Phase Space and Path Integral Methods in Seismic Wave Propagation and Imaging

Lou Fishman
MDF International
Slidell, LA 70461 USA
Currently visiting Depts. of Mathematics & Statistics and Geology & Geophysics, University of Calgary, Calgary, Alberta, Canada
lou@math.ucalgary.ca
Shidi53@aol.com

Pacific Institute for the Mathematical Sciences

POTSI PSEUDODIFFERENTIAL OPERATOR THEORY & SEISMIC IMAGING

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Goals of Lectures

1) Overview of wave equation seismic imaging
2) Application of modern mathematical physics methods to acoustic wave scattering and propagation
   a) Incorporation of well-posed, one-way methods into inherently two-way, global formulations
   b) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics
   c) Extension of Fourier analysis to inhomogeneous environments
3) Possible improvements to seismic imaging algorithms

Lecture 3

Phase Space and Path Integral Methods
Part 2

The focus on solving the composition equations for the operator symbols

Can the desired solutions be taken directly from the quantum mechanical and microlocal literatures?

Nonuniform operator symbol constructions

Exact operator symbol constructions

Uniform asymptotic operator symbol constructions

Uniform high-frequency wavefield extrapolator

Mathematical Framework

Transversely-Inhomogeneous Half-Space $K^2(x,z) = K^2(z)$

(1) Wave Equation

$$\left(\left(i/\bar{k} \right) \partial_x + \left(K^2(z) + \left(1/\bar{k}^2 \right) \partial_z^2 \right)^{1/2} \right) \phi^+(x,z) = 0$$

$$\left(i/\bar{k} \right) \partial_x \phi^+(x,z) + \frac{\bar{k}}{2\pi} \int_{\mathbb{R}^2} dp dz' \Omega_{\mathbf{B}} \left(p, \left(z + z' \right)/2 \right)$$

$$\bullet \exp\left(i\bar{k}p \left(z - z' \right) \right) \phi^+\left(x, z' \right) = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x,z\middle|0,z'\right)=\lim_{N\to\infty}\int_{\mathbb{R}^{2N-1}}\prod_{j=1}^{N-1}dz_j\prod_{j=1}^{N}\left(\frac{\bar{k}}{2\pi}\right)dp_j$$

$$\bullet\exp\left[i\bar{k}\sum_{j=1}^{N}\left(p_j\left(z_j-z_{j-1}\right)+\left(\frac{x}{N}\right)h_{\mathbf{B}}^{s}\left(p_j,z_j\right)\right)\right]$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi^{+}\left(x+\Delta x, z\right)\approx$$

$$\int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\,\Delta x h_{\mathbf{B}}^{s}\left(p,z\right)\right)\hat{\phi}^{+}\left(x,p\right)\right]$$

$$h_{\mathbf{B}}^{s}\left(p,q\right)=\left(\bar{k}/\pi\right)\int_{\mathbb{R}^2} ds\,dt\,\Omega_{\mathbf{B}}\left(s,t\right)\exp\left(-2i\bar{k}\left(q-t\right)\left(p-s\right)\right)$$

Mathematical Framework

Arbitrary transverse inhomogeneity

Operator symbols determined by appropriate composition equations, e.g.,

$$\Omega_{\mathbf{B}^2}(p,q) = K^2(q) - p^2 =$$

$$(\bar{k}/\pi)^2 \int_{\mathbb{R}^4} dt\,ds\,dv\,du\,\Omega_{\mathbf{B}}(t+p, s+q)$$

$$\bullet\,\Omega_{\mathbf{B}}(v+p, u+q)\exp\left(2i\bar{k}(sv-tu)\right)$$

supplemented with right-traveling-wave radiation condition

Mathematical Framework

Linear pde transformed into quadratically nonlinear, nonlocal, composition equation understood in terms of generalized functions

Why do this seemingly complicated transformation?

1) Approach allows for exploitation of correspondences between classical wave propagation, quantum physics, and modern mathematical asymptotics (microlocal analysis)
2) Operator symbol is natural physical quantity that encodes underlying profile and dynamical information in very convenient fashion (useful point for inverse formulations)
3) Operator symbol leads to explicit wavefield representation in terms of the path integral

Mathematical Framework

4) In addressing problem at level of infinitesimal generator, approximations made at level of operator symbol have a greater range of validity, in general, then corresponding approximations made at level of wavefield (important computational point)

   For example, high frequency on the operator symbol is a full-wave approximation valid in the moderate- to low-frequency wavefield regimes

5) Provides a systematic mathematical framework for addressing a comprehensive range of approximations (wide-angle parabolic, generalized phase screen, etc.) – in some sense, everyone is trying to approximate the symbols

6) Sharp, uniformly approximate solutions to composition equation can be constructed, making this entire discussion relevant in practice

Mathematical Framework

Natural Question

Can we take the desired exact and uniform approximate operator symbol constructions directly from the quantum mechanical and modern mathematical asymptotic literatures?

Answer – No!

Why?

(1) Mathematical analysis → Quantum mechanical results

(2) While the mathematical analysis provides the complete framework for the equations, microlocal analysis (asymptotics) only considers part of the solution – it is an approximation

Mathematical Framework

Why?

(3) Approximation appropriate for time-domain formulations not frequency-domain Helmholtz equation

(4) Will result in nonuniform, singular approximations for Helmholtz equation

Illustration – Weyl composition equation

$$\Omega_{\mathbf{B}^2}(p,q) = K^2(q) - p^2 = \left(\bar{k}/\pi\right)^2 \int_{\mathbb{R}^4} dt\,ds\,dv\,du\,\Omega_{\mathbf{B}}(t+p, s+q)$$

$$\bullet\,\Omega_{\mathbf{B}}(v+p, u+q)\exp\left(2i\bar{k}(sv - tu)\right)$$

Mathematical Framework

Illustration – Weyl composition equation

In the high-frequency $\left(\bar{k} \to \infty\right)$ limit, this takes the form

$$\Omega_{\mathbf{B}^2}\left(p,q\right) = K^2\left(q\right) - p^2 = \lim_{\substack{\eta\to p\\ y\to q}} \cos\left(\frac{1}{2\bar{k}}\left(\partial_\eta\partial_q - \partial_p\partial_y\right)\right)\Omega_{\mathbf{B}}\left(p,q\right)\Omega_{\mathbf{B}}\left(\eta,y\right)$$

Substituting the expansion

$$\Omega_{\mathbf{B}}\left(p,q\right) \sim \Omega_{\mathbf{B}}^{(0)}\left(p,q\right) + \left(1/\bar{k}^2\right)\Omega_{\mathbf{B}}^{(2)}\left(p,q\right) + \ldots$$

results in

$$\Omega_{\mathbf{B}}\left(p,q\right) \sim \left(K^2\left(q\right) - p^2\right)^{1/2} - \frac{K^3\left(q\right)\partial_q^2 K\left(q\right)}{8\bar{k}^2\left(K^2\left(q\right) - p^2\right)^{5/2}} + \ldots$$

Mathematical Framework

Illustration – Weyl composition equation

The corresponding result in the standard calculus is

$$h_{\mathbf{B}}^{s}(p,q) \sim \left(K^{2}(q)-p^{2}\right)^{1/2}+\frac{K(q)\partial_{q}K(q)\,p}{2i\bar{k}\left(K^{2}(q)-p^{2}\right)^{3/2}}+O\left(1/\bar{k}^{2}\right)$$

For the case of the DtN operator symbol, the result is

$$h_{\Lambda^{+}}^{s}(x,p,q) \sim -i\left(K^{2}(x,q)-p^{2}\right)^{1/2}+\frac{K(x,q)\partial_{x}K(x,q)}{2\bar{k}\left(K^{2}(x,q)-p^{2}\right)}+O\left(1/\bar{k}^{2}\right)$$

All are nonuniform, singular expansions.

Why?

Mathematical Framework

Illustration – Weyl composition equation

(1) Mathematically, the underlying elliptic ψDO calculus requires the symbol for $\mathbf{B}^2$ to be bounded away from 0, which it is not.

(2) Physically, modern mathematical asymptotics is about the propagation of singularities (e.g., wave fronts). This is appropriate for hyperbolic models such as the time-domain, plasma, and Klein-Gordon wave equations. The asymptotics will result in uniform expansions in these cases.

The Helmholtz equation is an “elliptic wave propagation” problem, characterized by smoothing (the evanescent modes). Singularities do not propagate in the Helmholtz case; they are damped out immediately.

Mathematical Framework

Illustration – Weyl composition equation

Applying the elliptic ψDO calculus (or an equivalent analysis) to the Helmholtz case results in the nonuniform, singular expansions.

Uniform operator symbol approximations over phase space must incorporate contributions from the omitted part of the solution (terms of exponential order from infinitely smooth part of kernel).

This requires going beyond the examples from quantum mechanics and the results from modern mathematical asymptotics.

Nonuniform Symbol Constructions

(1) High-Frequency Approximation (from ψDO result)

$$h_{\mathbf{B}}^{s}(p,q) \approx \left(K^2(q) - p^2\right)^{1/2}$$

This is just the GPSPI approximation

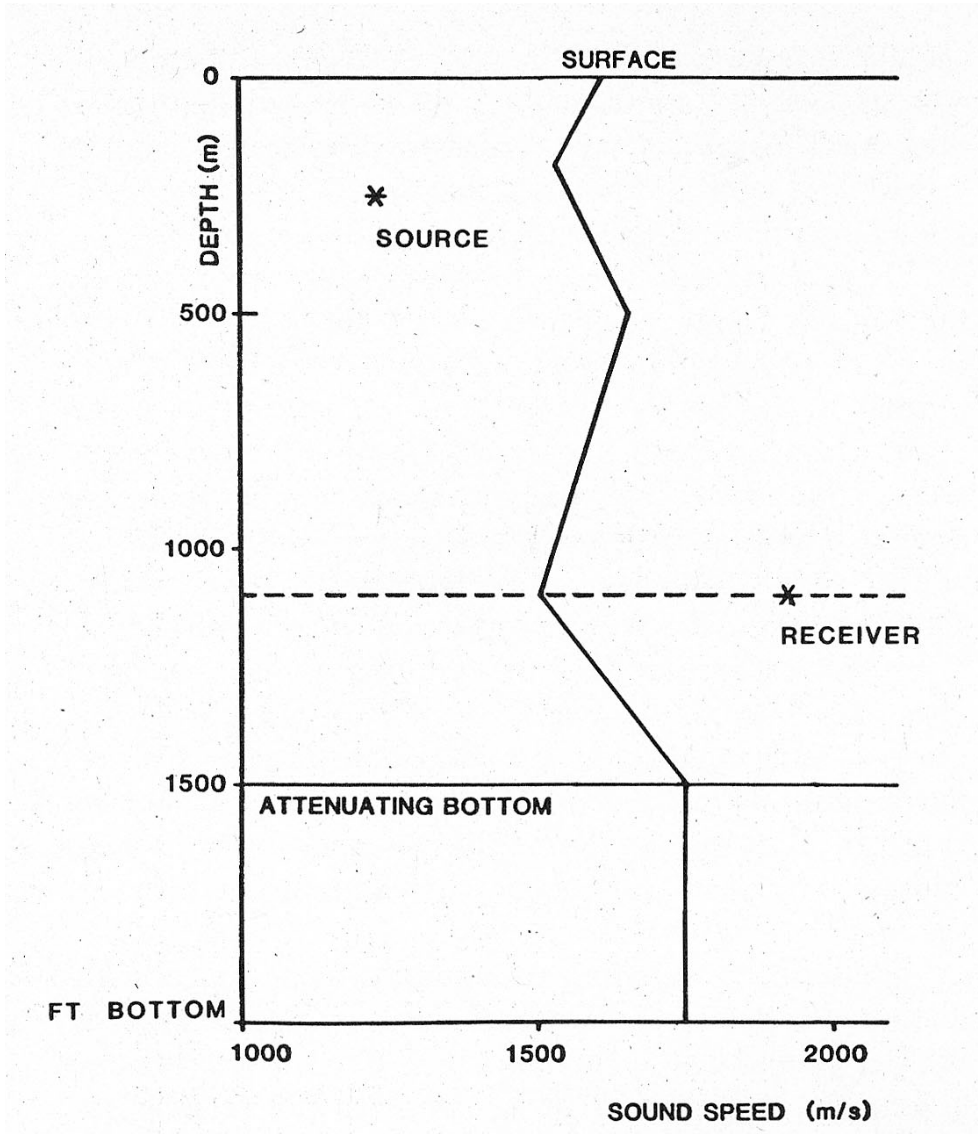
(2) Standard "Parabolic" Approximation (expanding (1))

$$h_{\mathbf{B}}^{s}(p,q) = \Omega_{\mathbf{B}}(p,q) \approx 1 + \frac{1}{2}\left(\left(K^2(q) - 1\right) - p^2\right)$$

(3) Range-Refraction Parabolic Approximation (Tappert, expanding nonuniform, singular ψDO result)

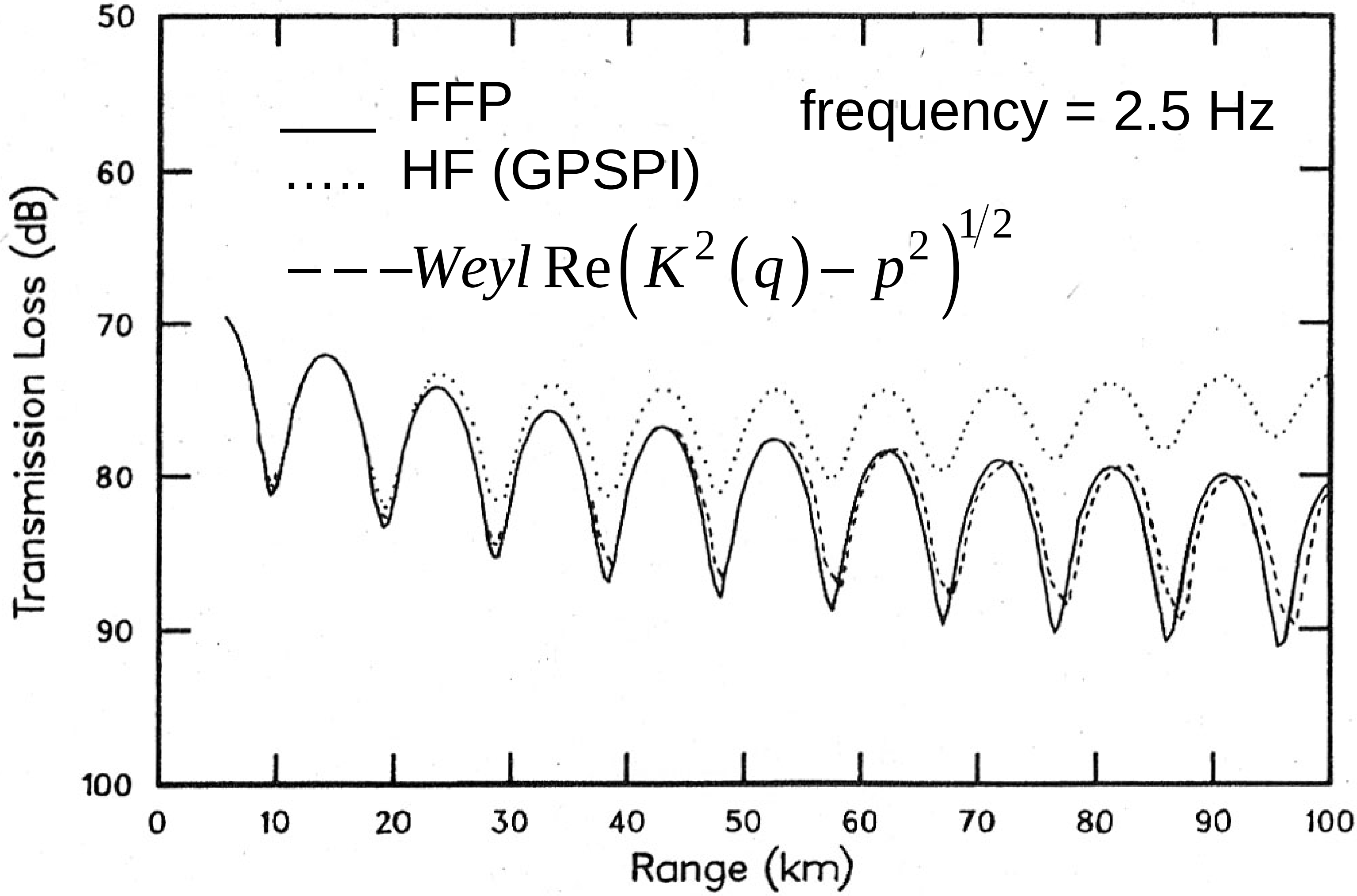
$$\Omega_{\mathbf{B}}(p,q) \approx K(q) - \left(\frac{p^2}{2K(q)} + \frac{\partial_q^2 K(q)}{8\bar{k}^2 K^2(q)}\right)$$

