB: First advice is: Just do it. You need somebody who is willing to devote, is capable, I would say, and willing to devote five years of his life, and his family, and so on, to this project. And you need to realize — at least in this level of financial support we had — you need also luck.

*Note: On May 5, 2006, Mathematikum welcomed its 500,000th visitor. Giessen's mayor was on hand to offer congratulations to the museum.*

# Asking the Right Question

by David Leeming, University of Victoria

One of the many keen readers of *Pi in the Sky*, Danesh Forouhari, a software engineer from Sunnyvale, Calif., submitted the following problem to our Challenge Problems section.

"Let A and P denote the area and perimeter of a triangle respectively.

Prove that $A < \frac{1}{4}P^2$."

Danesh's proof uses Heron's formula for the area of a triangle. If we let $a$, $b$ and $c$ be the sides of the triangle and let $S = \frac{1}{2}(a+b+c) = \frac{P}{2}$

Heron's formula for the area $A$ of a triangle is

$A = \sqrt{S(S-a)(S-b)(S-c)}$ so

$A = \frac{1}{16}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}$,

and since $a > 0$, $b > 0$ and $c > 0$, we have

$(a+b+c)(b+c-a)(a+c-b)(a+b-c) < (a+b+c)^4$

Therefore, $A\frac{\sqrt{(a+b+c)^4}}{16} = \frac{P^2}{4}$ QED.

Danesh posed this inequality involving the relationship between the area and perimeter (squared) of an arbitrary triangle. The constant ¼ in the inequality follows from the method of proof, which replaces $-a$ by $a$, etc. to achieve the result.

The *Pi in the Sky* editors who read the problem immediately focused on the constant ¼. Can we do better? In other words, can we find a smaller constant than ¼ for which the inequality is true for all triangles. Indeed, what is the *smallest possible constant c* such that the inequality $A \le cP^2$ (1) holds for all possible triangles?

Our Challenge Problems editor, Dragos Hrimiuc, came up with an elegant solution which uses Heron's formula and the arithmetic-geometric mean (AGM) inequality to show that the correct constant $c$ in inequality (1) is $\frac{1}{12\sqrt{3}}$ which is much smaller than ¼. He proves that $\frac{1}{12\sqrt{3}}$ is the *smallest possible* value of $c$.

Here is his proof.

Using the same notations we get

$$\frac{A^2}{p^4} = \frac{s(s-a)(s-b)(s-c)}{16s^4} = \frac{(s-a)(s-b)(s-c)}{16s^3}.$$

By the AGM inequality

$$(s-a)(s-b)(s-c) \le \left(\frac{s-a+s-b+s-c}{3}\right)^3 = \frac{s^3}{27}$$

Thus

$$\frac{A^2}{p^4} \le \frac{1}{16 \cdot 27} \qquad (2)$$

i.e

$$A \le \frac{1}{12\sqrt{3}} p^2.$$

(equality iff $a=b=c$).

(iff=if and only if)

Now, (only) for equilateral triangles ($a=b=c$) does equality hold in (2). You can show this for yourself. That will complete the proof that $\frac{1}{12\sqrt{3}}$ is the smallest possible constant in inequality (2).

Thanks to Danesh Forouhari we have an interesting problem relating the area A and the perimeter P of an arbitrary triangle. With some minor massaging by the mathematicians, we are able to ask the right question about the relationship between A and P. This is what mathematicians strive to do! They want the best possible inequality for a particular problem, or the optimal possible result that one can achieve.

Footnote: Let $a_1$, $a_2$, ..., $a_n$ be positive numbers and let $A = \frac{1}{n}(a_1 + a_2 + ... + a_n)$ and $G = (a_1 a_2 ... a_n)^{\frac{1}{n}}$. The arithmetic - geometric mean (AGM) inequality states that G≤A with equality if and only if the $n$ numbers are equal. A good reference for more details, including the proof of the AGM inequality, is N.D. Kazarinoff, *Geometric Inequalities*, MMA, 1961.

# Divisibility by Seven

by David Leeming and Jeremy Tatum, University of Victoria

We had many, many responses to the "Divisibility by Seven" article in our December 2005 issue of *Pi in the Sky*, with some responses coming to us from as far away as Germany, Hawaii and New Zealand. We are unable to list all who responded but we are grateful to everyone who took the time to reply. Obviously, *Pi in the Sky* has captured the interest of readers around the world.

