Phase Space and Path Integral Methods in Seismic Wave Propagation and Imaging

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POTSI PSEUDODIFFERENTIAL OPERATOR THEORY & SEISMIC IMAGING

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Goals of Lectures

1) Overview of wave equation seismic imaging
2) Application of modern mathematical physics methods to acoustic wave scattering and propagation
    a) Incorporation of well-posed, one-way methods into inherently two-way, global formulations
    b) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics
    c) Extension of Fourier analysis to inhomogeneous environments
3) Possible improvements to seismic imaging algorithms

Lecture 4

Applications to Seismic Imaging

Review of the key points of Lectures 1 – 3

Potential theoretical developments

Overview of the theoretical constructions

Applications to seismic imaging

A seismic image

Conclusions

GIMMC presentation

Open discussion

Locally Homogeneous Medium Wavefield Extrapolation (GPSPI)
(physics formulation)

$$\phi^{+}\left(x+\Delta x, z\right) \approx \int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\Delta x\left(K^{2}\left(z\right)-p^{2}\right)^{1/2}\right)\hat{\phi}^{+}\left(x, p\right)\right]$$

$$\left(i/\bar{k}\right)\partial_{x}\phi^{+}\left(x,z\right)+\frac{\bar{k}}{2\pi}\int_{\mathbb{R}^{2}} dpdz'\left(K^{2}\left(z\right)-p^{2}\right)^{1/2} \bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^{+}\left(x,z'\right)=0$$

$$K\left(z\right)=\frac{c_{0}}{c\left(z\right)} \qquad \bar{k}=\frac{\omega}{c_{0}} \qquad c\left(z\right)=v\left(z\right)$$

Three Common Misconceptions About GPSPI

1) The one-way wave equation corresponding to the limiting form of the GPSPI algorithm

$$\left(i/\bar{k}\right)\partial_x\phi^+(x,z)+\frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2}dp\,dz'\,\overbrace{\left(K^2(z)-p^2\right)^{1/2}}^{\text{symbol}}\bullet\exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^+\left(x,z'\right)=0$$

is believed exact for a range-independent medium. The square-root function is believed to be the correct function (symbol) for infinitesimal wavefield extrapolation.

Three Common Misconceptions About GPSPI

2) The wavefield growth problems, which can develop for finite, range step-size, are believed to vanish in the limit of zero, range step-size, since, in that limit, the theory is believed to be exact. The amplitude problems are assumed a result of the numerical discretization, and are not viewed as fundamental in nature.

3) Since, in the typical derivation of GPSPI, the up- and down-going wavefields are assumed to be independent, it is thought to be impossible to extrapolate a full, two-way wavefield by well-posed, one-way marching methods.

Classical Wave Theory

Modern Approaches – Principal Themes

1) Incorporation of well-posed, one-way methods into inherently two-way, global formulations

2) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics

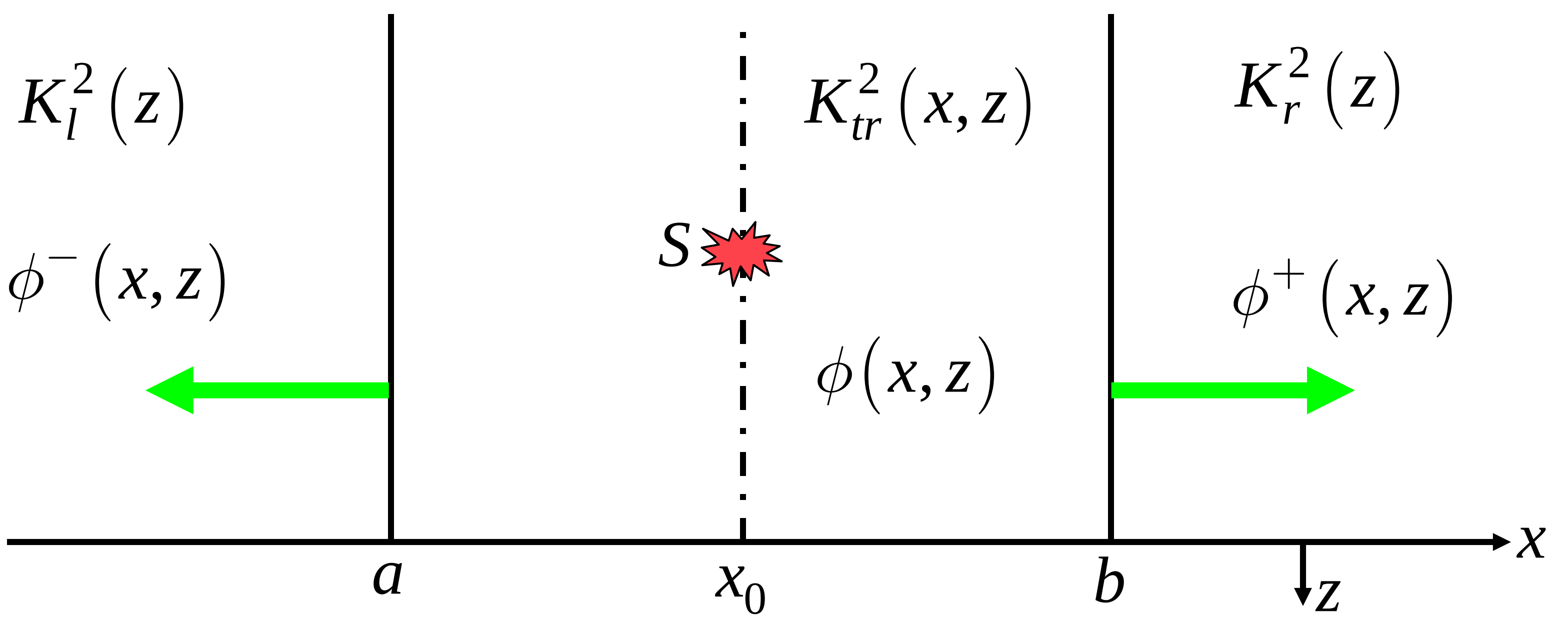
3) Extension of Fourier analysis to inhomogeneous environments

Mathematical Illustration

Scalar Helmholtz Equation

$$\left(\partial_x^2 + \underbrace{\partial_z^2 + \bar{k}^2 K^2(x,z)}_{\mathbf{B}^2}\right)\phi(x,z) = -\delta(x-x_s)\delta(z-z_s)$$

General Radiation Formulation



Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

Given $\phi\left(x_0, z\right)$, then propagation from x_0 is given by

$$\left(\left(1/\bar{k}\right)\partial_x + \mathbf{\Lambda}^{+}\left(x,b\right)\right)\phi\left(x,z\right) = 0$$

$$\left(\left(1/\bar{k}\right)\partial_x - \mathbf{\Lambda}^{-}\left(a,x\right)\right)\phi\left(x,z\right) = 0$$

where

$$\left(1/\bar{k}\right)\partial_x \mathbf{\Lambda}^{+}\left(x,b\right) = \left(\mathbf{\Lambda}^{+}\left(x,b\right)\right)^2 + \mathbf{B}^2\left(x\right)$$

with the initial condition

$$\mathbf{\Lambda}^{+}\left(b,b\right) = -i\mathbf{B}\left(b\right)$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

and

$$-\left(1/\bar{k}\right)\partial_x \boldsymbol{\Lambda}^-(a,x) = \left(\boldsymbol{\Lambda}^-(a,x)\right)^2 + \mathbf{B}^2(x)$$

with the initial condition

$$\boldsymbol{\Lambda}^-(a,a) = -i\mathbf{B}(a)$$

with

$$\mathbf{B}(x) = \left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

and where the initial wavefield is given by

$$\phi(x_0,z) = \left(1/\bar{k}\right)\left(\boldsymbol{\Lambda}^+(x_0,b) + \boldsymbol{\Lambda}^-(a,x_0)\right)^{-1}\delta(z-z_s)$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

(1) Now have two, well-posed marching problems, in opposite directions, done in secession

(2) Must cover both directions for the two-way “elliptic wave propagation” problem: one direction to get the DtN operator and the other direction to propagate the total wavefield with the DtN operator

(3) The idea is not to do the first marching procedure (DtN operator construction) computationally, but, rather, to solve the problem asymptotically. Thus, there will be only one, one-way marching computational procedure.