Model Computational Environment



Model Computation

Non-conservation of energy/flux in GPSPI



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Exact Symbols – Focusing Quadratic

Profile

$$K^2(q) = K_0^2 - \omega^2 q^2, n = 2, K_0, \omega > 0, q \varepsilon \mathbb{R}$$

Weyl operator symbol

$$\Omega_{\mathbf{B}}(p,q) = -\exp(i\pi/4)\left(\frac{\varepsilon}{2}\right)^{1/2}\left(\frac{1}{\pi}\right)\int_L d\tau \varsigma\left(1/2, (-i/2\pi)\tau, \exp(2i\pi Y)\right)$$

$$\bullet \exp\left(Y\tau - \mathrm{X}\tanh\tau\right)\operatorname{sec}h\tau\left(Y - \mathrm{X}\operatorname{sec}h^2\tau - \tanh\tau\right), Y \neq 0, 1, 2, \ldots$$

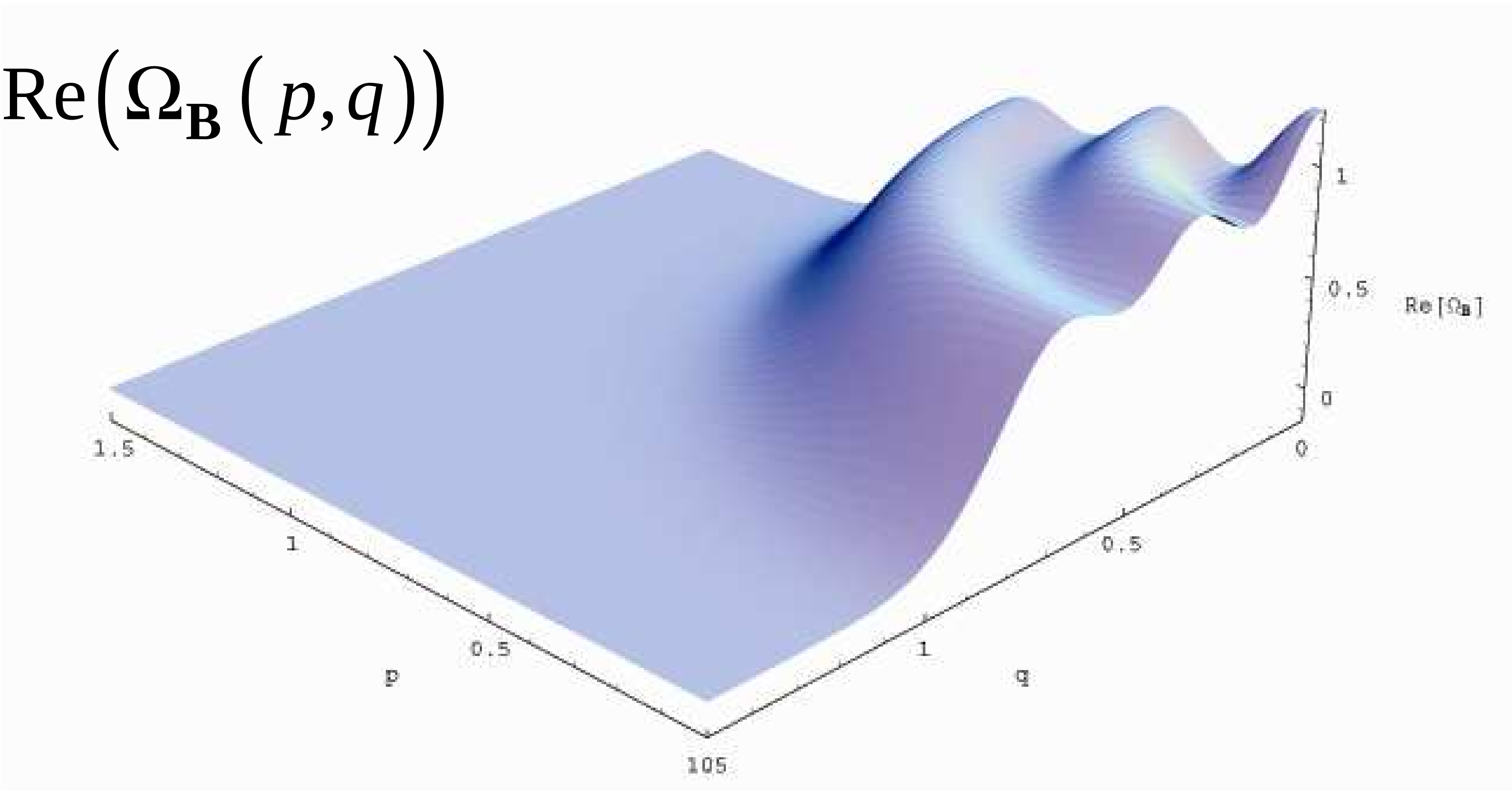
$$\mathrm{X} = (1/\varepsilon)\left(\omega^2 q^2 + p^2\right), Y = K_0^2/\varepsilon, \varepsilon = \omega/\overline{k}$$

$$\varsigma(\sigma, \Delta, \xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{(n+\Delta)^{\sigma}}, \Delta \neq 0, -1, -2, \ldots, |\xi| < 1, +$$ anal. conts.

(Lerch transcendental function)

Exact Operator Symbol
Focusing Quadratic Profile

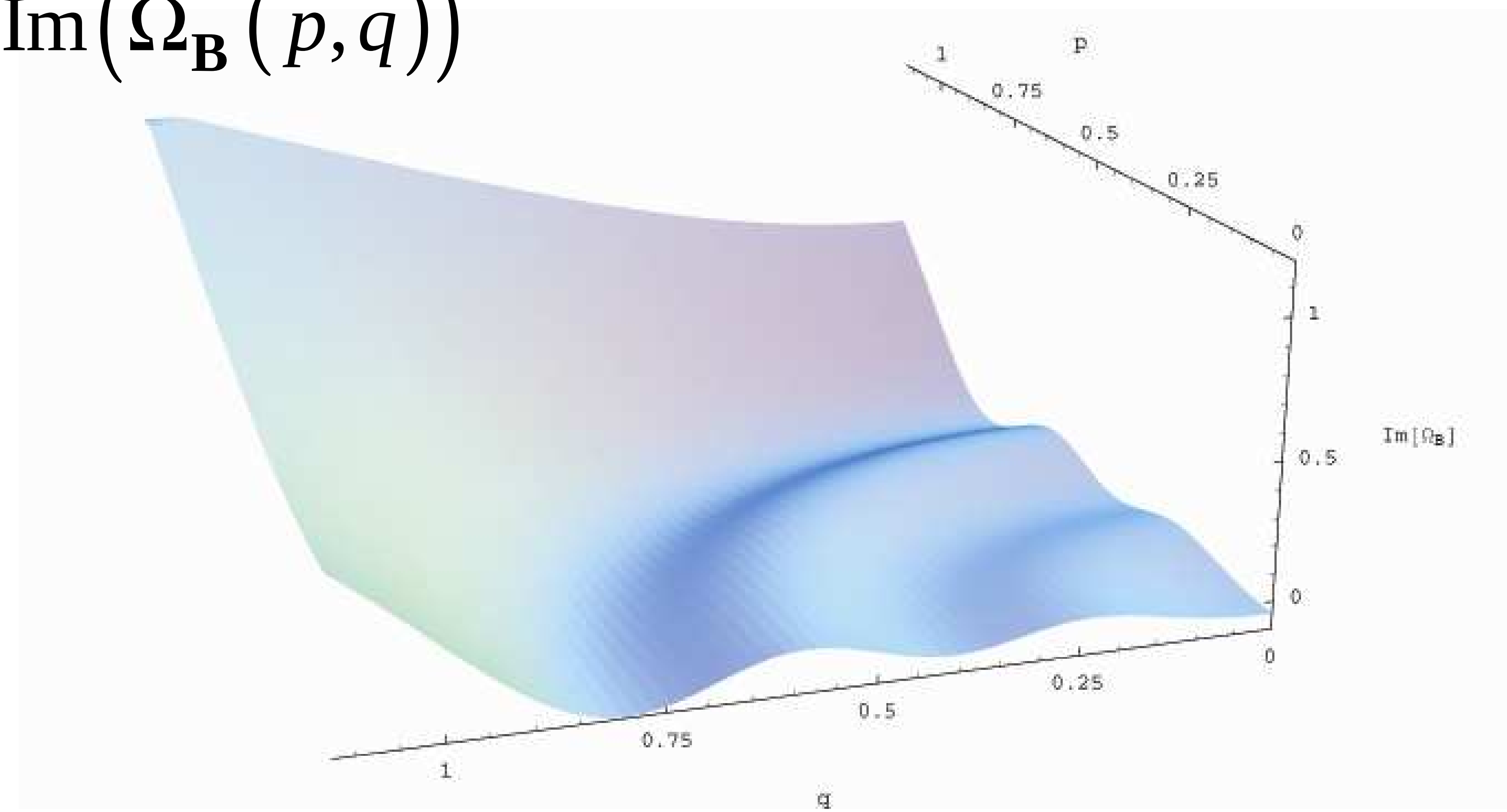
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



$$K^2\left(q\right)=1-q^2,\overline{k}=10.5$$

Exact Operator Symbol
Focusing Quadratic Profile

$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



$$K^2\left(q\right)=1-q^2, \bar{k}=10.5$$

Exact Symbols – Defocusing Quadratic

Profile

$$K^2(q) = K_0^2 + \omega^2 q^2, n = 2, K_0, \omega > 0, q \varepsilon \mathbb{R}$$

Weyl operator symbol

$$\Omega_{\mathbf{B}}(p,q) = -\left(\frac{\varepsilon}{2}\right)^{1/2}\left(\frac{1}{\pi}\right)\int_L d\tau \varsigma\left(1/2, (-i/2\pi)\tau, \exp(-2\pi Y)\right)$$

$$\bullet \exp\left(i\left(Y\tau + X\ \tanh\tau\right)\right)\operatorname{sec} h\tau\left(iY + iX\ \operatorname{sec} h^2\tau - \tanh\tau\right), Y \neq 0$$

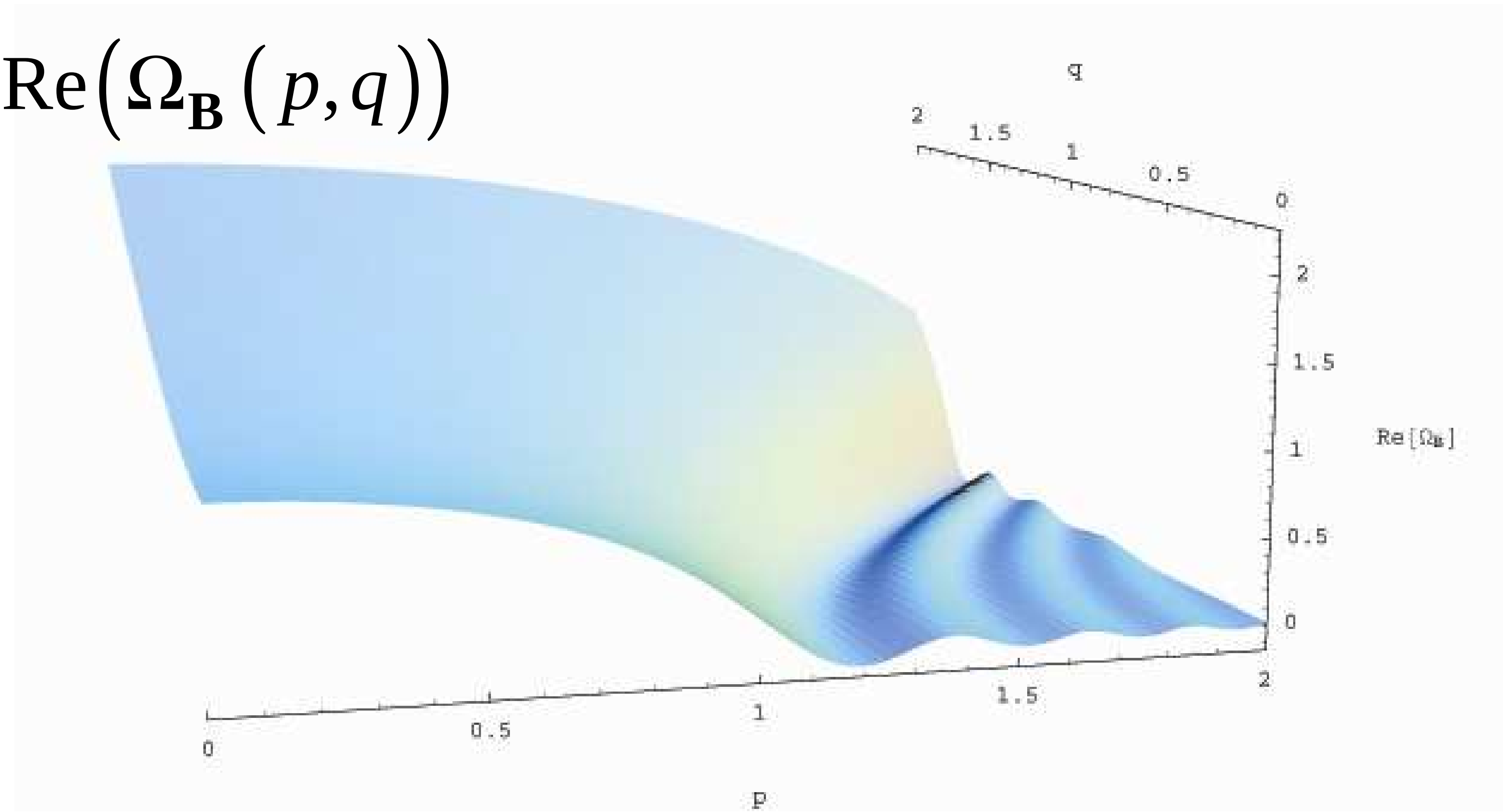
$X = (1/\varepsilon)\left(\omega^2 q^2 - p^2\right), Y = K_0^2/\varepsilon, \varepsilon = \omega/\overline{k}, \varsigma$ is Lerch trans. fn.

$$\Omega_{\mathbf{B}}(p,q) = -\exp(i\pi/4)(\varepsilon/\pi)^{1/2}\int_0^\infty dt \exp\left(i\left(Yt + X\ \tanh t\right)\right)$$

$$\bullet t^{-1/2} \operatorname{sec} ht\left(iY + iX\ \operatorname{sec} h^2 t - \tanh t\right)$$

Exact Operator Symbol Defocusing Quadratic Profile

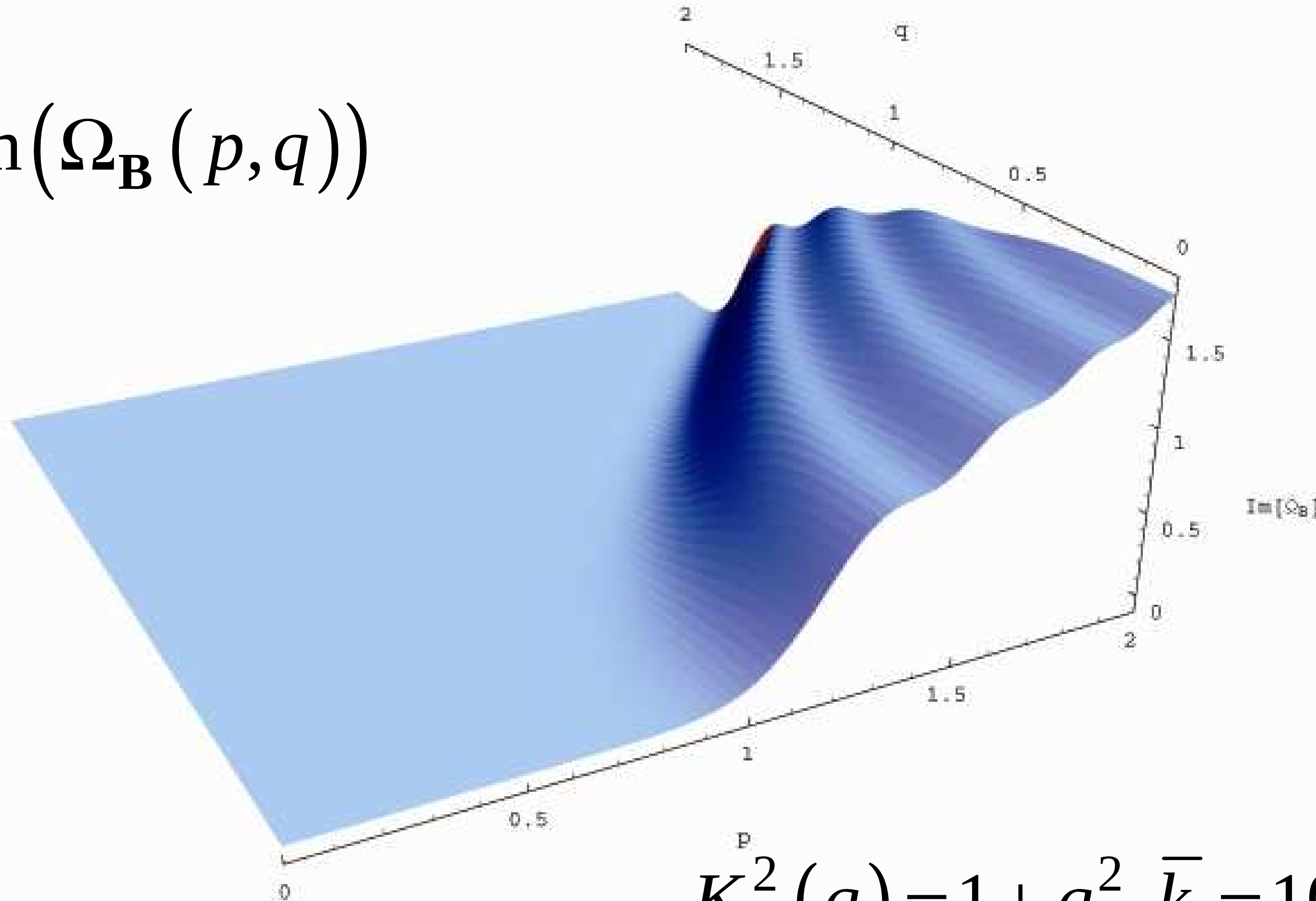
$\mathrm{Re}\left(\Omega_{\mathbf{B}}(p,q)\right)$



$$K^2(q) = 1 + q^2, \bar{k} = 10.5$$

Exact Operator Symbol Defocusing Quadratic Profile

$$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



$$K^2\left(q\right)=1+q^2, \bar{k}=10.5$$

Exact Symbols – Delta Distribution

Profile

$$K^2(q) = K_0^2 + 2\lambda\delta(q), n = 2$$

$K_0 > 0, \lambda, q \varepsilon \mathbb{R}, \lambda > 0$ (focusing) , $\lambda < 0$ (defocusing)