All respondents presented more or less the same explanation that the numbers 1, 3, 2, 6, 4, 5 are the remainders when you divide $10^0$, $10^1$, $10^2$, $10^3$, $10^4$ and $10^5$ by 7 (i.e. $10^n \pmod 7$), $0 \le n \le 5$). For higher powers of 10, the sequence of six numbers repeats.

A particularly nice solution was given by Bruce Shawyer, which we present here:

> Jeremy Tatum's test for divisibility by 7 in *Pi in the Sky* issue 9, December 2005, uses the sequence $\{a_n\}=\{1,3,2,6,4,5, \ldots\}$, which repeats itself cyclically.
>
> He asked, "Where does this come from, and are there tests for divisibility by 13, 17 and 19?"
>
> The answer to his 7 question, which also give the repeating sequence of 13, 17 and 19 is (we call the number $p$)
>
> $$a_n = 10^n - (\text{the largest multiple of } p \le 10^n)$$
>
> For example, if $p=7$, we get $\{a_n\} = \{1,3,2,6,4,5, \ldots\}$, repeating cyclically.
>
> If $p=13$, we get $\{a_n\}=\{10,9,12,3,4,1, \ldots\}$, repeating cyclically.
>
> If $p=17$, we get $\{a_n\}=\{10,15,14,4,6,9,5,16,7,2,3,13,11,8,12,1, \ldots\}$, repeating cyclically.
>
> And so on.
>
> Now, for the method of proof, I will illustrate the principle with a three-digit number, *abc*, and divisibility by 7.
>
> If $100a+10b+c=7x$ (integer $x$), we also know that $98a+7b=7y$ (integer $y$).
>
> Subtracting gives $2a+3b+1c=7(x-y)$.
>
> The definitions of $a_n$ show how to extend this.
>
> *Bruce Shawyer (Memorial University of Newfoundland)*

If we allow the remainders to be *negative*, then we obtain an interesting pattern, and a simplification of the sequence $\{a_n\}$. If the (positive) remainder is greater than $1/2\ (p-1)$, subtract $p$. For $p=7$, the sequence becomes 1, 3, 2, -1, -3, -2; for $p=13$, the sequence becomes -3, -4, -1, 3, 4, 1; and for $p=17$, the sequence becomes -7, -2, -3, 4, 6, 8, 5, -1, 7, 2, 3, -4, -6, -8, -5, 1 all repeating cyclically. This is a big gain since it not only reduces the amount of work required to produce the sequence $\{a_n\}$, it also reduces the size of the number we need in order to test for divisibility. For example, to test the number 80 339 773 for divisibility by 17, we use the modified sequence:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 0 | 3 | 3 | 9 | 7 | 7 | 3 |
| -1 | 5 | -8 | 6 | 4 | -3 | -2 | -7 |
| -8 | +0 | -24 | +18 | +36 | -21 | -14 | -21 |

The sum of these numbers is -34, which is divisible by 17, so 80 339 773 is divisible by 17. Now, you test the number 3 569 236 229 for divisibility by 13 using the modified sequence.

For more on divisibility of large numbers, please see the article by Charles and Tatum in the March 2003 issue of *Pi in the Sky*.

# Solutions to Math Challenges in Issue 9 of *Pi in the Sky* (December 2005)

## Problem 1

Find the largest value of $x_{m,n} = \dfrac{1}{m+n+1} - \dfrac{1}{(m+1)(n+1)}$ where $m, n$ are positive integers.

## Solution 1

The AGM inequality yields

$(m+1)(n+1) \leq \dfrac{(m+n+2)^2}{4}$, so that $\dfrac{1}{(m+1)(n+1)} \geq \dfrac{4}{(m+n+2)^2}$ and

$x_{m,n} = \dfrac{1}{m+n+1} - \dfrac{1}{(m+1)(n+1)} \leq \dfrac{1}{m+n+1} - \dfrac{4}{(m+n+2)^2}$, with equality if and only if $m = n$.

Letting $k = m+n+2$. we can determine that $f(k) = \dfrac{1}{k-1} - \dfrac{4}{k^2}$ is decreasing for $k \geq 6$ – that is, for $m + n \geq 4$.