(4) The initial field calculation will also be done asymptotically.

Mathematical Framework

Transversely-Inhomogeneous Half-Space $K^2(x,z) = K^2(z)$

(1) Wave Equation

$$\left((i/\bar{k})\partial_x + \left(K^2(z) + (1/\bar{k}^2)\partial_z^2 \right)^{1/2} \right) \phi^+(x,z) = 0$$

$$(i/\bar{k})\partial_x \phi^+(x,z) + \frac{\bar{k}}{2\pi} \int_{\mathbb{R}^2} dp\,dz' \Omega_{\mathbf{B}}\left(p, (z+z')/2\right)$$

$$\bullet \exp\left(i\bar{k}p(z-z')\right)\phi^+(x,z') = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x,z\middle|0,z'\right)=\lim_{N\to\infty}\int_{\mathbb{R}^{2N-1}}\prod_{j=1}^{N-1}dz_{j}\prod_{j=1}^{N}\left(\frac{\bar{k}}{2\pi}\right)dp_{j}$$

$$\bullet\exp\left[i\bar{k}\sum_{j=1}^{N}\left(p_{j}\left(z_{j}-z_{j-1}\right)+\left(\frac{x}{N}\right)h_{\mathbf{B}}^{s}\left(p_{j},z_{j}\right)\right)\right]$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi^{+}\left(x+\Delta x, z\right) \approx$$

$$\int_{\mathbb{R}} dp \exp\left(i\overline{k}pz\right)\left[\exp\left(i\overline{k}\,\Delta x h_{\mathbf{B}}^{s}\left(p, z\right)\right)\hat{\phi}^{+}\left(x, p\right)\right]$$

$$h_{\mathbf{B}}^{s}\left(p,q\right)=\left(\overline{k}\big/\pi\right)\int_{\mathbb{R}^2} dsdt\Omega_{\mathbf{B}}\left(s,t\right)\exp\left(-2i\overline{k}\left(q-t\right)\left(p-s\right)\right)$$

Mathematical Framework

Arbitrary transverse inhomogeneity

Operator symbols determined by appropriate composition equations, e.g.,

$$\Omega_{\mathbf{B}^2}\left(p,q\right)=K^2\left(q\right)-p^2=$$

$$\left(\bar{k}/\pi\right)^2\int_{\mathbb{R}^4} dtdsdvdu\,\Omega_{\mathbf{B}}\left(t+p,s+q\right)$$

$$\bullet\,\Omega_{\mathbf{B}}\left(v+p,u+q\right)\exp\left(2i\bar{k}\left(sv-tu\right)\right)$$

supplemented with right-traveling-wave radiation condition

Mathematical Framework

Natural Question

Can we take the desired exact and uniform approximate operator symbol constructions directly from the quantum mechanical and modern mathematical asymptotic literatures?

Answer – No!

Why?

(1) Mathematical analysis → Quantum mechanical results

(2) While the mathematical analysis provides the complete framework for the equations, microlocal analysis (asymptotics) only considers part of the solution – it is an approximation

Mathematical Framework

Why?

(3) Approximation appropriate for time-domain formulations not frequency-domain Helmholtz equation

(4) Will result in nonuniform, singular approximations for Helmholtz equation

Illustration – Weyl composition equation

$$\Omega_{\mathbf{B}^2}\left(p,q\right)=K^2\left(q\right)-p^2=\left(\bar{k}/\pi\right)^2\int_{\mathbb{R}^4}dtdsdvdu\,\Omega_{\mathbf{B}}\left(t+p,s+q\right)$$

$$\bullet\Omega_{\mathbf{B}}\left(v+p,u+q\right)\exp\left(2i\bar{k}\left(sv-tu\right)\right)$$

Exact Symbols – Focusing Quadratic

Profile

$$K^2(q) = K_0^2 - \omega^2 q^2, n = 2, K_0, \omega > 0, q \varepsilon \mathbb{R}$$

Weyl operator symbol

$$\Omega_{\mathbf{B}}(p,q) = -\exp(i\pi/4)\left(\frac{\varepsilon}{2}\right)^{1/2}\left(\frac{1}{\pi}\right)\int_L d\tau \varsigma\left(1/2, (-i/2\pi)\tau, \exp(2i\pi Y)\right)$$

$$\bullet \exp\left(Y\tau - \mathrm{X}\tanh\tau\right)\sec h\tau\left(Y - \mathrm{X}\sec h^2\tau - \tanh\tau\right), Y \neq 0,1,2,\ldots$$

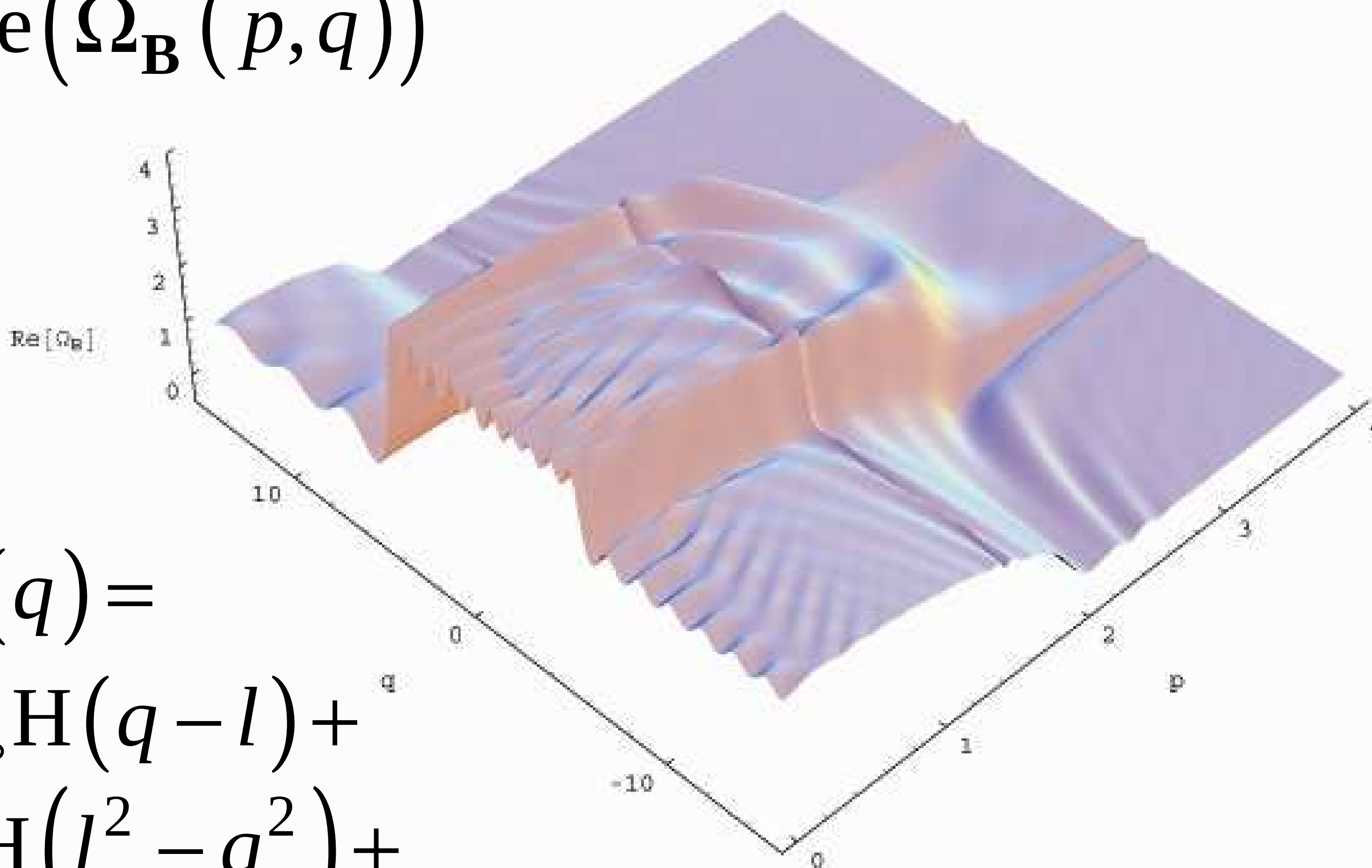
$$\mathrm{X} = (1/\varepsilon)\left(\omega^2 q^2 + p^2\right), Y = K_0^2/\varepsilon, \varepsilon = \omega/\overline{k}$$

$$\varsigma(\sigma, \Delta, \xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{(n+\Delta)^\sigma}, \Delta \neq 0, -1, -2, \ldots, |\xi| < 1, +$$ anal. conts.