Weyl operator symbol

$$\Omega_{\mathbf{B}}(p,q) = \left(K_0^2 - p^2\right)^{1/2} - \left(2i\bar{k}\lambda/\pi\right)\exp\left(2iK_0Q\right)$$

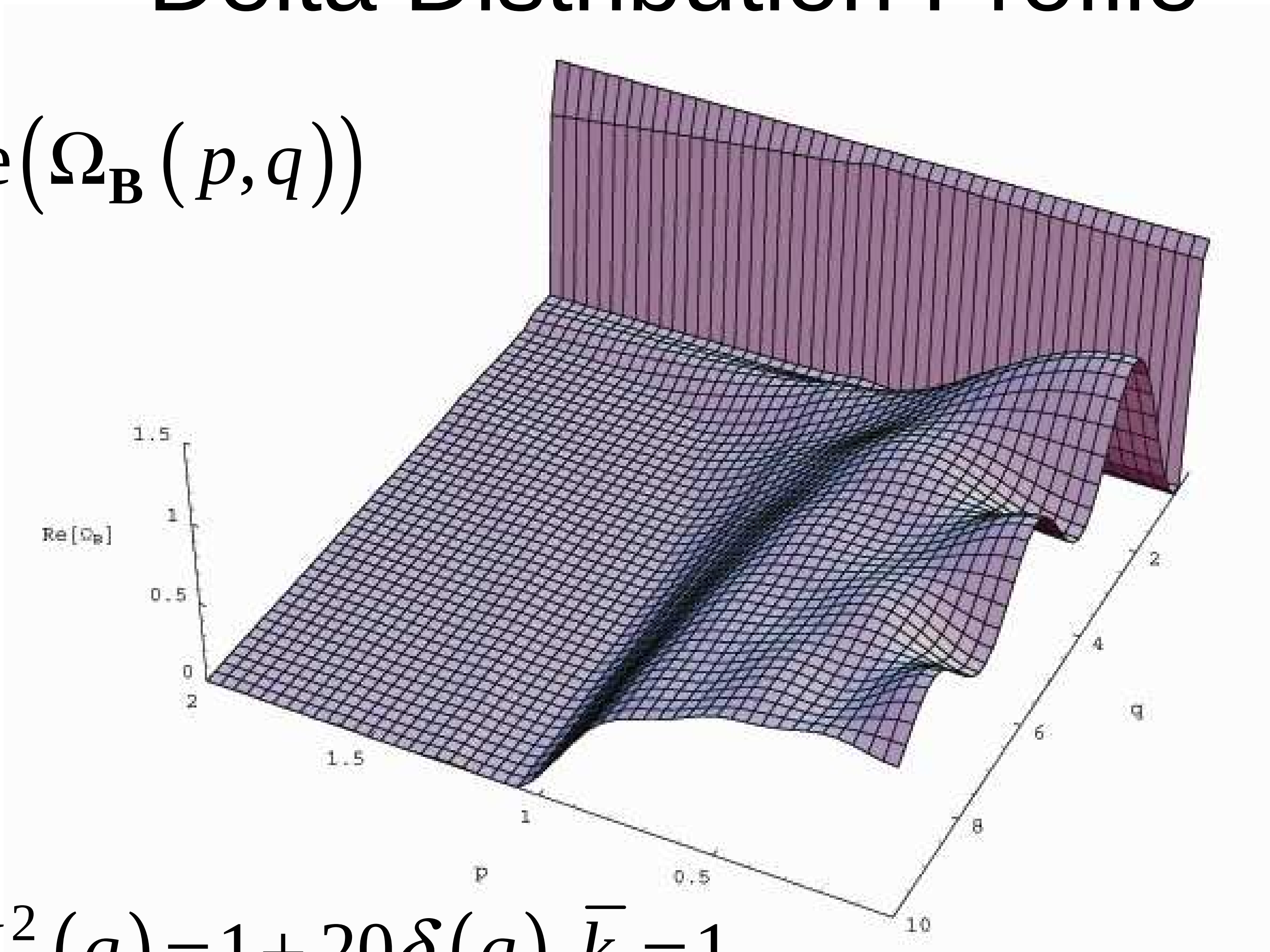
$$\bullet\int_0^\infty dt t^{1/2}\left(t - 2iK_0\right)^{1/2}\left(t - \left(\bar{k}\lambda + iK_0\right)\right)^{-1}\exp\left(-2Qt\right)$$

$$\bullet\left(\frac{\sin\left(2pQ\right)}{p} + \frac{1}{2}\left(\frac{\exp\left(2ipQ\right)}{\left(t - i\left(K_0 + p\right)\right)} + \frac{\exp\left(-2ipQ\right)}{\left(t - i\left(K_0 - p\right)\right)}\right)\right)$$

$$Q = \bar{k}\left|q\right|$$

Exact Operator Symbol
Delta Distribution Profile

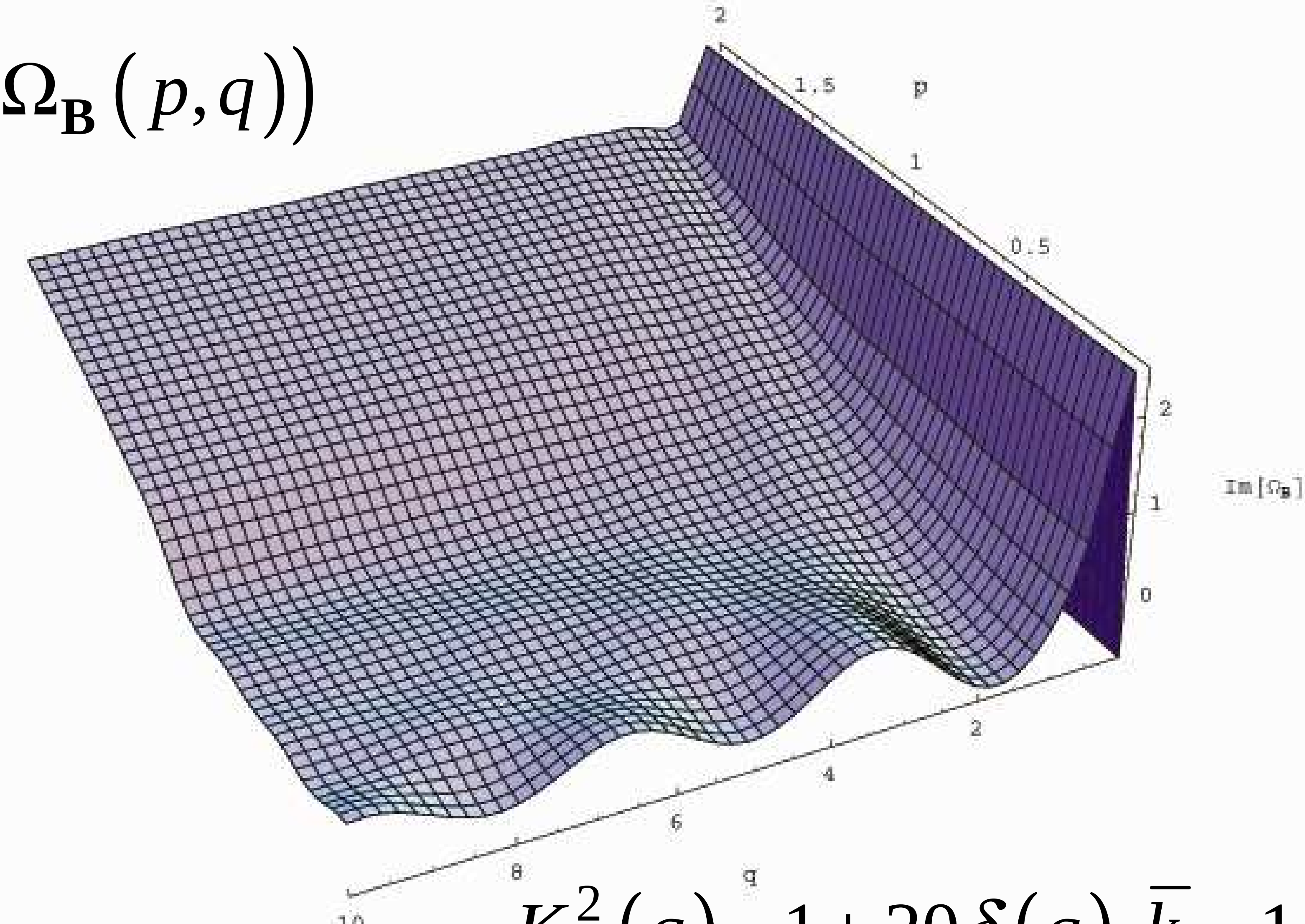
$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



$$K^2\left(q\right)=1+20\delta\left(q\right),\overline{k}=1$$

Exact Operator Symbol
Delta Distribution Profile

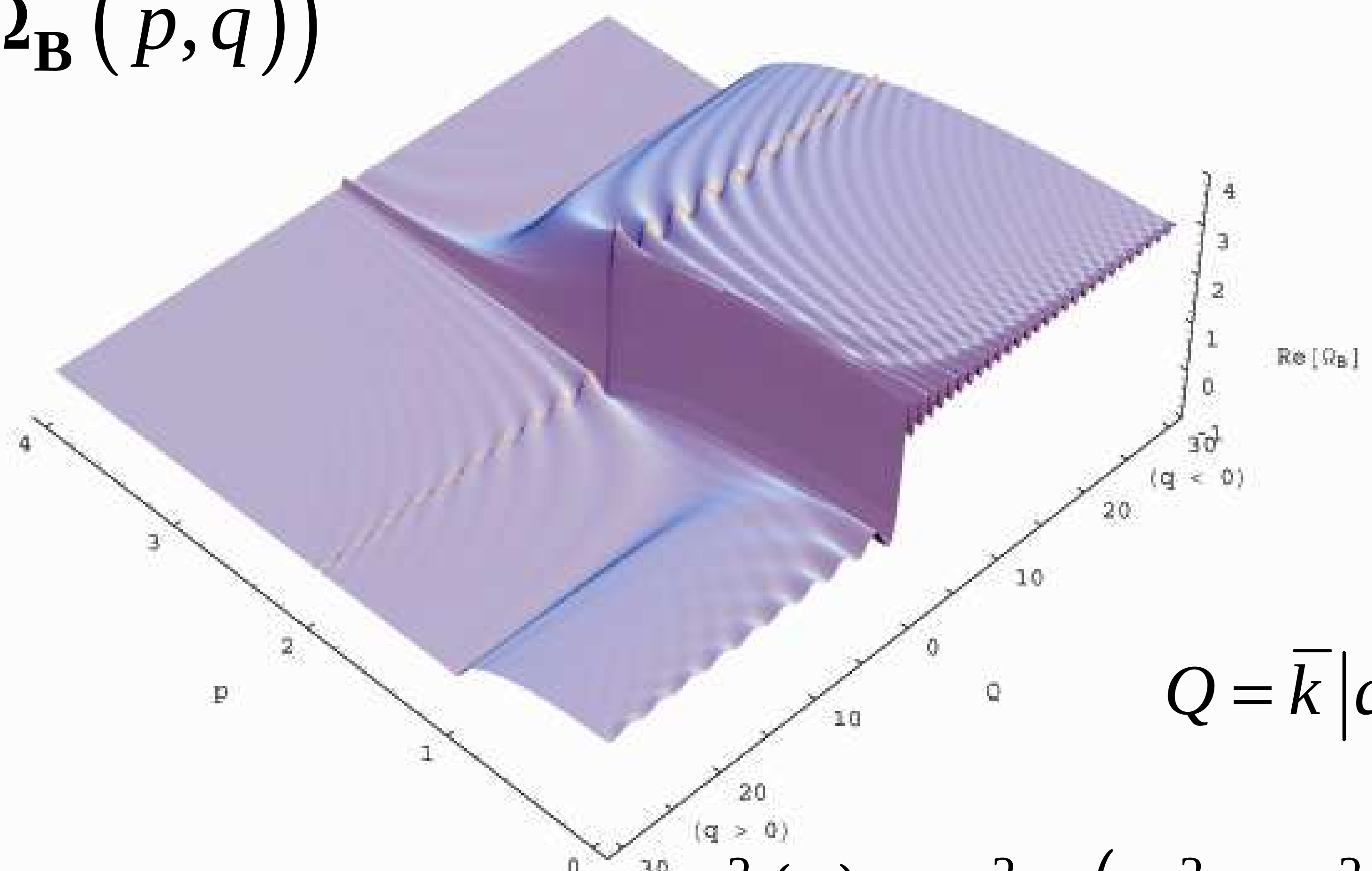
$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



$K^2\left(q\right)=1+20\delta\left(q\right),\overline{k}=1$

Exact Operator Symbol
2-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



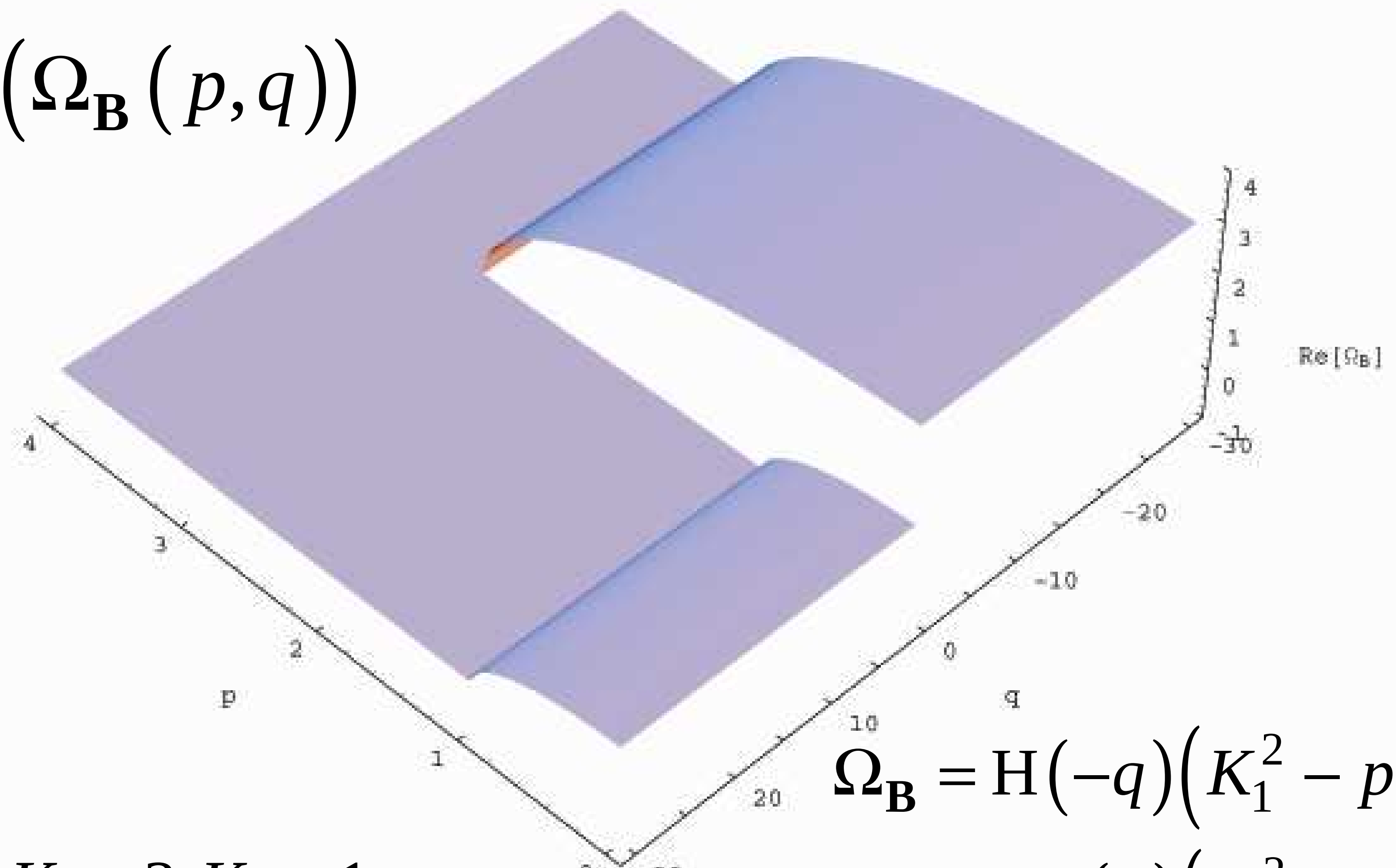
$$Q=\bar{k}\left|q\right|$$

$$K_1=3, K_2=1$$

$$K^2\left(q\right)=K_1^2+\left(K_2^2-K_1^2\right)\mathrm{H}\left(q\right)$$

Locally Homogeneous Approximation
2-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$