Noting that $x_{1,1} = x_{1,2} = x_{2,1} = \dfrac{1}{12} = 0.08\bar{3}$, $x_{3,1} = x_{1,3} = 0.075$, and $x_{2,2} = \dfrac{4}{45} = 0.0\bar{8}$,

we conclude that $x_{m,n} \leq \dfrac{4}{45}$ for $m, n \in \mathbb{N}$.

Notice: Henry Ricardo from Medgar Evers College (CUNY), Brooklyn, New York, has mentioned that the inequality is attributed to G. Grüßß. The above solution, which was sent to us, is essentially that of E. Landau and can be found on pages 190-191 of Analytical Inequalities by D.S. Mitrinović (Springer, 1970).
We received an excellent solution to this problem from Yuning Chen and Edward T.H. Wang from Wilfrid Laurier University.

## Problem 2

For any positive integer $n$, let $S(n)$ denote the sum of its digits in decimal notation. If $S(n)$=50 and $S(15n)$=300, find $S(4n)$.

## Solution 2

One can easily prove that $S(m+n) \leq S(m) + S(n)$ for any positive integers $m, n$.
Now

$300 = S(15n) = S(10n + 5n) \leq S(10n) + S(n) + S(4n) = 100 + S(4n)$

Thus,

$S(4n) \geq 200.$

On the other hand,

$S(4n) \leq 4S(n) = 200$

Therefore

$S(4n) = 200$

Notice: There are positive integers $n$ such that $S(n)$=50 and $S(15n)$=300. We can take $n$=1,...,1 with fifty digits of 1.

## Problem 3

Let $A$ be a nonempty set of positive integers such that if $a \in A$ then $4a$ and $[\sqrt{a}] \in A$ (here $[a]$ denotes the greatest integer less than or equal to $a$).
Prove that $A$ is the set of all positive integers.

## Solution 3

(i) Let us first prove that $1 \in A$. Let $a \in A$. Then we have

$$[a^{1/2}] \in A, \left[[a^{1/2}]^{1/2}\right] \in A, \ldots, \underbrace{\left[\ldots\left[[a^{1/2}]^{1/2}\right]^{1/2}\ldots\right]}_{n-brackets} \in A, \ldots$$

Also, the following inequalities are true:

$$1 \leq [a^{1/2}] \leq a^{1/2}, 1 \leq \left[[a^{1/2}]^{1/2}\right] \leq a^{1/2^2}, \ldots, 1 \leq \underbrace{\left[\ldots\left[[a^{1/2}]^{1/2}\right]^{1/2}\ldots\right]}_{n-brackets} \leq a^{1/2^n}$$

On the other hand, there is a postive integer $k$, sufficiently large, such that $a^{1/2^k} \leq 1.5$.

## Sudoku solutions from page 2

### Easy

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 7 | 4 | 1 | 8 | 9 | 5 | 3 | 6 |
| 1 | 8 | 6 | 3 | 2 | 5 | 4 | 7 | 9 |
| 3 | 9 | 5 | 7 | 6 | 4 | 8 | 1 | 2 |
| 7 | 6 | 8 | 2 | 5 | 3 | 9 | 4 | 1 |
| 9 | 5 | 1 | 8 | 4 | 7 | 6 | 2 | 3 |
| 4 | 3 | 2 | 6 | 9 | 1 | 7 | 5 | 8 |
| 8 | 4 | 3 | 9 | 7 | 2 | 1 | 6 | 5 |
| 6 | 2 | 7 | 5 | 1 | 8 | 3 | 9 | 4 |
| 5 | 1 | 9 | 4 | 3 | 6 | 2 | 8 | 7 |

### Hard

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 9 | 7 | 2 | 8 | 1 | 6 | 4 | 5 | 3 |
| 1 | 8 | 4 | 3 | 5 | 9 | 6 | 7 | 2 |
| 6 | 5 | 3 | 4 | 2 | 7 | 8 | 1 | 9 |
| 7 | 4 | 1 | 9 | 6 | 5 | 3 | 2 | 8 |
| 5 | 2 | 9 | 7 | 8 | 3 | 1 | 6 | 4 |
| 3 | 6 | 8 | 1 | 4 | 2 | 5 | 9 | 7 |
| 2 | 3 | 6 | 5 | 7 | 8 | 9 | 4 | 1 |
| 4 | 9 | 7 | 6 | 3 | 1 | 2 | 8 | 5 |
| 8 | 1 | 5 | 2 | 9 | 4 | 7 | 3 | 6 |