(Lerch transcendental function)

Exact Operator Symbol
3-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



$$K^2\left(q\right)=$$
$$K_{+\infty}^2\mathrm{H}\left(q-l\right)+$$
$$K_1^2\mathrm{H}\left(l^2-q^2\right)+$$
$$K_{-\infty}^2\mathrm{H}\left(-q-l\right)$$

$$K_{-\infty}=2, K_1=3, K_{+\infty}=1$$
$$\bar{k}=1, l=5$$

Uniform High-Frequency Expansion

Operator Symbol

(1) n = 2

(2) Refractive index field varies "slowly" on wavelength scale

$$\Omega_{\mathbf{B}}(p,q) \sim \left(K^2(q) - p^2\right)^{1/2} + \int_0^\infty du \cos\left(\bar{k}pu\right)$$

$$\bullet \left(K^2(q) \left(A \left(\frac{H_1^{(1)}\left(\bar{k}I_0\right)}{I_0} + \frac{C}{\bar{k}} H_0^{(1)}\left(\bar{k}I_0\right) \right) - \frac{H_1^{(1)}\left(\bar{k}K(q)u\right)}{K(q)u} \right) \right)$$

where

$$A = \left(\frac{I_0}{I_1}\right)^{3/2} \left(\frac{1}{K^2(q)} \left(K(q+u/2)K(q-u/2)\right)^{-1/2} \right)$$

Uniform High-Frequency Expansion

Operator Symbol

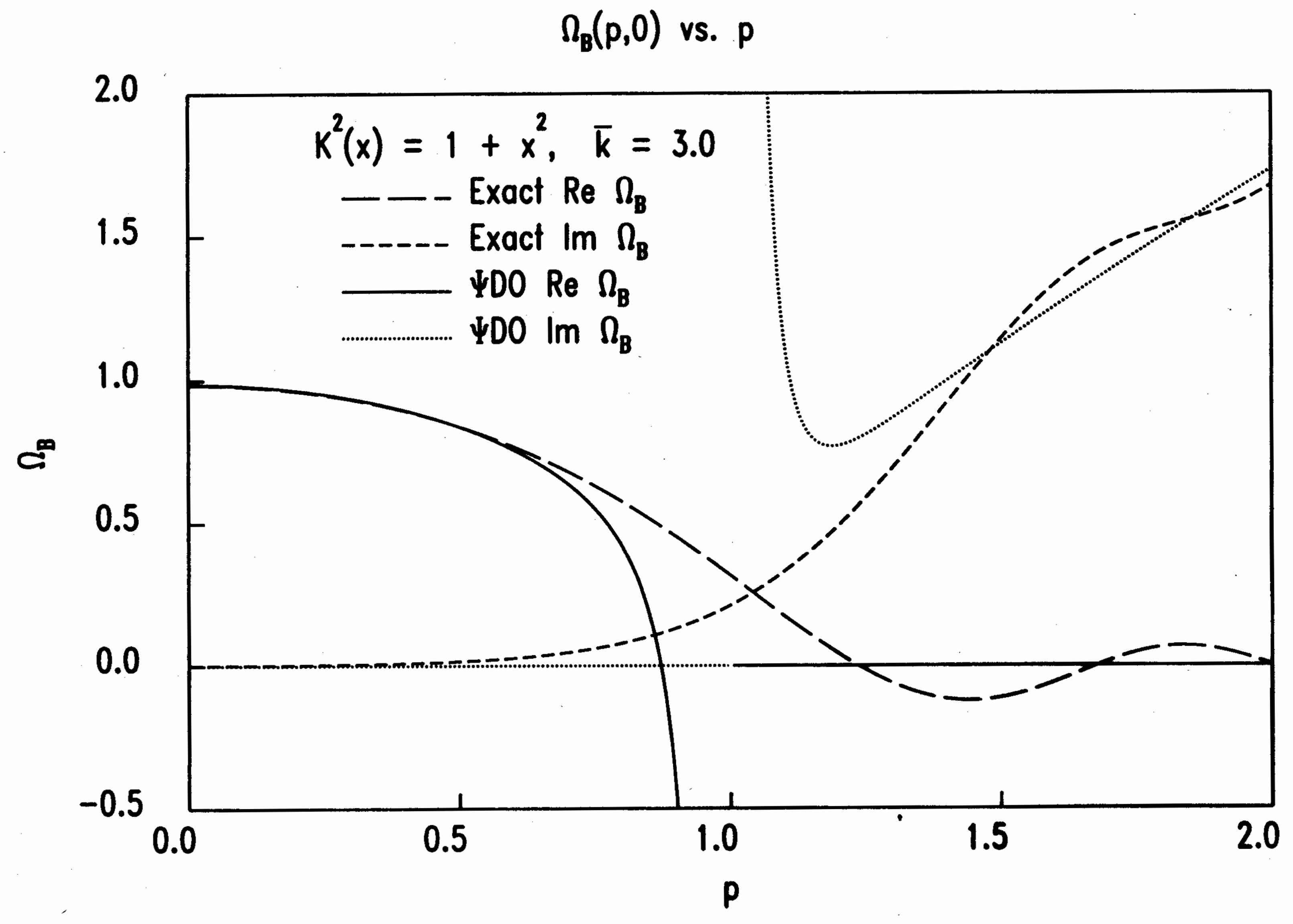
$$C = \left(\frac{1}{8I_0}\right)\left(\frac{15I_2}{I_1^2} - \frac{3}{I_0} - \frac{6}{I_1}\left(\frac{1}{K^2(q+u/2)} + \frac{1}{K^2(q-u/2)}\right.\right.$$

$$\left.\left. +2\left(\frac{K'(q-u/2)}{K^2(q-u/2)} - \frac{K'(q+u/2)}{K^2(q+u/2)}\right) - \tilde{I}\right)\right)$$