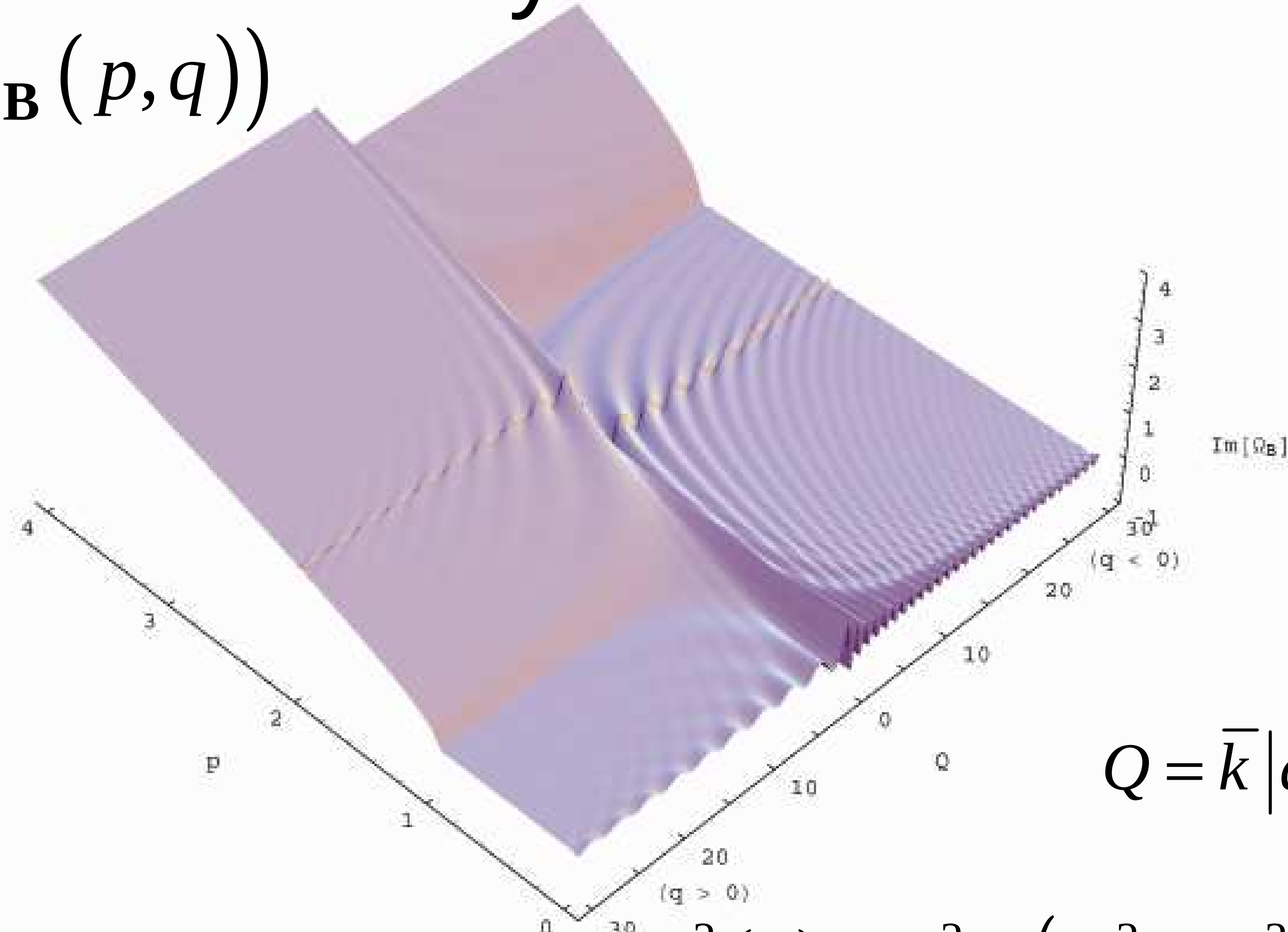


$$K_1 = 3, K_2 = 1$$

$$\Omega_{\mathbf{B}} = \mathrm{H}(-q)\left(K_1^2 - p^2\right)^{1/2} + \mathrm{H}(q)\left(K_2^2 - p^2\right)^{1/2}$$

Exact Operator Symbol
2-Layer Profile

$$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



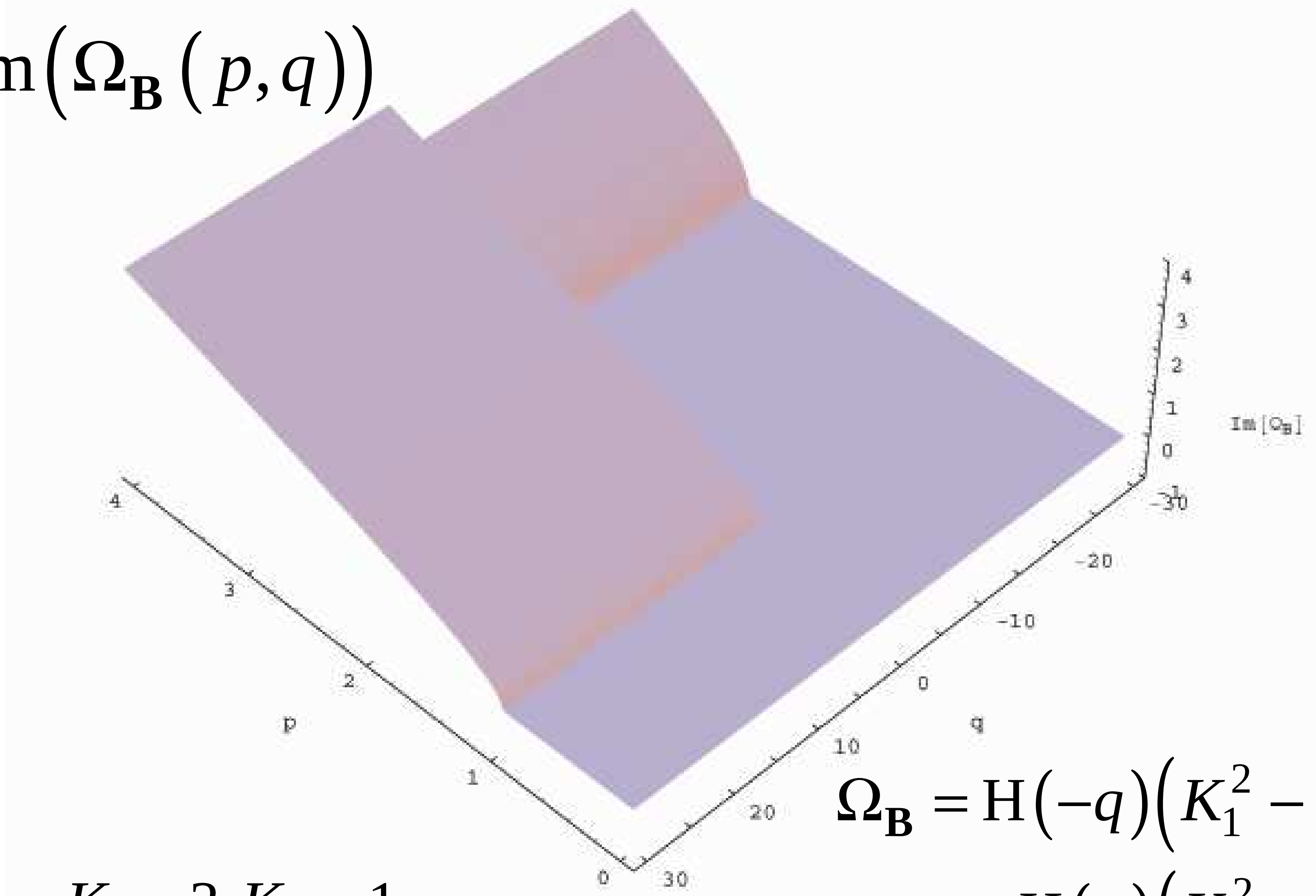
$$Q = \bar{k}\left|q\right|$$

$$K_1 = 3, K_2 = 1$$

$$K^2\left(q\right) = K_1^2 + \left(K_2^2 - K_1^2\right)\mathrm{H}\left(q\right)$$

Locally Homogeneous Approximation
2-Layer Profile

$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$

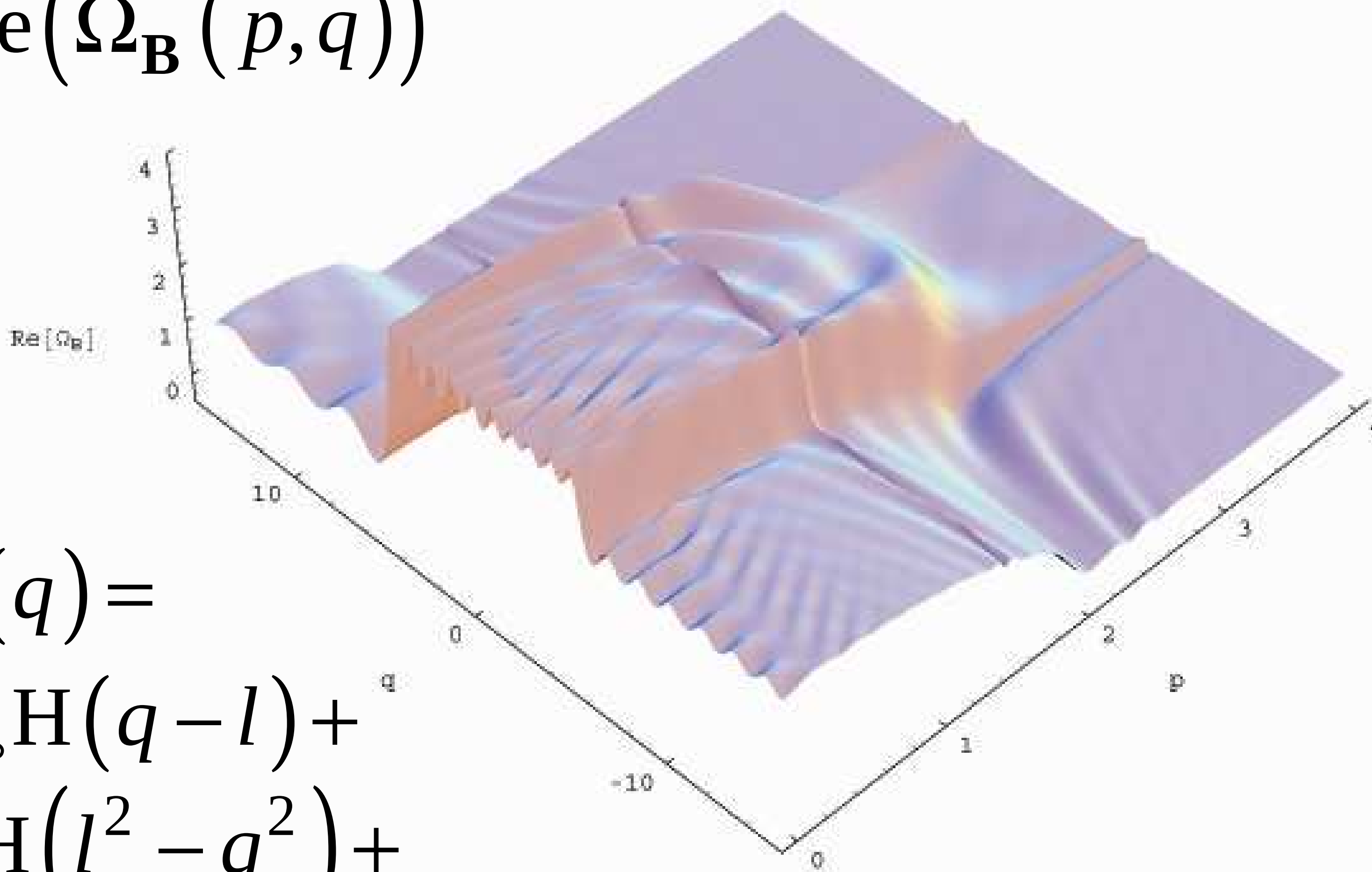


$K_1 = 3, K_2 = 1$

$$\Omega_{\mathbf{B}} = \mathrm{H}(-q)\left(K_1^2 - p^2\right)^{1/2} + \mathrm{H}(q)\left(K_2^2 - p^2\right)^{1/2}$$

Exact Operator Symbol 3-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}(p,q)\right)$$

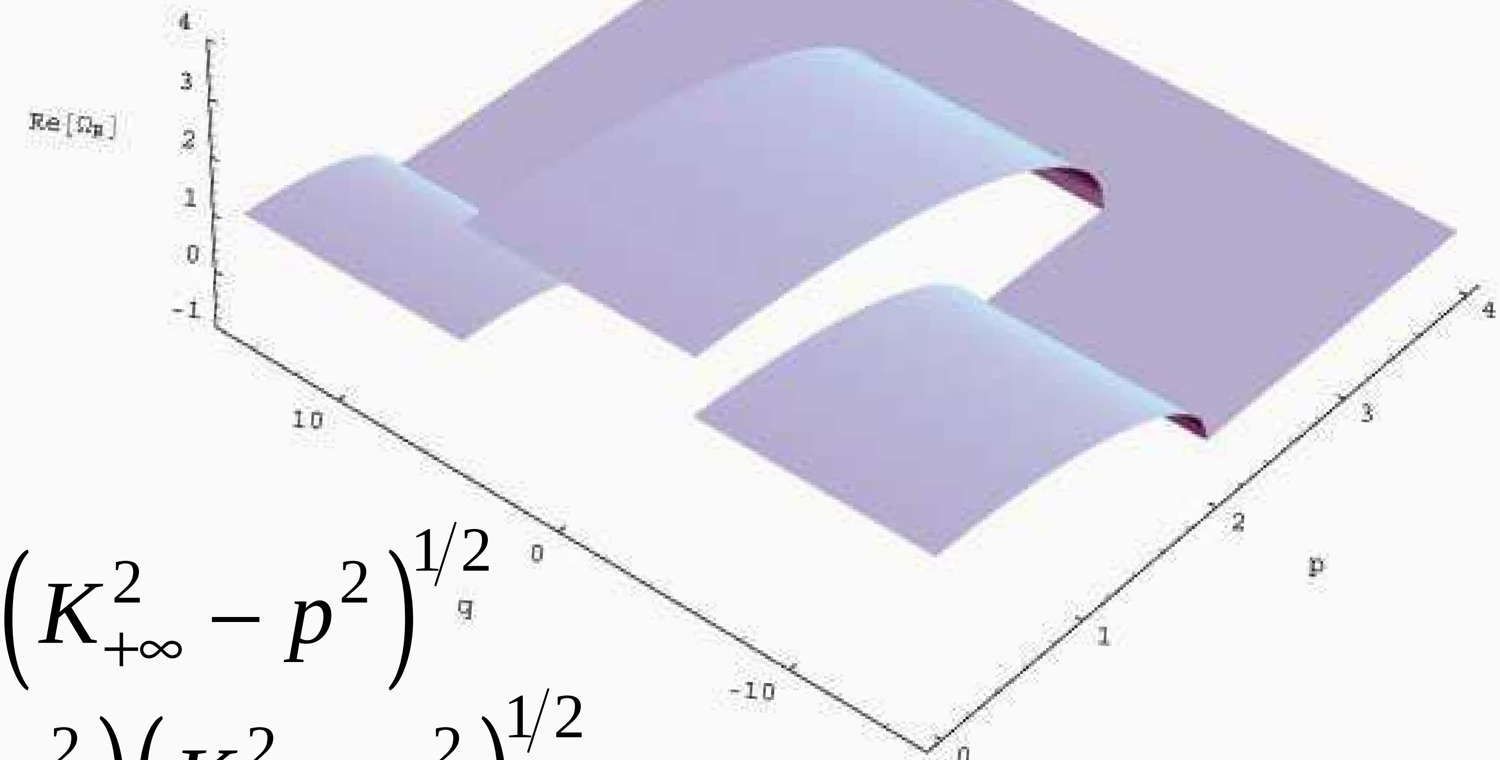


$$K^2(q) =$$
$$K_{+\infty}^2 \mathrm{H}(q-l) +$$
$$K_1^2 \mathrm{H}\left(l^2 - q^2\right) +$$
$$K_{-\infty}^2 \mathrm{H}(-q-l)$$