Therefore $1 \le \underbrace{\left[\ldots\left[[a^{1/2}]^{1/2}\right]^{1/2}\ldots\right]}_{k-\text{brackets}} \le a^{1/2^k} \le 1.5$

and thus $\underbrace{\left[\ldots\left[[a^{1/2}]^{1/2}\right]^{1/2}\ldots\right]}_{k-\text{brackets}} = 1.$

(ii) Now, let us prove that $2^n \in A$ for $n = 1, 2, \ldots$
Indeed, since $1 \in A$, we obtain
$2^2 \in A, 2^4 \in A, \ldots, 2^{2n} \in A, \ldots$
Therefore $\left[\sqrt{2^{2n}}\right] = 2^n \in A, n = 1, 2, \ldots$
(iii) Take an arbitrarily positive integer $m$ and let's prove that $m \in A$. It is sufficient to prove that there is a positive integer $k$ such that $m^{2^k} \in A$.
For each positive integer $k$, there is a positive interger $p_k$ such that
$2^{p_k} \le m^{2^k} < 2^{p_k+1}$ (we can take $p_k = [\log_2 m^{2^k}]$).
Now for $k$ sufficiently large, we have the inequality

$$\left(1+\frac{1}{m}\right)^{2^k} \ge 1 + \frac{1}{m}2^k > 4$$

that combined with the above inequality produces
(*) $2^{p_k} \le m^{2^k} < 2^{p_k+1} < 2^{p_k+2} < (m+1)^{2^k}$
Now let $k$ be a positive integer that satisfies (*)
Since $2^{2(p_k+1)+1} \in A$, we have

$$\left[\sqrt{2^{2(p_k+1)+1}}\right] = [2^{p_k+1}\sqrt{2}] \in A$$

Hence

$$\underbrace{\left[\ldots\left[[2^{(p_k+1)}\sqrt{2}]^{1/2}\right]^{1/2}\ldots\right]}_{(k+1)-\text{brackets}} \in A.$$

On the other hand, using (*), we get

$$m^{2^k} < 2^{p_k+1} \le \left[2^{(p_k+1)}\sqrt{2}\right] < (m+1)^{2^k}$$

and then

$$m \le \underbrace{\left[\ldots\left[[2^{(p_k+1)}\sqrt{2}]^{1/2}\right]^{1/2}\ldots\right]}_{(k+1)-\text{brackets}} < m+1$$

which shows that

$$m = \underbrace{\left[\ldots\left[[2^{(p_k+1)}\sqrt{2}]^{1/2}\right]^{1/2}\ldots\right]}_{(k+1)-\text{brackets}} \in A.$$

## Problem 4

Let $\mathbb{N}$ denote the set of positive integers and $f: \mathbb{N} \to \mathbb{N}$ a function such that $f(f(n)) + 2f(n) = 3n + 4$ for every $n \in \mathbb{N}$. Find $f(2006)$.

## Solution 4

Let us find $f(1)$. Let $f(1) = K \ge 1$.
Then $f(K) = 7 - 2K \ge 1$, hence $K \in \{1, 2, 3\}$.
If $K = 1$ then $f(1) = 1$ and also $f(1) + 2f(1) = 7$, hence $f(1) = \frac{7}{3}$, a contradiction.
If $K = 3$ then $f(1) = 3$ and also $f(3) + 2f(1) = 7$, hence $f(3) = 1$. Now we get $f(f(3)) = f(1) = 3$, which implies that $3 + 2f(3) = 13$: that is, $f(3) = 5$, a contradiction.
We conclude that $f(1) = 2$.

Let us prove by induction that $f(n) = n + 1$, for all $n \in \mathbb{N}$.
The statement is true for $n = 1$. Let us assume that it is true for $n = k$, *i.e.* $f(k) = k + 1$.
We have:

$$f(f(k)) + 2f(k) = 3k + 1 \Rightarrow f(k+1) + 2(k+1) = 3k + 4$$
$$\Rightarrow f(k+1) = k + 2$$

Therefore, by induction, $f(n) = n + 1$ for all $n \in \mathbb{N}$,
Now, $f(2006) = 2007$.