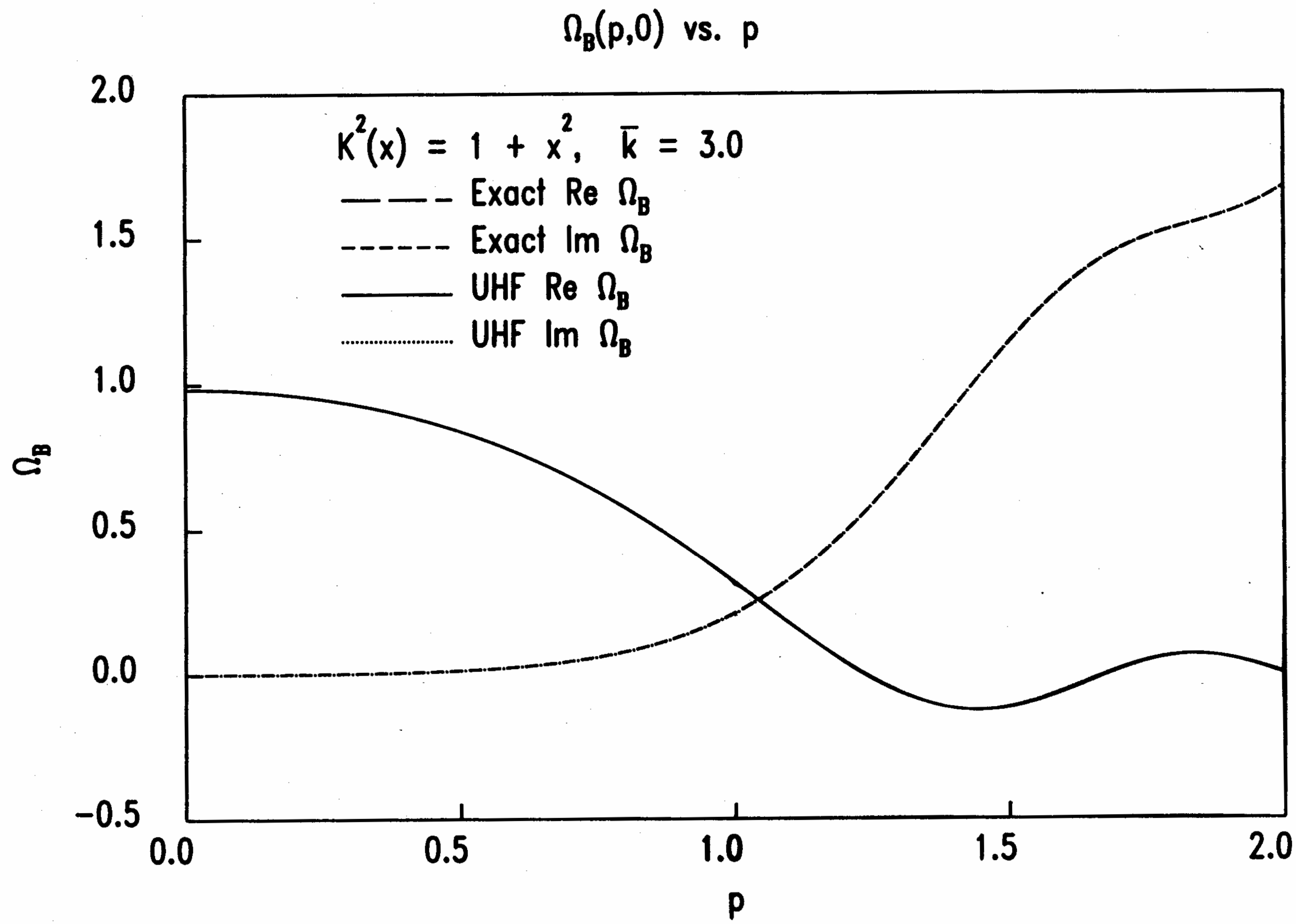
$$I_m = \int_{q-u/2}^{q+u/2} dt \left(K(t)\right)^{1-2m}, m = 0,1,2$$

$$\tilde{I} = \int_{q-u/2}^{q+u/2} dt \left(\frac{\left(K'(t)\right)^2}{K^3(t)}\right)$$

Pseudodifferential Versus Exact



UHF Versus Exact



Uniform High-Frequency Wave Theory

Path Integral + UHF Standard Operator Symbol

(1) Distinct from high-frequency approximations made directly on the wave field, including the globally uniform, high-frequency constructions of Maslov (Fourier transform) and Klauder (coherent-state transform)

(2) Incorporates wave (diffraction) effects via “sum over paths” in phase space and uniform approximation of phase functional (operator symbol) in path integral

(3) Since there is functional stationary phase approximation, there are no caustic-related phenomena

(4) Essentially the difference of classical, high-frequency asymptotic wavefield as (1) global solution and (2) local solution repeatedly composed to produce a global solution

Uniform High-Frequency Wave Theory

Path Integral + UHF Standard Operator Symbol

(5) Correct incorporation of both high-propagating-angle and post-critical wave phenomena

(6) Since uniform asymptotic construction, approximation is independent of the reference sound speed

(7) More accurate incorporation of energy flux conservation

(8) Since approximations are made at the “infinitesimal” (operator symbol) level, far greater range of computational validity

(9) Algorithm runs just like GPSPI computational algorithm

(10) Reformulation of path integral in terms of coherent states (Gabor basis) should result in faster computational algorithms

Potential Theoretical Developments

1) More operator symbol asymptotics

   (a) Multiscale asymptotics

   (b) Complete DtN operator symbol asymptotics

2) Path integral constructions

   (a) Exact Feynman/Kac constructions
   (Budaev and Bogy)
   (Connect HE with diffusion/drift process)

$$\left(\nabla^2+\bar{k}^2K^2\left(\underline{x}\right)\right)\Psi\left(\underline{x}\right)=-F\left(\underline{x}\right)$$

$$\Psi\left(\underline{x}\right)=A\left(\underline{x}\right)\exp\left(i\bar{k}S\left(\underline{x}\right)\right)$$ (Liouville representation)

Potential Theoretical Developments

2) Path integral constructions

(a) Exact Feynman/Kac constructions

$$\Psi(\underline{x}) = \mathrm{E}\left(\left(i/2\bar{k}\right)\int_0^\infty dt \exp\left(-i\bar{k}S\left(\underline{\xi}_t\right)\right)F\left(\underline{\xi}_t\right)\exp\left(-(1/2)\int_0^t ds \nabla^2 S\left(\underline{\xi}_s\right)\right)\right)$$

$$\bullet \exp\left(i\bar{k}S\left(\underline{x}\right)\right)$$

where

$$d\underline{\xi}_t = \left(i/\bar{k}\right)^{1/2} d\underline{w}_t - \nabla S\left(\underline{\xi}_t\right)dt \quad \text{(diffusion+drift process)}$$

$$\left|\nabla S\left(\underline{x}\right)\right|^2 = K^2\left(\underline{x}\right) \quad \text{(eikonal equation)}$$

Apply standard stochastic numerical methods

Potential Theoretical Developments

2) Path integral constructions

(b) Feynman/DeWitt-Morette representation

Green's function as coordinate space path integral

Motivated by WKB form of Green's function and the Feynman/Garrod approximate path integral from physics

$$G\left(\underline{x},\underline{x}_s\right)=-\left(1/2\bar{k}^2\right)\int_{\mathrm{E}} D\left(\underline{\xi}\right)\exp\left(i\bar{k}W\left(\underline{\xi}\right)\right)$$

where

$$W=\int_{\underline{x}_s}^{\underline{x}}\left\|d\underline{\xi}\right\|\left(1-2V\left(\underline{\xi}\right)\right)^{1/2}$$

is the analog of the action of a "free particle" on a space

Potential Theoretical Developments

2) Path integral constructions

(b) Feynman/DeWitt-Morette representation

with metric

$$d\underline{\eta}^2 = \left(1 - 2V\left(\underline{\xi}\right)\right)\left\|d\underline{\xi}\right\|^2$$

and where E represents a path space such that

$$\frac{1}{2} = (1/\tau)\int_0^\tau dt\left(\left(1/2\right)\left\|d\underline{\xi}(t)/dt\right\|^2 + V\left(\underline{\xi}(t)\right)\right)$$

with the constraints $\underline{\xi}(0) = \underline{x}_s, \underline{\xi}(\tau) = \underline{x}$ and where τ is a stochastic variable, with $V(\underline{x}) = -(1/2)\left(K^2(\underline{x}) - 1\right)$