$$K_{-\infty} = 2, K_1 = 3, K_{+\infty} = 1$$
$$\bar{k} = 1, l = 5$$

Locally Homogeneous Approximation
3-Layer Profile

$$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



$$\Omega_{\mathbf{B}} =$$

$$\mathrm{H}\left(q-l\right)\left(K_{+\infty}^2 - p^2\right)^{1/2}$$

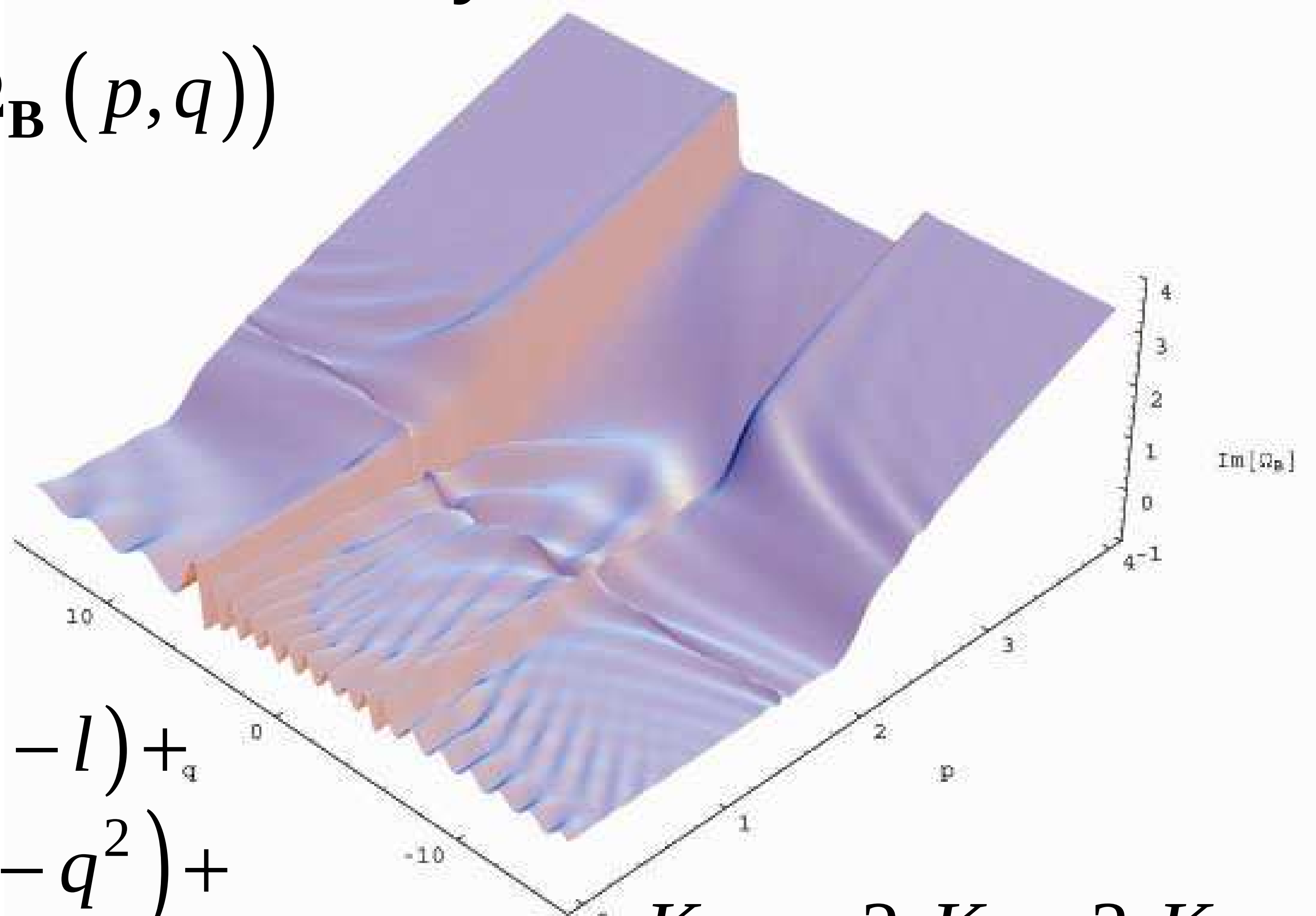
$$+\mathrm{H}\left(l^2-q^2\right)\left(K_1^2 - p^2\right)^{1/2}$$

$$+\mathrm{H}\left(-q-l\right)\left(K_{-\infty}^2 - p^2\right)^{1/2}$$

$$K_{-\infty}=2, K_1=3, K_{+\infty}=1$$
$$l=5$$

Exact Operator Symbol
3-Layer Profile

$$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$

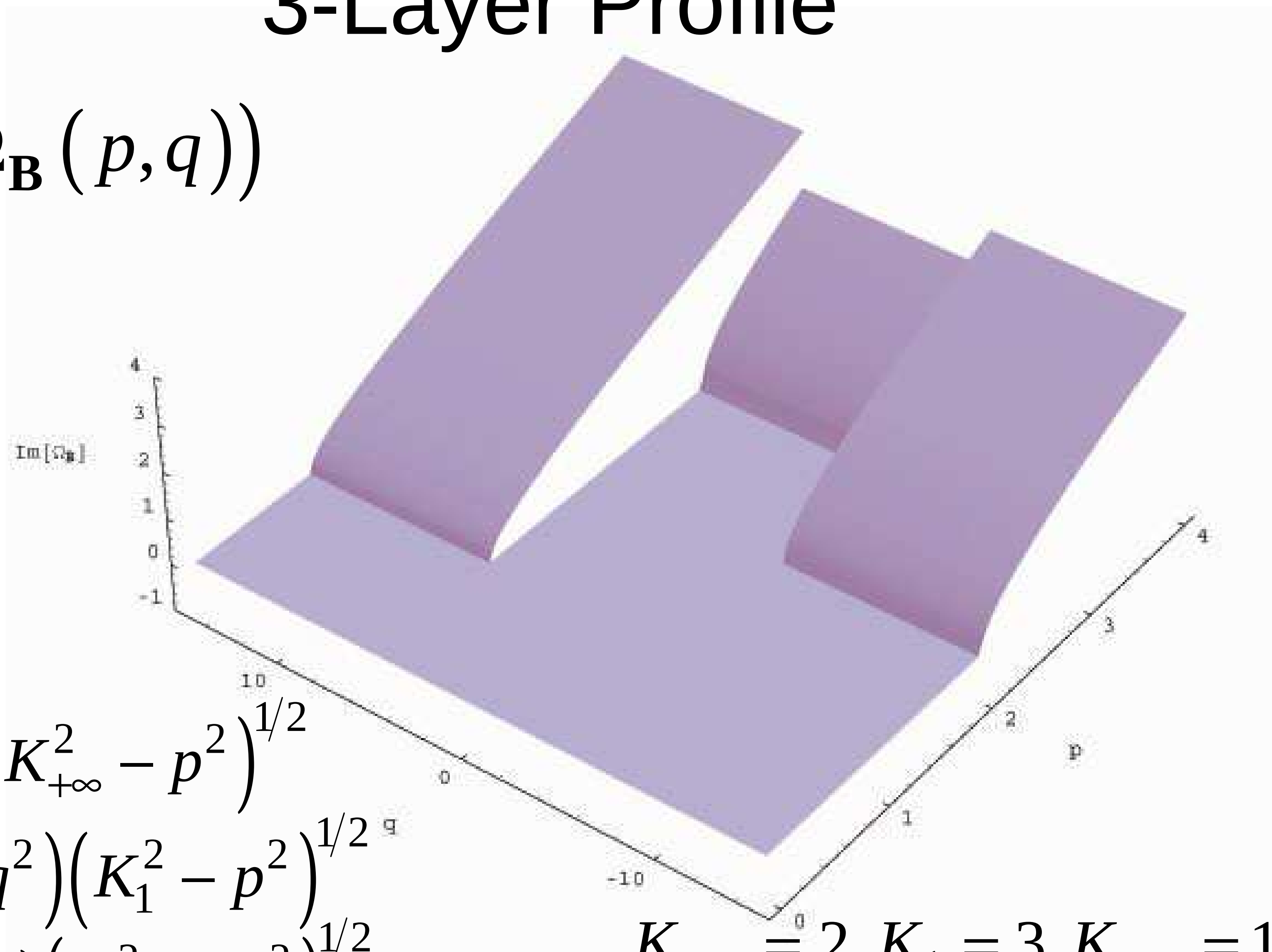


$$K^2(q) = K_{+\infty}^2 \mathrm{H}(q-l) + K_1^2 \mathrm{H}\left(l^2 - q^2\right) + K_{-\infty}^2 \mathrm{H}(-q-l)$$

$$K_{-\infty} = 2, K_1 = 3, K_{+\infty} = 1$$
$$\bar{k} = 1, l = 5$$

Locally Homogeneous Approximation
3-Layer Profile

$$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$

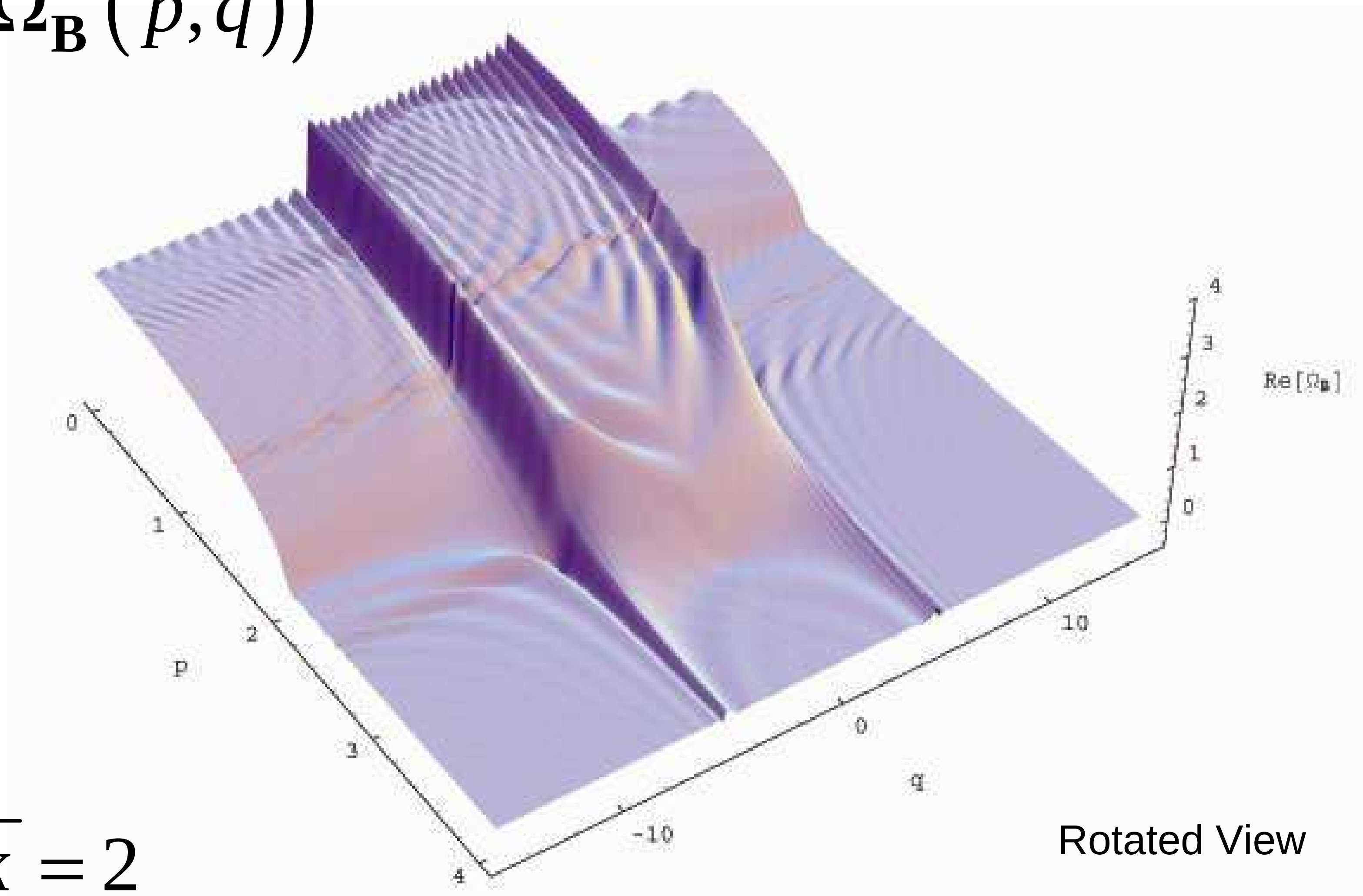


$$\Omega_{\mathbf{B}} =$$
$$\mathrm{H}\left(q-l\right)\left(K_{+\infty}^{2}-p^{2}\right)^{1/2}$$
$$+\mathrm{H}\left(l^{2}-q^{2}\right)\left(K_{1}^{2}-p^{2}\right)^{1/2}$$
$$+\mathrm{H}\left(-q-l\right)\left(K_{-\infty}^{2}-p^{2}\right)^{1/2}$$