A similar solution to this problem was submitted by Yuning Chen and Edward T.H. Wang from Wilfrid Laurier University. Killian Miller and Steven Lowdon from the University of Victoria have found that $f$ assuming that $f(n) = an + b$. However, their solution does not assume the uniqueness of $f$.

## Problem 5

Let $AA'$, $BB'$, $CC'$ be the angle bisectors of $\Delta ABC$. If $\widehat{B'A'C'} = 90°$, find $\widehat{BAC}$.

## Solution 5

Let $BC = a, AC = b, AB = c, \widehat{BAA'} = \widehat{CAA'} = \alpha$. Since $AA'$ is the line bisector of $\widehat{A}$, we have $\frac{A'B}{A'C} = \frac{c}{b}$, from which we obtain

$$A'B = \frac{ac}{b+c} \text{ and } A'C = \frac{ab}{b+c} (*)$$

Similiarly

$$AC' = \frac{cb}{a+b}, AB' = \frac{bc}{a+c} (**)$$

By using Stewart's theorem, (*) and the law of cosines we get

$$AA' = \frac{2bc\cos\alpha}{b+c} (***)$$

In $\Delta B'AA'$, $\Delta C'AA'$, and $\Delta AC'B'$ by using cosine law, (**) and (***) we get

$$A'B'^2 = \frac{b^2c^2}{(a+c)^2} + \frac{4b^2c^2(a-b)}{(b+c)^2(a+c)}\cos^2\alpha$$

$$A'C'^2 = \frac{b^2c^2}{(a+b)^2} + \frac{4b^2c^2(a-c)}{(b+c)^2(a+b)}\cos^2\alpha$$

$$B'C'^2 = \frac{b^2c^2}{(a+b)^2} + \frac{b^2c^2}{(a+c)^2} - \frac{2b^2c^2}{(a+b)(a+c)}\cos 2\alpha$$

Since $\Delta A'C'B'$ is a right triangle, we must have
$B'C'^2 = A'B'^2 + A'C'^2$
which, after some computations, can be written as

$$\frac{4(a^2+bc)}{(b+c)^2}\cos^2\alpha = 1$$

or

$$16bc\cos^4\alpha - 4(b^2+c^2+3bc)\cos^2\alpha + (b+c)^2 = 0$$

Solving for $\cos\alpha$, we can obtain $\cos\alpha = \frac{1}{2}$ and thus $A = 120°$

Past math challenge answers can be found on our website,
***www.pims.math.ca/pi***

# *Pi in the Sky* Math Challenges

**Problem 1**

Prove that there are infinitely many rational numbers *r* such that $\sqrt{r}$ and $\sqrt{r+1}$ are simultaneously rational numbers.

*Recall: A **rational number** is a fraction p/q where p and q are both integers (q≠0).*

**Problem 2**

Consider the number 3025. If we split the digits into 30 and 25, add these two numbers and square the result, we get 3025 back. Find another four digit number, all four digits distinct, which has the same property.

**Problem 3**

Let $m, n$ be positive integers such that $n\sqrt{23} - m > 0$. Prove that $n\sqrt{23} - m > \frac{2}{m}$.

**Problem 4**

Find all real numbers $x_1, \ldots, x_{2007}$ such that $x_1 + \ldots + x_{2007} = 2007$ and $|x_1 - x_2| = |x_2 - x_3| = \ldots = |x_{2007} - x_1|$.

**Problem 5**

Find all $a,b,c \in \mathbb{R}$ such that $|ax^2 + bx + c| \le 1$ for every $-0.5 \le x \le 0.5$ and $a^2+b^2+c^2$ is maximum.

**Problem 6**

Two altitudes of a triangle are of length 1 and 2. If $h$ denotes the length of the third altitude, then show that $2/3 < h < 2$.

Send your solutions to the math challenges to *pi@pims.math.ca*, or mail them to us at:

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## Pi in the Sky call for submissions

*Pi in the Sky* is seeking submissions for the Summer 2007 edition. We accept materials on any subject related to mathematics or its applications. Submissions are subject to editorial review.

Please send all submissions, art work, letters to the editor, and questions to *pi@pims.math.ca*.