Stochastic process embodying fixed “average energy” paths

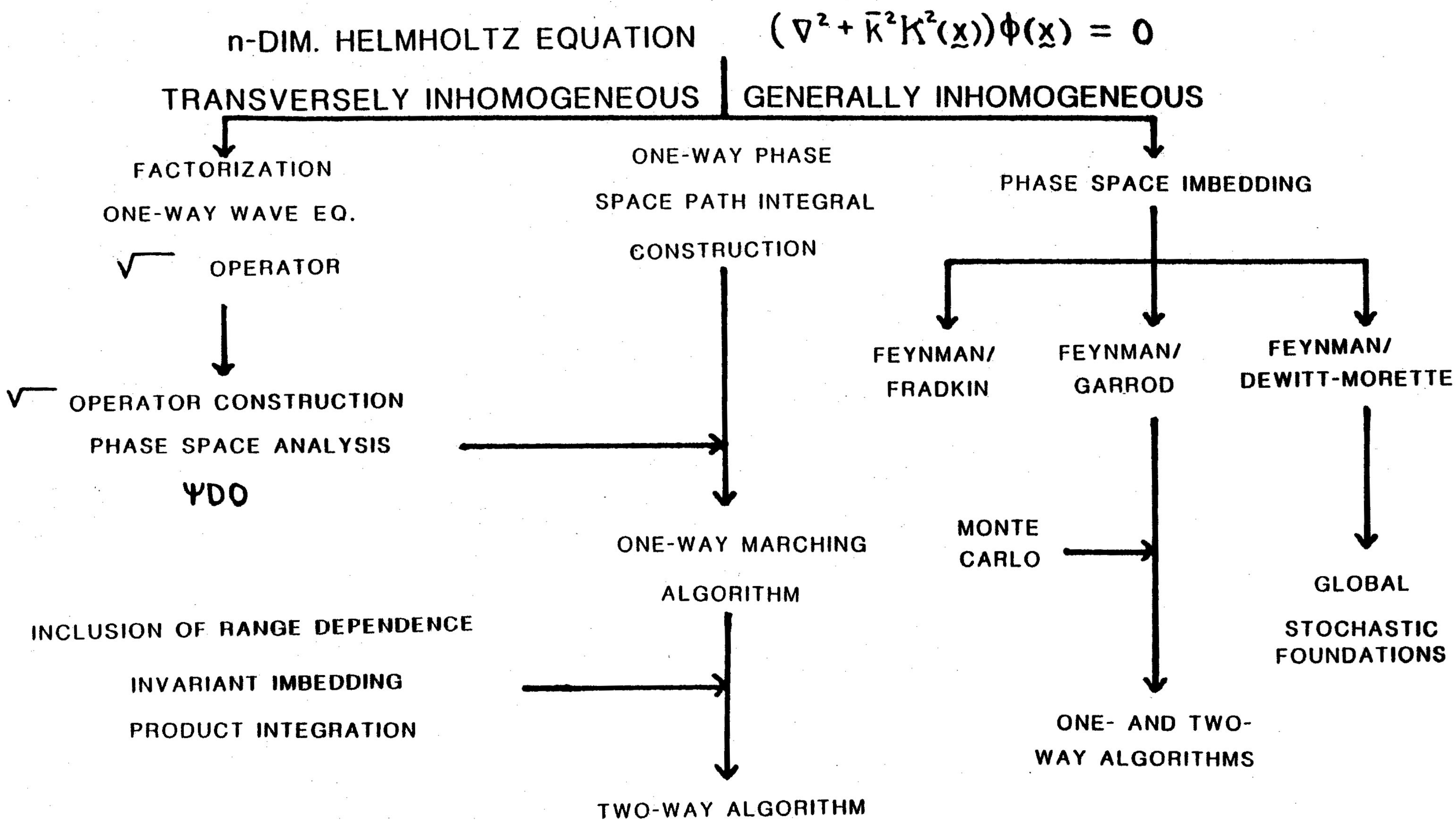
Potential Theoretical Developments

2) Path integral constructions

(c) Coherent state path integral (Klauder)

Should be instrumental in constructing fast algorithms

Theoretical Construction Overview



Applications to Seismic Imaging

(1) Migration of primaries and multiples
Exact, well-posed, one-way reformulation
Corresponding migration condition

(2) High-angle imaging in heterogeneous media

(3) Improved resolution through more accurate representation of transverse scattering

(4) Better amplitude estimation through flux conservation and improved accuracy

Applications to Seismic Imaging

(1) Starting point – GPSPI

$$h^s_{\mathbf{B}}(p,q) \approx \left(K^2(q) - p^2\right)^{1/2}$$

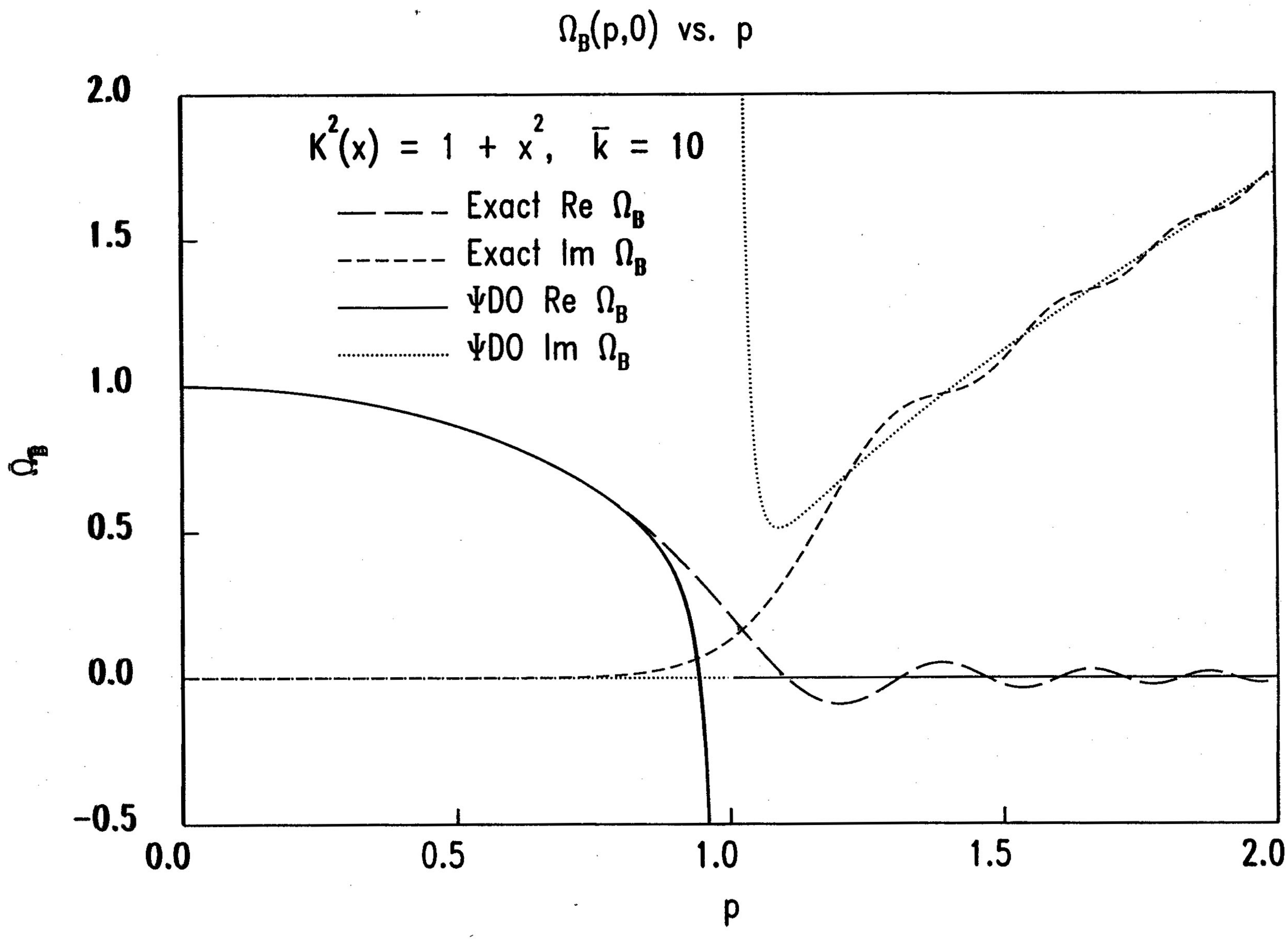
(2) Stolk imaging

$$h^s_{\Lambda^+}(x,p,q) \sim -i\left(K^2(x,q) - p^2\right)^{1/2} + \frac{K(x,q)\partial_x K(x,q)}{2\bar{k}\left(K^2(x,q) - p^2\right)} + O\left(1/\bar{k}^2\right)$$