$$K_{-\infty}=2, K_{1}=3, K_{+\infty}=1$$
$$l=5$$

Exact Operator Symbol
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$

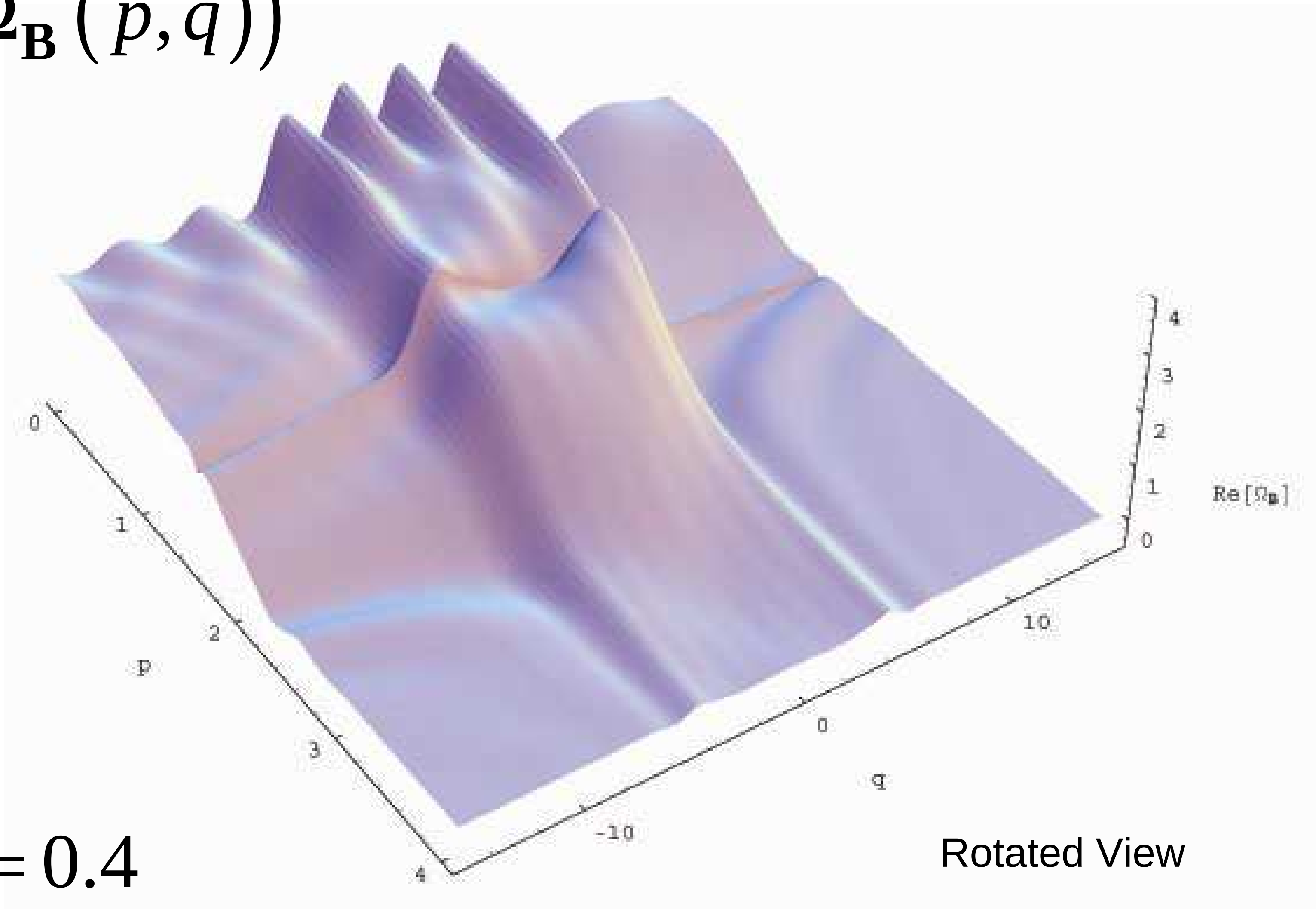


Rotated View

$\bar{k}=2$

Exact Operator Symbol
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$

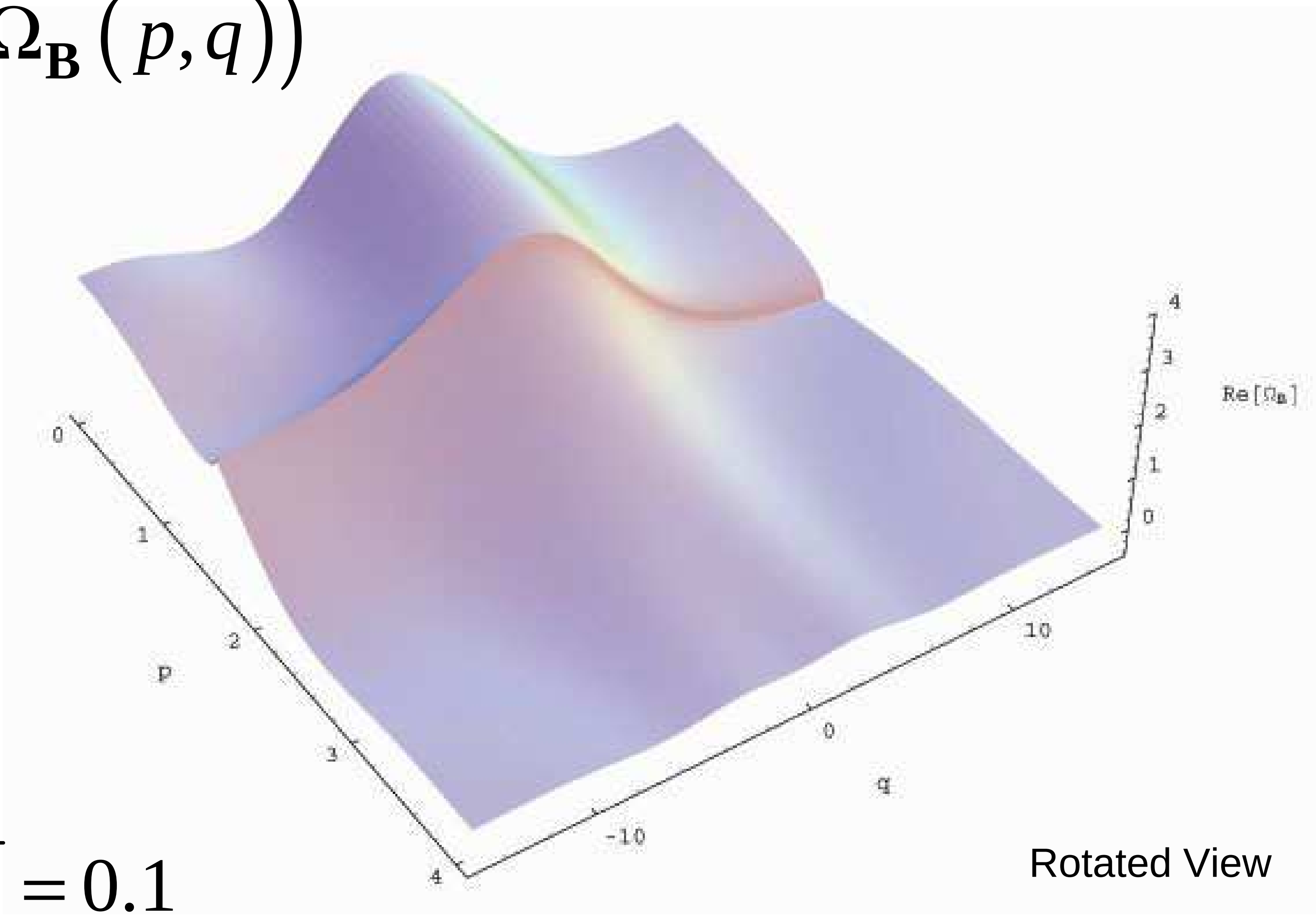


$\overline{k} = 0.4$

Rotated View

Exact Operator Symbol
3-Layer Profile

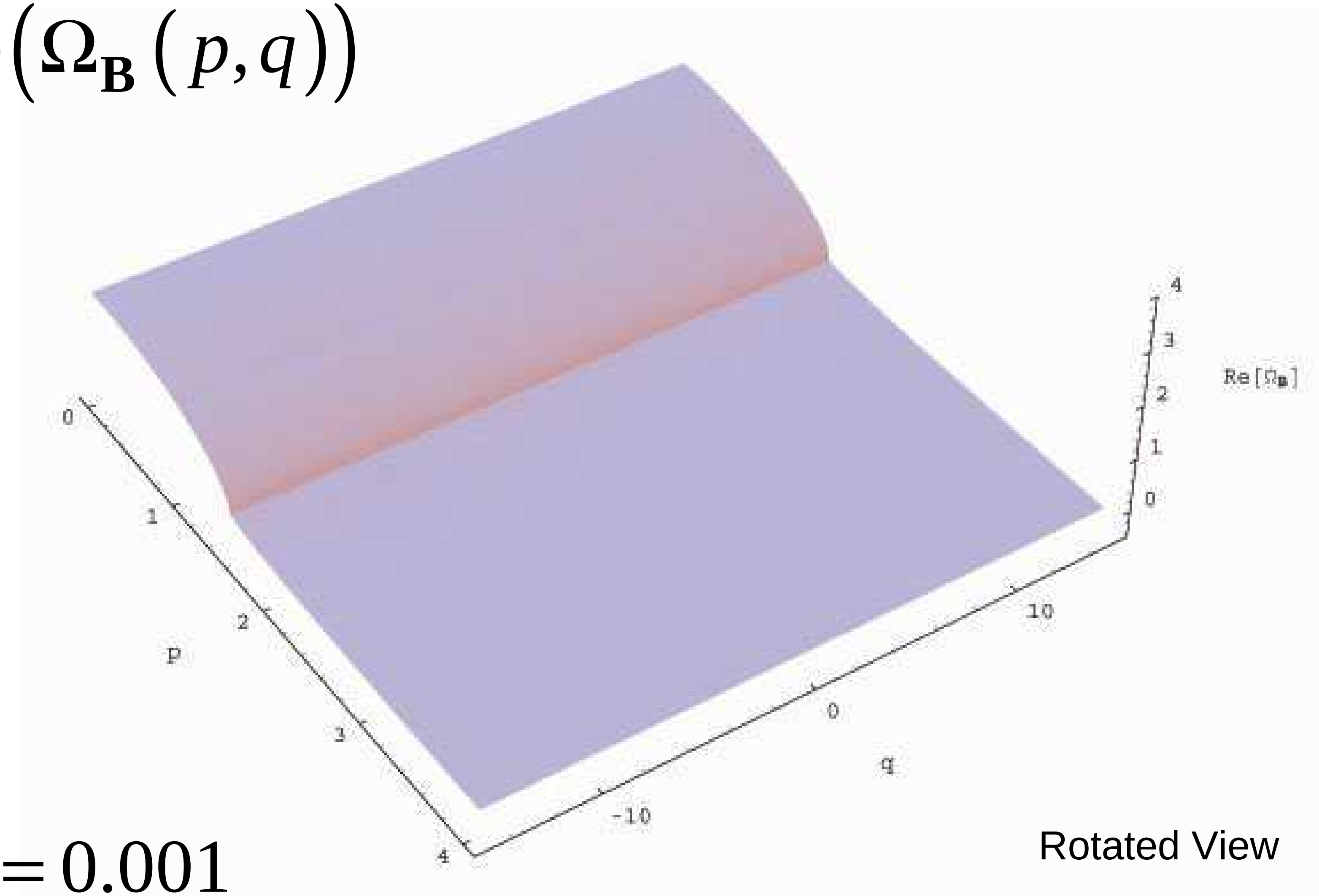
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



$\bar{k} = 0.1$

Exact Operator Symbol
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



$\bar{k} = 0.001$

Rotated View

Uniform High-Frequency Asymptotics

Derivation Outline

(1) Identity – add and subtract singularity

$$\Omega_{\mathbf{B}}(p,q)=\left(K^2(q)-p^2\right)^{1/2}+\left(\Omega_{\mathbf{B}}(p,q)-\left(K^2(q)-p^2\right)^{1/2}\right)$$

(2) n-d Helmholtz in (n-1)-d Helmholtz imbedding

$$\Omega_{\mathbf{B}}(p,q)=\left(K^2(q)-p^2\right)^{1/2}+\int_{\mathbb{R}}du\,\exp\left(i\bar{k}pu\right)$$

$$\bullet\left(i\int_0^{\infty}d\xi\,\xi^2\left(-4i\bar{k}\,\hat{\Phi}\left(\xi^2,q-u/2;0,q+u/2\right)\right.\right.$$

$$\left.\left.-\left(\bar{k}/\pi\right)\left(K^2(q)+\xi^2\right)^{-1/2}\exp\left(i\bar{k}\left(K^2(q)+\xi^2\right)^{1/2}|u|\right)\right)\right)$$

Uniform High-Frequency Asymptotics

Derivation Outline

where

$$\left(\partial_z^2 + \bar{k}^2\left(K^2(z) + \xi^2\right)\right)\hat{\Phi}\left(\xi^2, z; 0, z'\right) = -\left(\bar{k}/2\pi\right)\delta\left(z - z'\right)$$

supplemented with outgoing-wave radiation condition

Removal of turning, focal, and higher-order points for ξ ε [0,∞) from one-dimensional Helmholtz Green's function

(3) WKB expansion

$$-4\pi i\hat{\Phi}\left(\xi^2, z; 0, z'\right) \sim \sum_{n=0}^{2} \hat{\Phi}_n\left(\xi^2, z; 0, z'\right)\left(i\bar{k}\right)^{-n} + O\left(1/\bar{k}^3\right)$$

(a) WKB expansion now provides uniform approximation due to removal of turning, focal, and higher-order points for ξ ε [0,∞)

Uniform High-Frequency Asymptotics

Derivation Outline

(b) Number of terms retained ensures recovery of ψDO expansion in outer phase space region through $O\left(1/\bar{k}^2\right)$

(4) Additional asymptotics

(a) Stationary phase approximation

(b) Method of matched asymptotic expansions

Result:

(1) Uniform evaluation of the ξ-integral

(2) Uniform high-frequency operator symbol approximation as a single integral

Uniform High-Frequency Expansion

Operator Symbol

(1) n = 2

(2) Refractive index field varies "slowly" on wavelength scale

$$\Omega_{\mathbf{B}}(p,q) \sim \left(K^2(q) - p^2\right)^{1/2} + \int_0^\infty du \cos\left(\bar{k}pu\right)$$

$$\bullet \left(K^2(q) \left(A \left(\frac{H_1^{(1)}\left(\bar{k}I_0\right)}{I_0} + \frac{C}{\bar{k}} H_0^{(1)}\left(\bar{k}I_0\right) \right) - \frac{H_1^{(1)}\left(\bar{k}K(q)u\right)}{K(q)u} \right) \right)$$

where

$$A = \left(\frac{I_0}{I_1}\right)^{3/2} \left(\frac{1}{K^2(q)} \left(K(q+u/2) K(q-u/2) \right)^{-1/2} \right)$$

Uniform High-Frequency Expansion

Operator Symbol

$$C = \left(\frac{1}{8I_0}\right)\left(\frac{15I_2}{I_1^2} - \frac{3}{I_0} - \frac{6}{I_1}\left(\frac{1}{K^2(q+u/2)} + \frac{1}{K^2(q-u/2)}\right)\right.$$

$$\left. +2\left(\frac{K'(q-u/2)}{K^2(q-u/2)} - \frac{K'(q+u/2)}{K^2(q+u/2)}\right) - \tilde{I}\right)$$

$$I_m = \int_{q-u/2}^{q+u/2} dt\,\big(K(t)\big)^{1-2m}, m = 0,1,2$$

$$\tilde{I} = \int_{q-u/2}^{q+u/2} dt\left(\frac{\big(K'(t)\big)^2}{K^3(t)}\right)$$

Uniform High-Frequency Symbol

Characterization

(1) The UHF operator symbol is dimensionless as it must be.

(2) The UHF operator symbol is finite; the singular behavior associated with the ψDO symbol has been removed.

(3) In the homogeneous medium limit, the correct limiting form,

$$\Omega_{\mathbf{B}}(p,q)=\left(K_0^2-p^2\right)^{1/2}$$

is recovered.

(4) In the high-frequency limit, the correct limiting form,

$$\Omega_{\mathbf{B}}(p,q)=\left(K^2(q)-p^2\right)^{1/2}$$

is recovered.

Uniform High-Frequency Symbol

Characterization

(5) In the ψDO limit, the correct limiting form,

$$\Omega_{\mathbf{B}}\left(p,q\right)=i\left|p\right|-\frac{iK^2\left(q\right)}{2\left|p\right|}$$

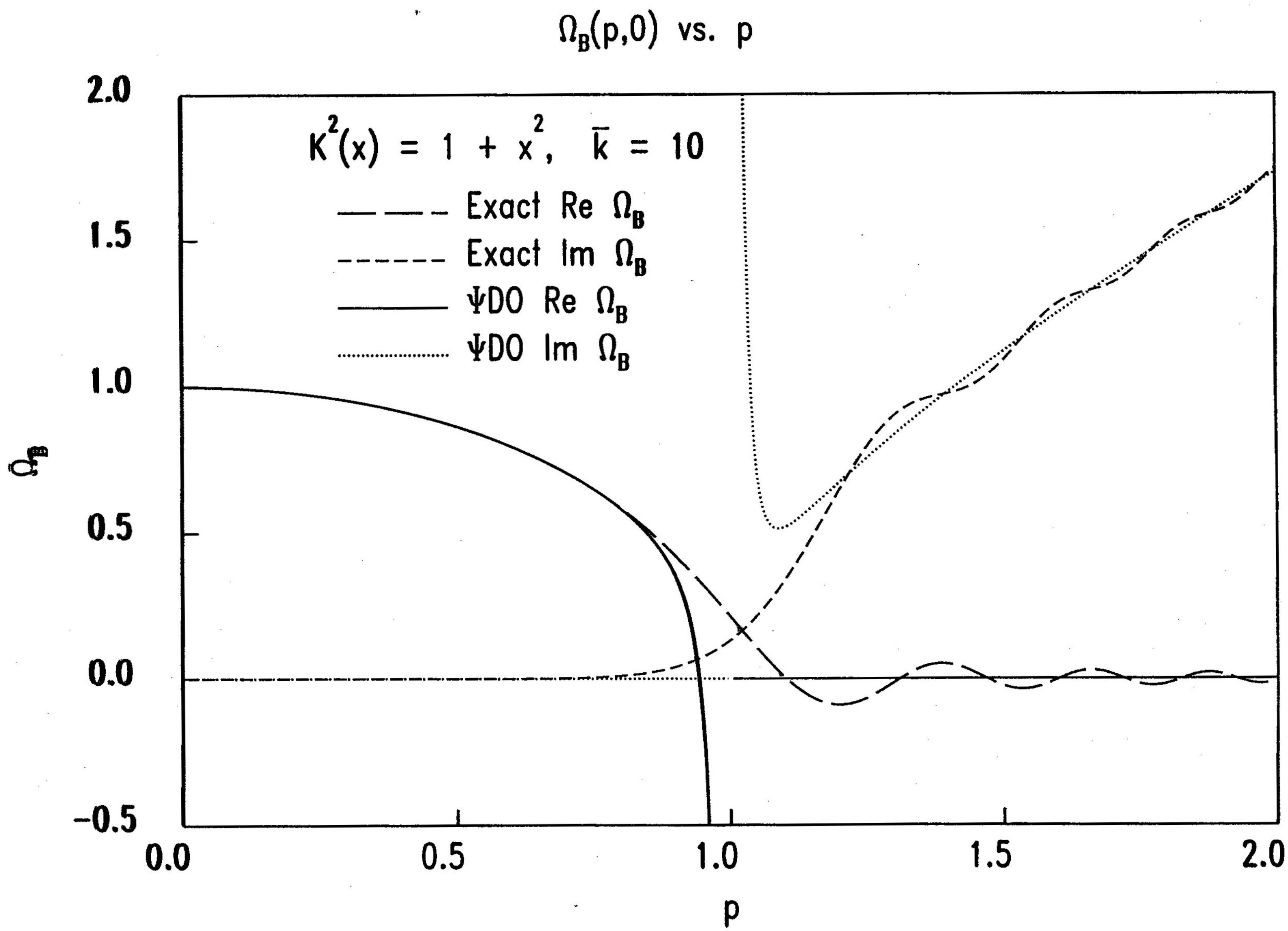
is recovered.

(6) The expansion is correct through terms of $O\left(1/\bar{k}^2\right)$.

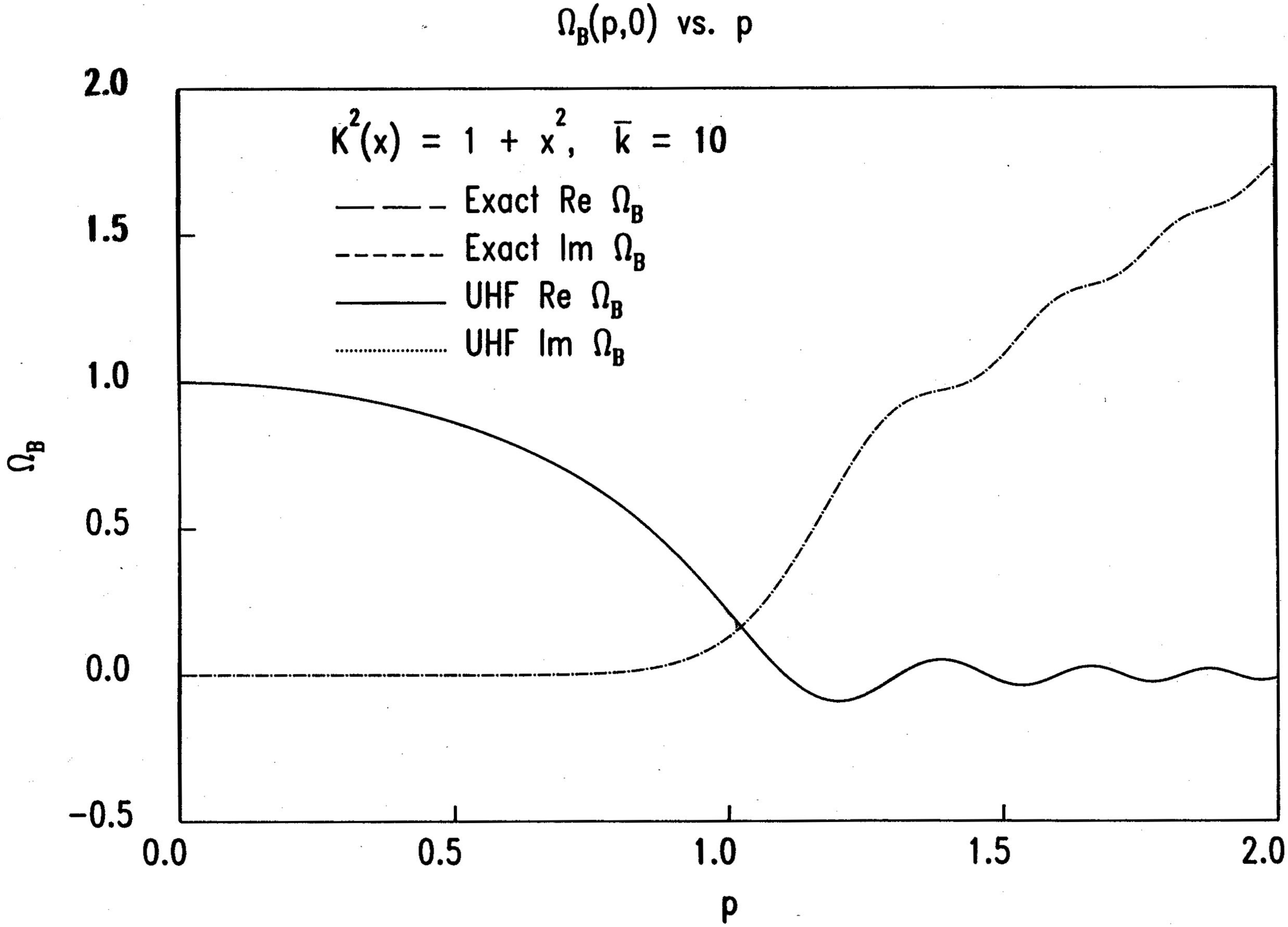
(7) The full ψDO expansion is recovered from an asymptotic evaluation of the integral about the end point at u = 0.

(8) For the defocusing quadratic profile, the UHF operator symbol reproduces both the algebraic and oscillatory asymptotic branches derived from the exact operator symbol construction.