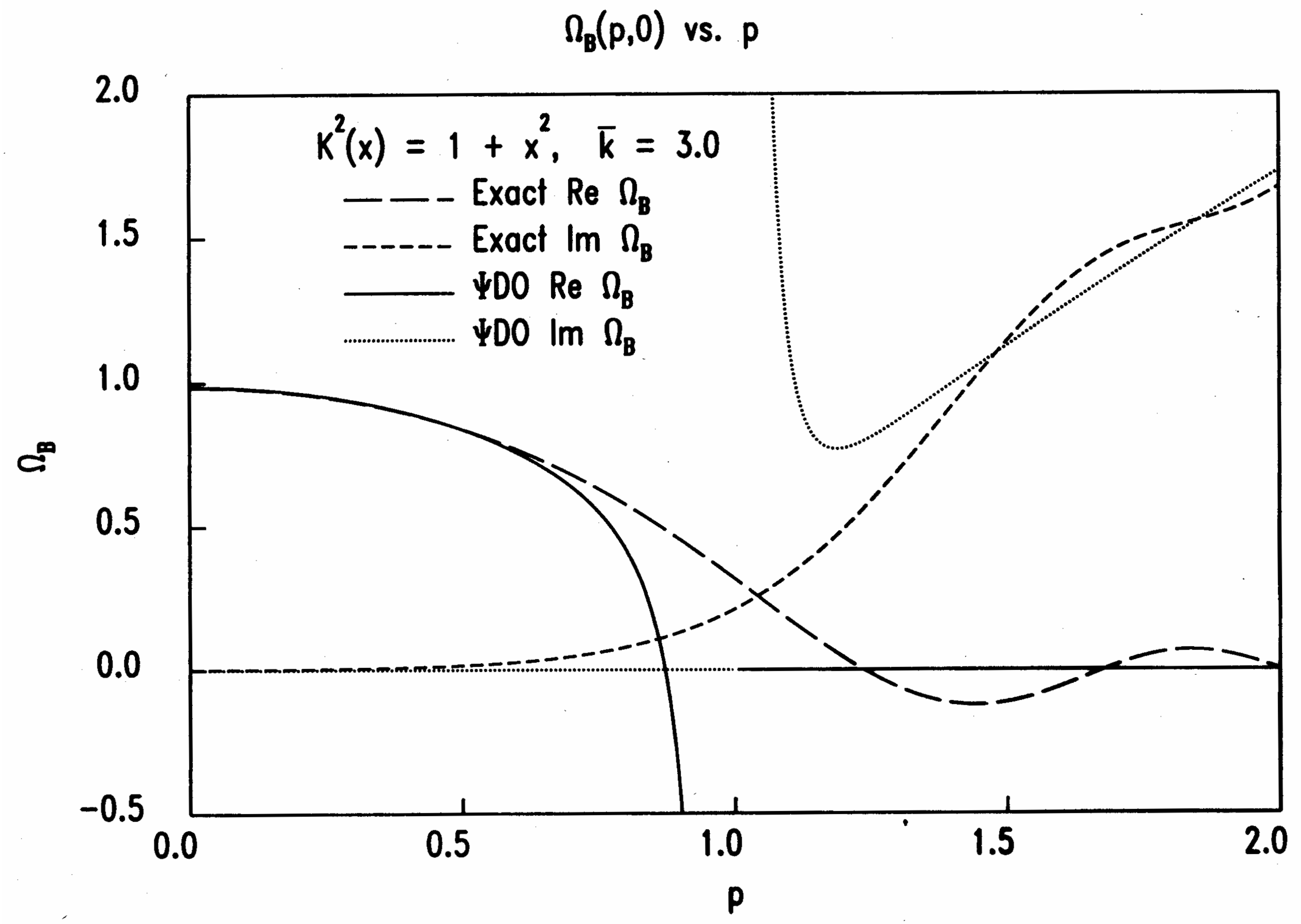
(a) Start with nonuniform, singular expansion

(b) Filter phase space regions, in the singular terms, which represent the “high-angle rays” incorrectly (frequency-dependent filter)

Pseudodifferential Versus Exact



Pseudodifferential Versus Exact



Applications to Seismic Imaging

(3) UHF symbol imaging

- (a) Full symbol – accuracy assessment

- (b) Simplified symbol approximation for increased computational speed

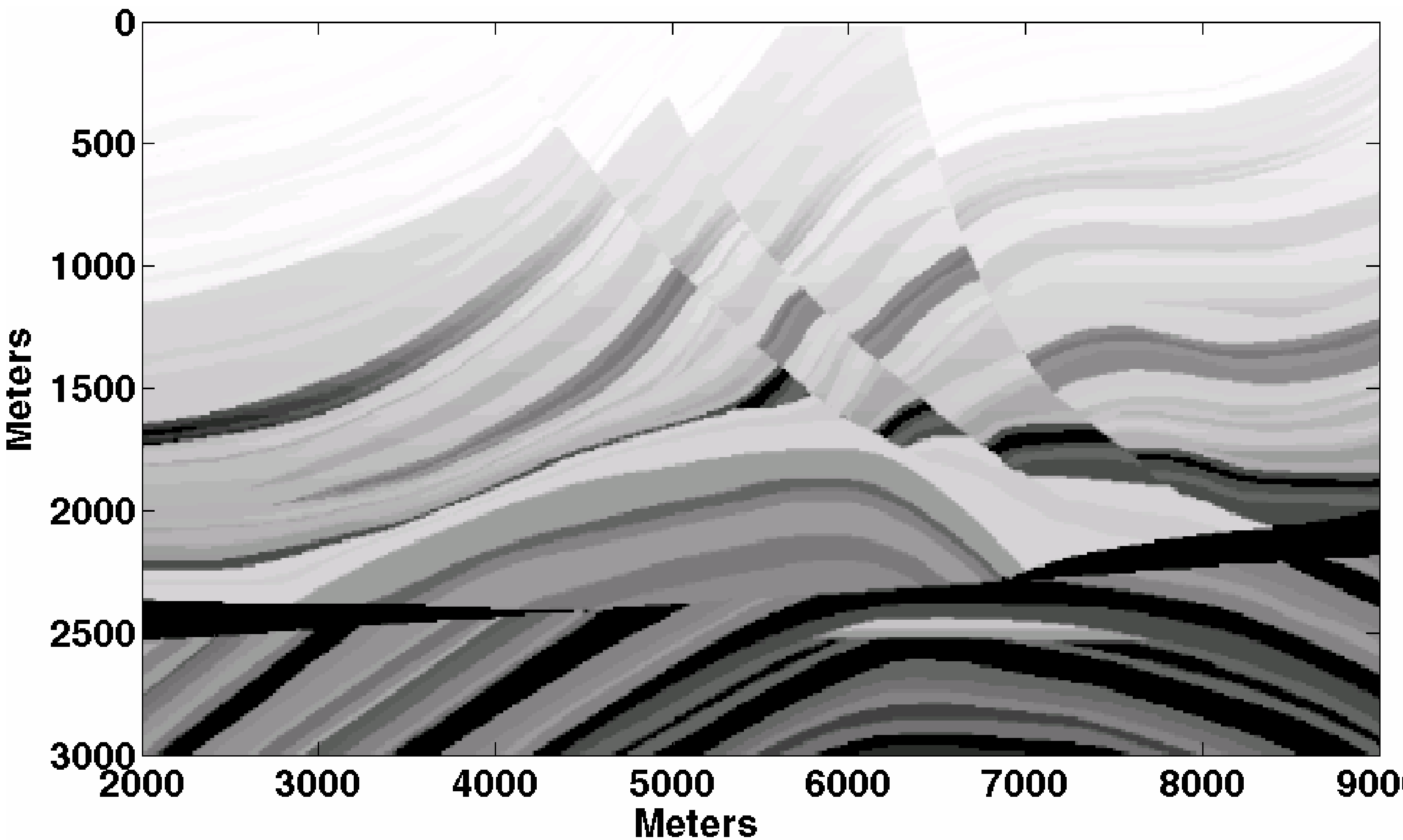
POTSI PhD Research

- Richard Bale: Elastic-wave migration using a GPSPI approach.
- Chad Hogan: Wavefield extrapolators using advanced symbols.
- Yongwang Ma: A Gabor implementation of GPSPI.
- Saleh al-Saleh: Migration velocity analysis and Schwartz kernel implementation of wavefield extrapolators.

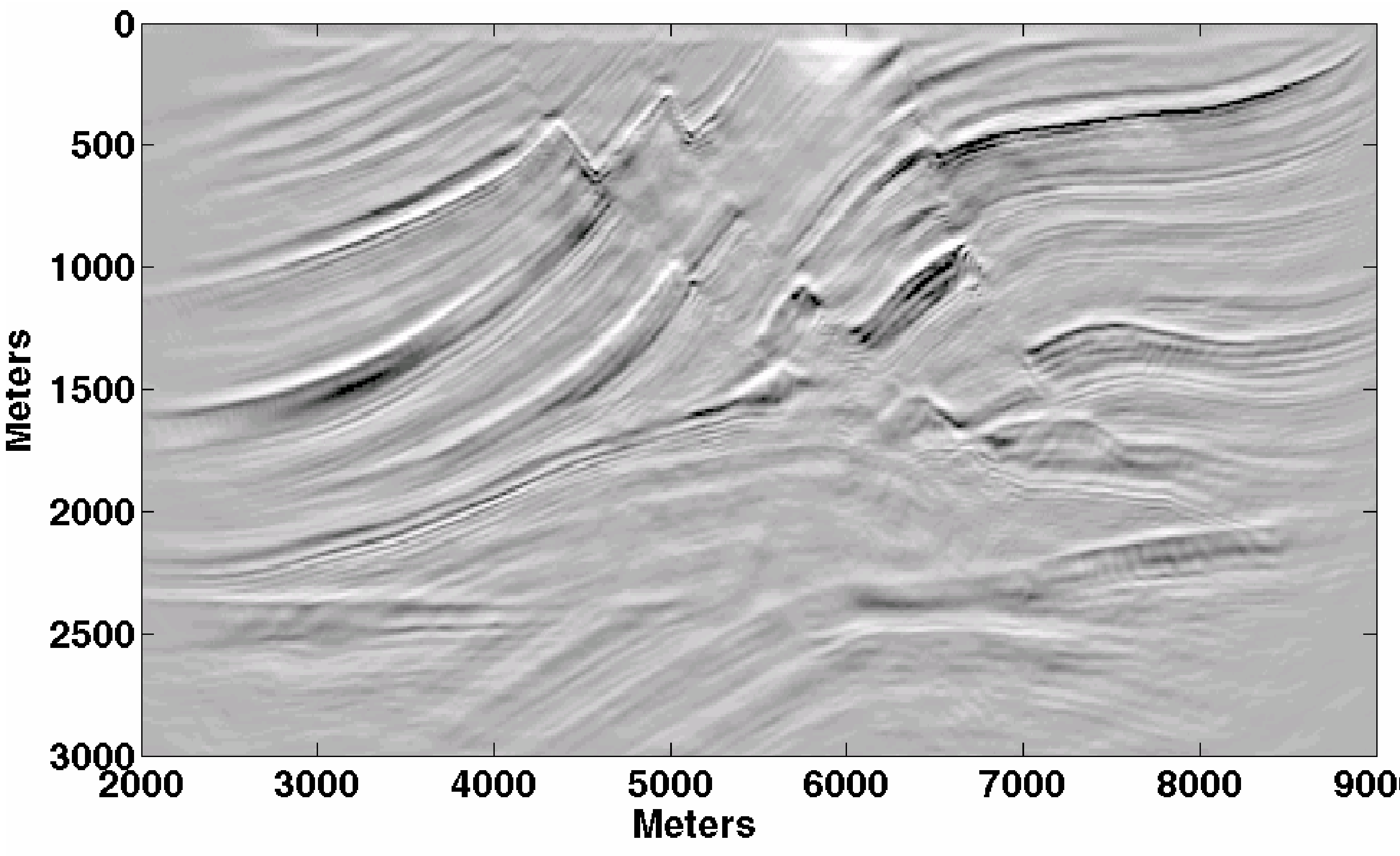
Challenges

- Rapid calculation of symbols
- Rapid application of extrapolator
- Appropriate, robust parameterization
- Adaptation to exploration geometries
- Extension to elastic waves

Marmousi Velocity Model

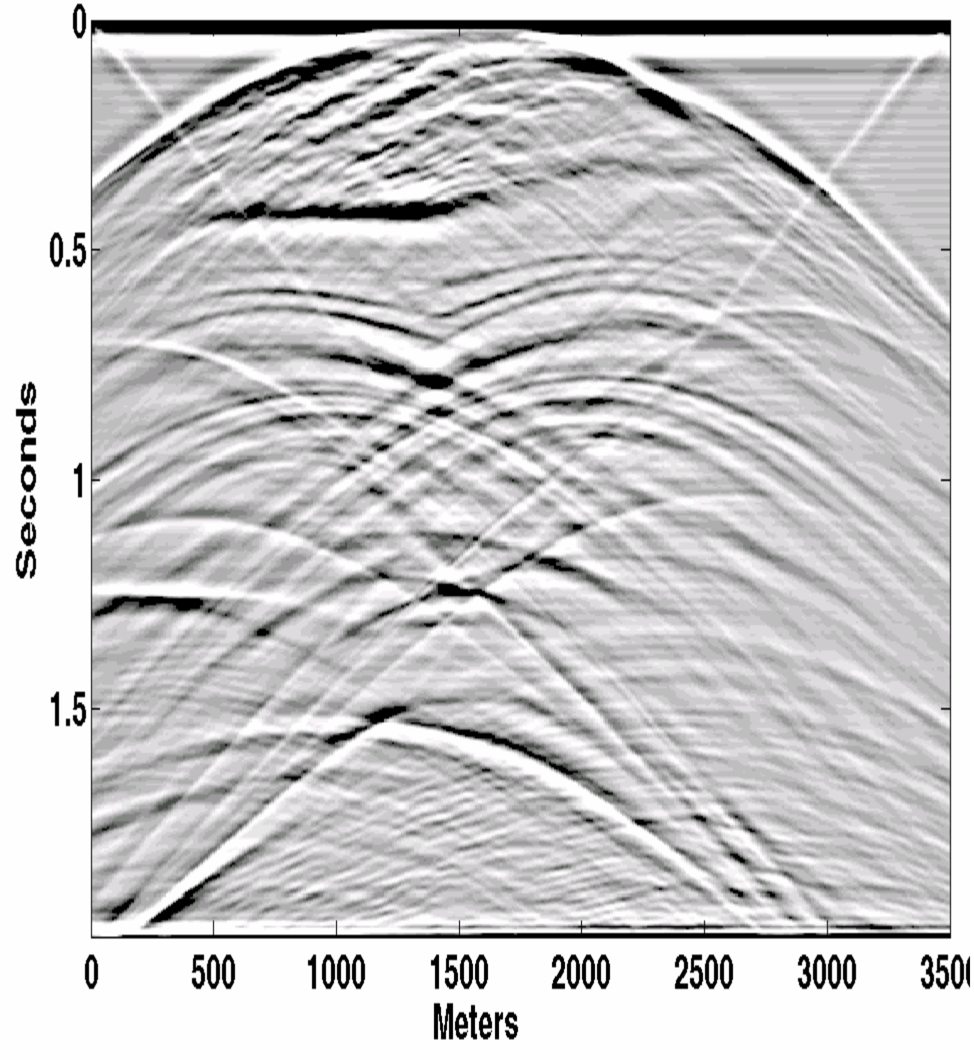


Pre-Stack Depth Migration

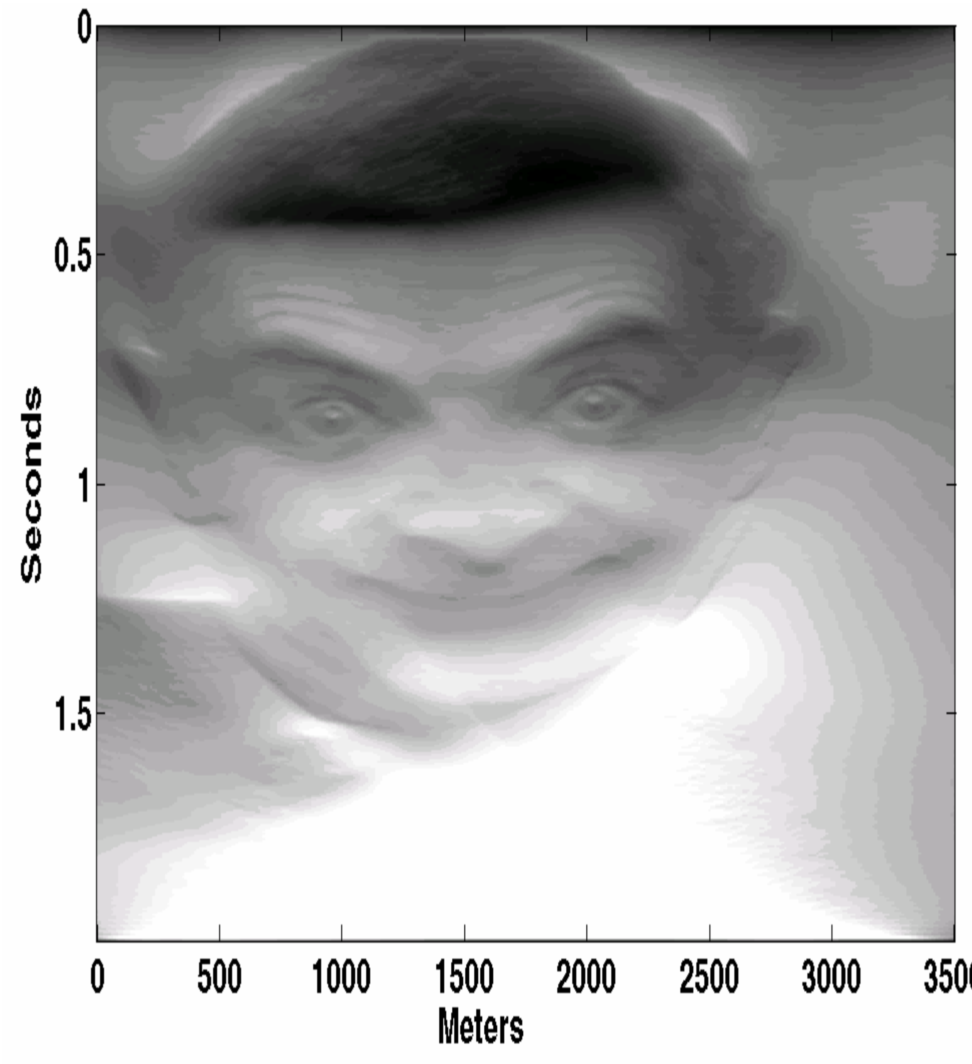


Final Seismic Example

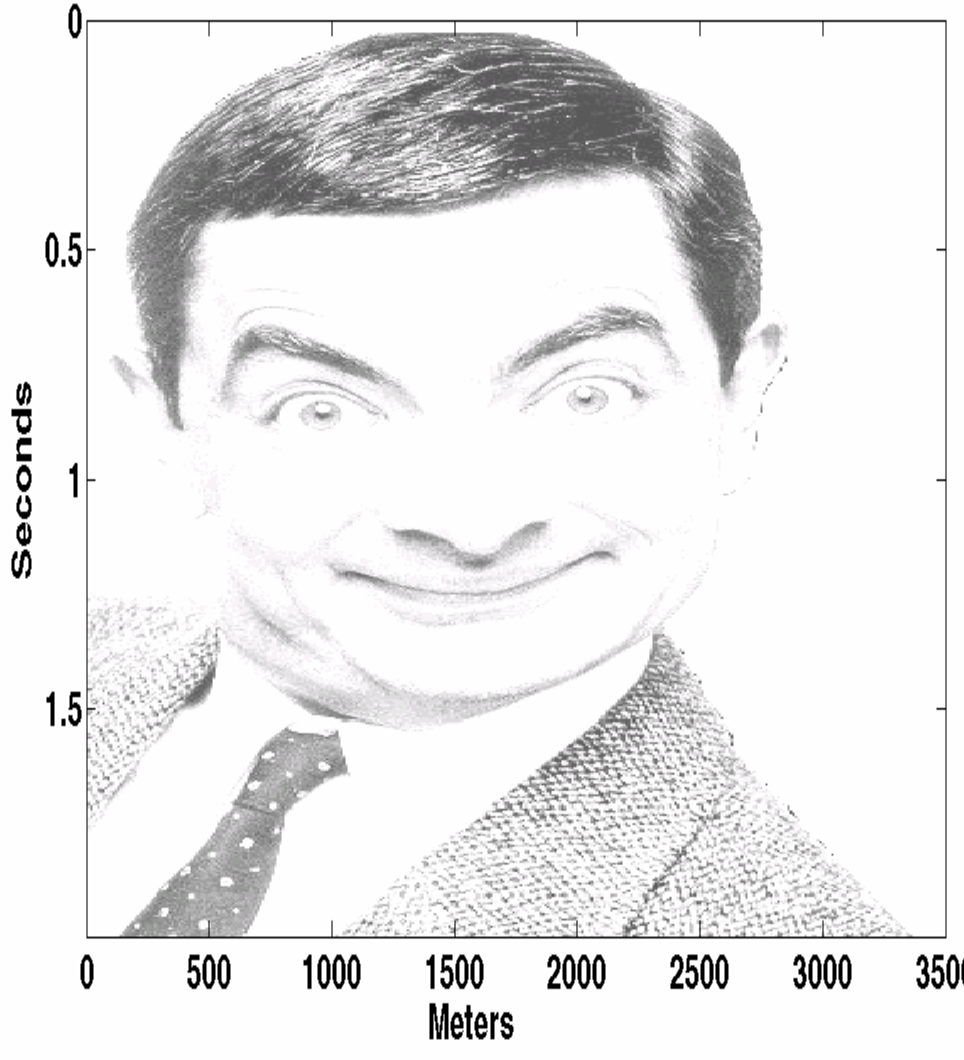
What will this become after imaging?



Result



Mr. Bean



Conclusions

1) One-way wave equations can be constructed that incorporate all of the forward and backward scattering inherent in the two-way wave equations.

2) The above one-way formulations can be made explicit by exploiting the correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics.

3) Effectively, these constructions extend Fourier analysis to inhomogeneous environments.

4) Uniform high-frequency asymptotic operator symbol approximations extend GPSPI.

5) Everything just runs like the GPSPI algorithm.