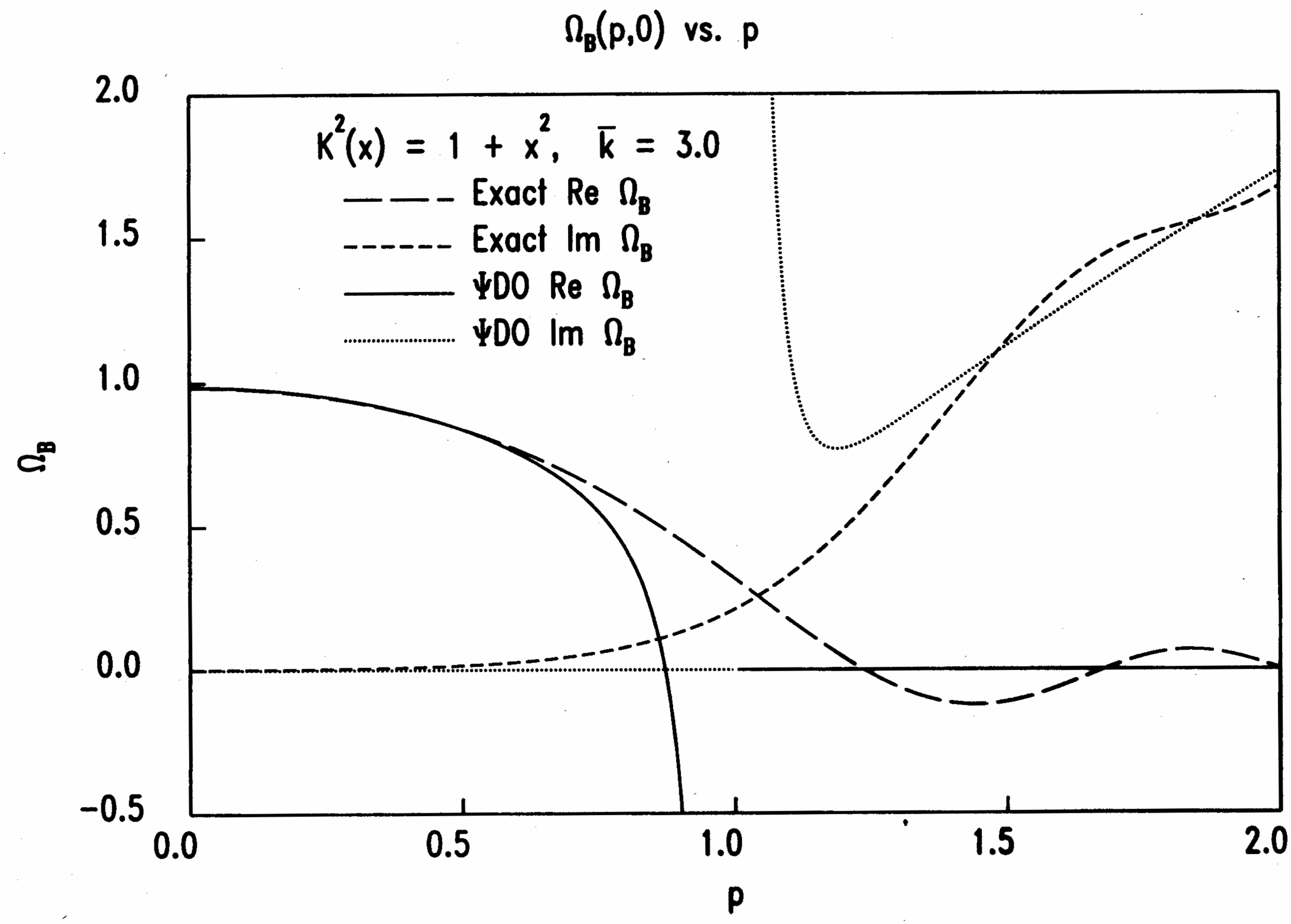
Pseudodifferential Versus Exact



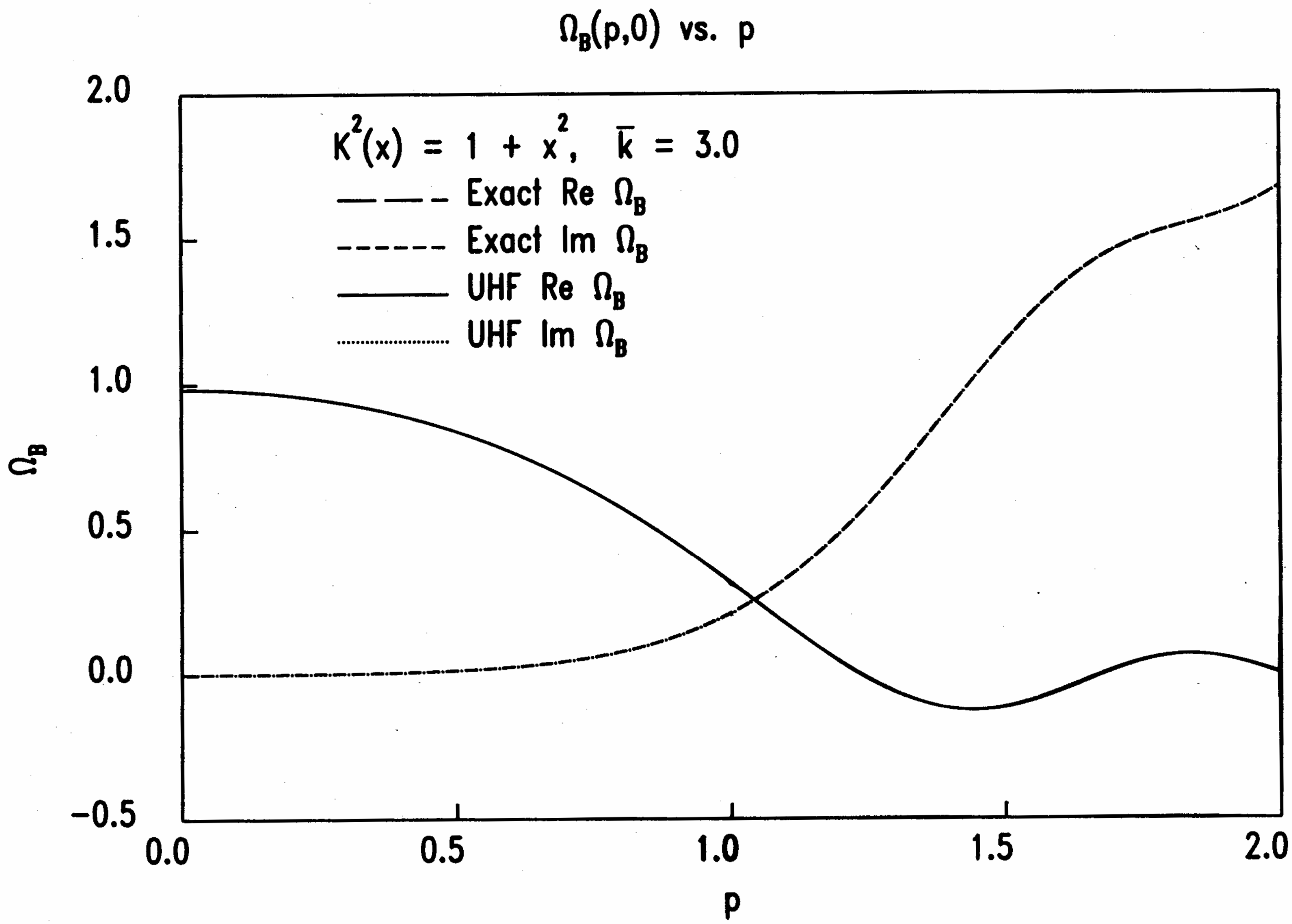
UHF Versus Exact



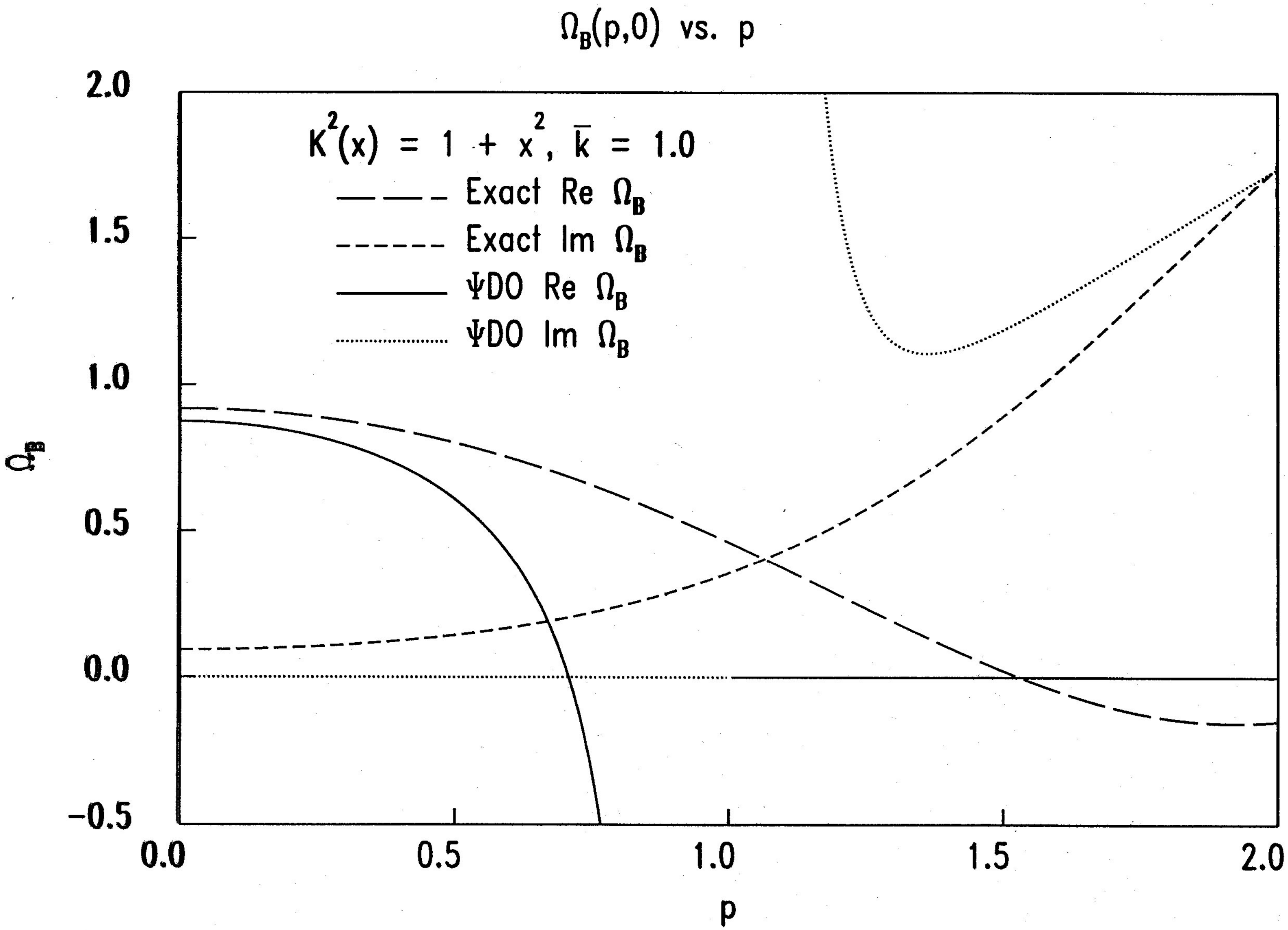
Pseudodifferential Versus Exact



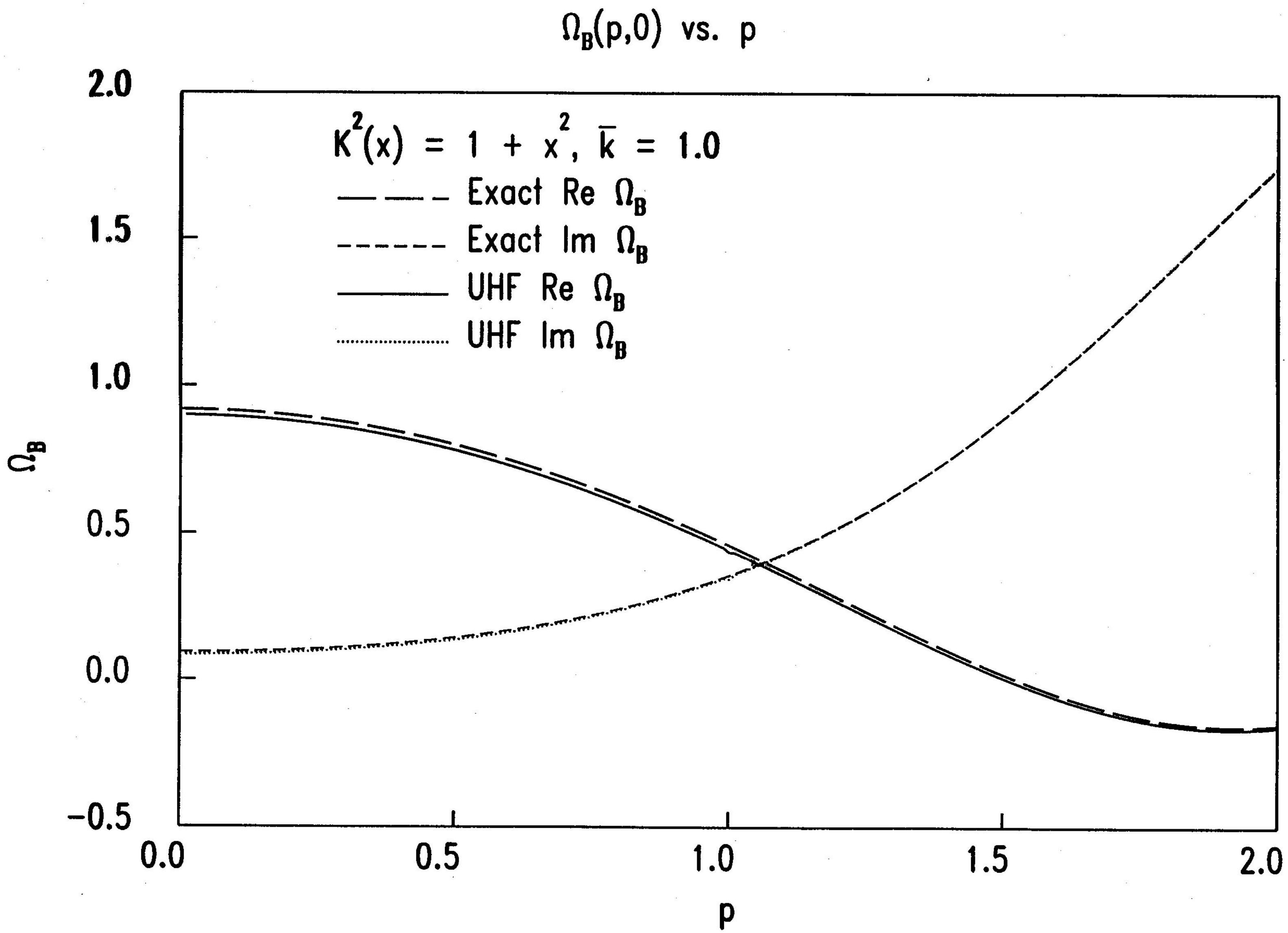
UHF Versus Exact



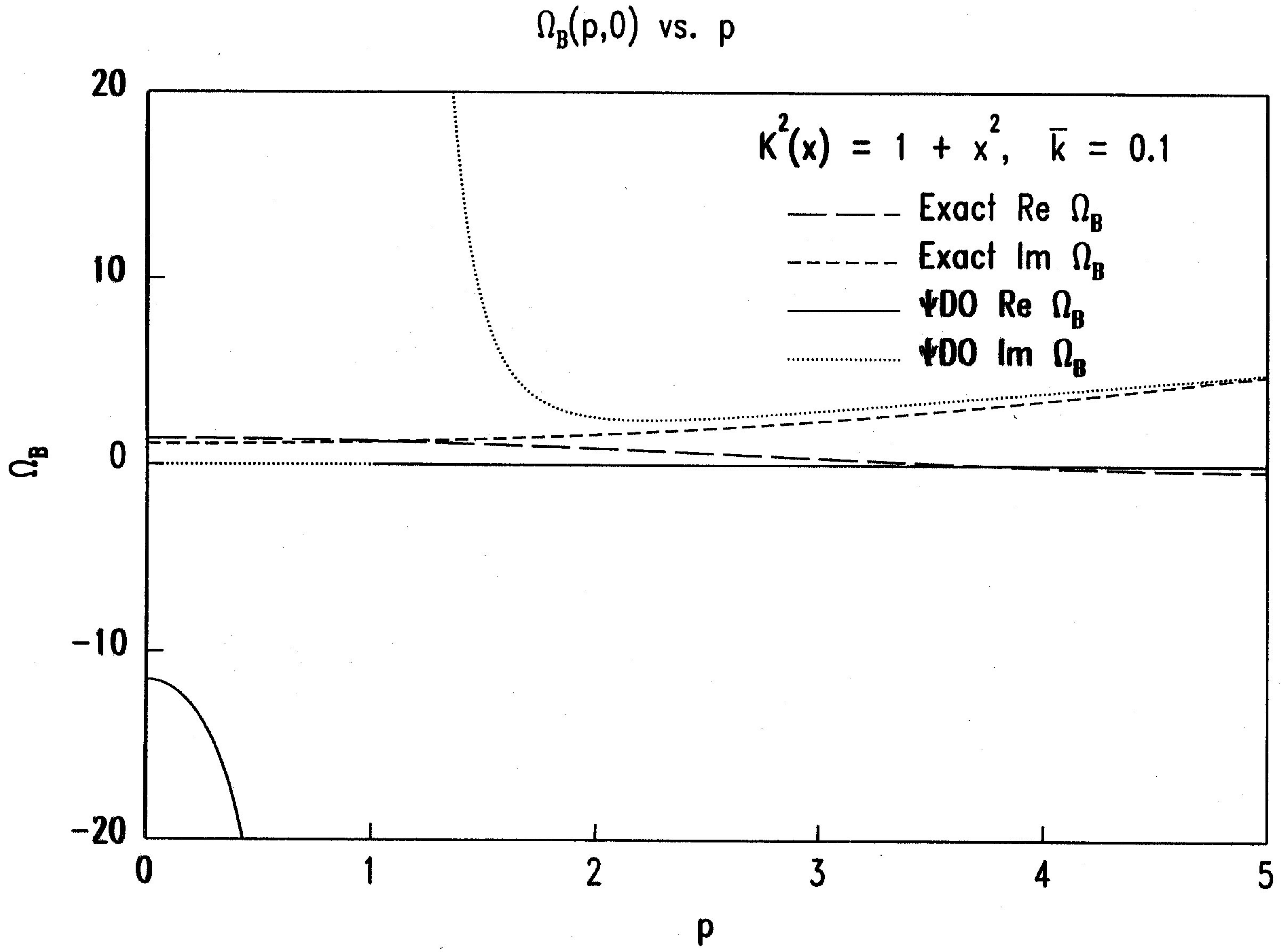
Pseudodifferential Versus Exact



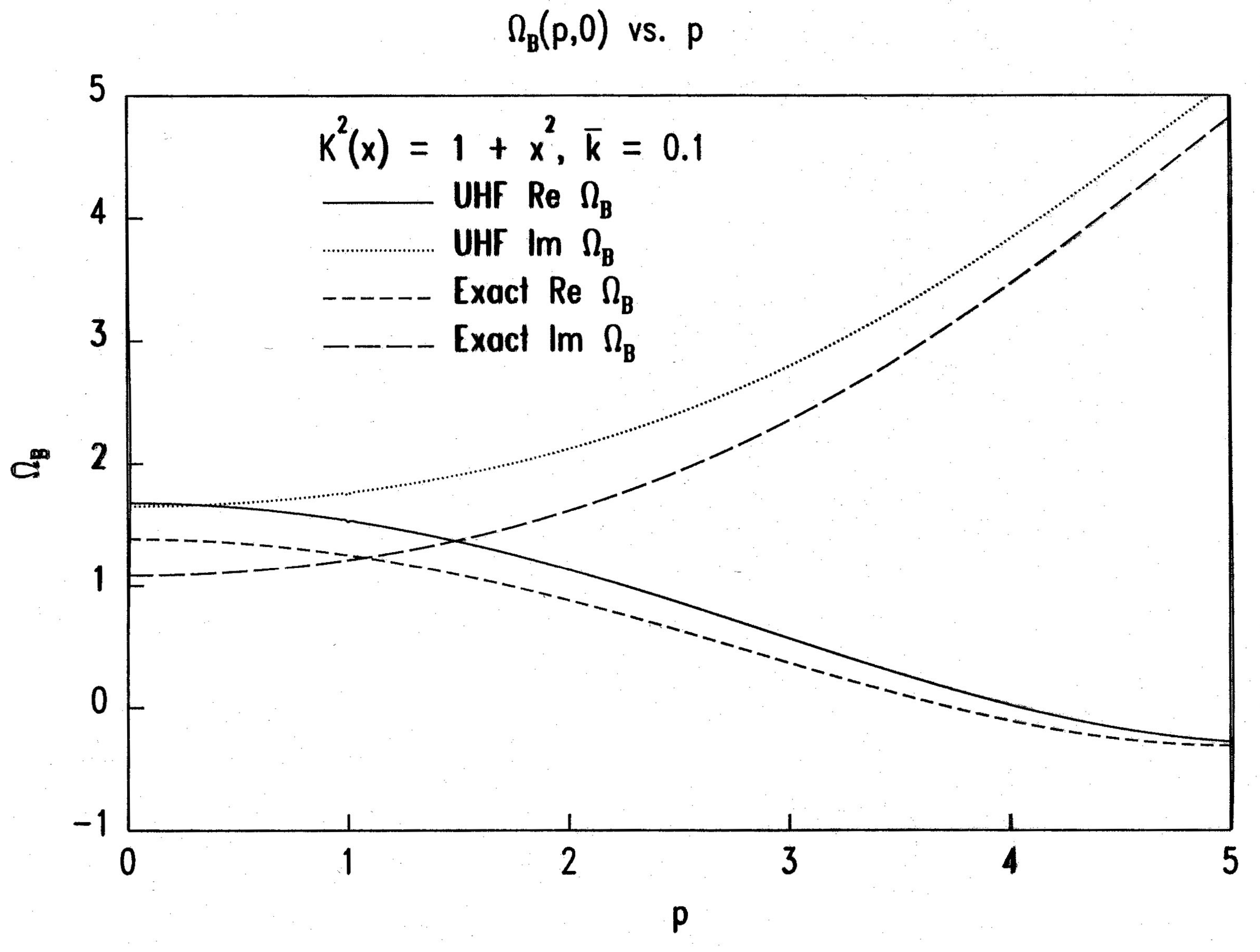
UHF Versus Exact



Pseudodifferential Versus Exact



UHF Versus Exact



Low-Frequency Symbol Asymptotics

Profile

$$K^2(q) \to \left\{ K_2^2, q \to \infty; K_1^2, q \to -\infty \right\}$$

Leading asymptotic term in limit $\bar{k} \to 0$

$$\Omega_{\mathbf{B}}(p,q) \sim \Omega_{\mathbf{B}}^{\mathbf{D}}(p, Q=0), Q = \bar{k}|q|$$

where $\Omega_{\mathbf{B}}^{\mathbf{D}}(p,Q)$ is the symbol for the discontinuity, or 2-layer, profile, and

Low-Frequency Symbol Asymptotics

$$\Omega_{\mathbf{B}}^{\mathbf{D}}\left(p,Q=0\right)=\mathrm{H}\left(p^2-K_AK_D\right)\left(K_A^2-p^2\right)^{1/2}\left(1-K_D^2/p^2\right)^{1/2}$$

$$\bullet\left(1-\left(K_AK_D\right)^2/p^4\right)+\left(2^3/\pi\right)\left(K_A+K_D\right)\left(K_AK_D\right)^2\left(p^2+K_AK_D\right)^{-2}$$

$$\bullet\int_0^1 dt\,t^{1/2}\left(1-t\right)^{1/2}\left(1-\alpha t\right)^{1/2}\left(1-\beta t\right)^{-1}$$

where

$$K_A=\left(1/2\right)\left(K_>+K_<\right),K_D=\left(1/2\right)\left(K_>-K_<\right),K_>=\max\left(K_1,K_2\right)$$

$$K_<=\min\left(K_1,K_2\right),\alpha=\frac{4K_AK_D}{\left(K_A+K_D\right)^2},\beta=\frac{4K_AK_Dp^2}{\left(p^2+K_AK_D\right)^2}$$

$\mathrm{H}\left(\bullet\right)=$ Heaviside function

Low-Frequency Limiting Symbol

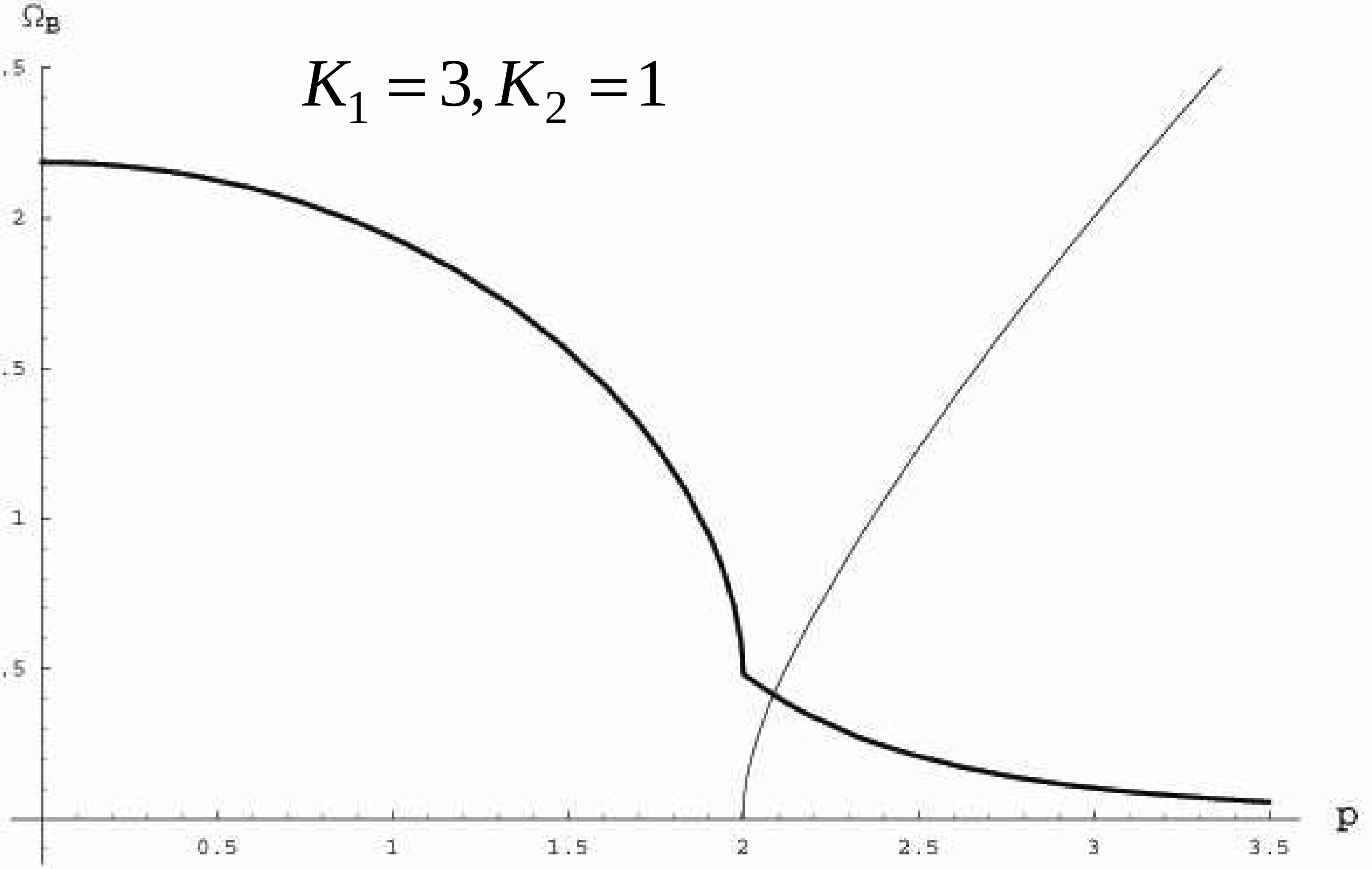


Fig. 8

Propagation of Singularities

Plasma wave equation

$$\left(\partial_t^2 - \partial_z^2 + \bar{k}^2 K^2(z)\right)\psi(t,z) = 0$$

and its relevant square-root operator

$$\tilde{\mathbf{B}} = \left(K^2(z) - \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

Defocusing quadratic medium

$$K^2(q) = K_0^2 + \omega^2 q^2, n = 2$$

Weyl operator symbol

$$\Omega_{\mathbf{B}}(p,q) = (\varepsilon/\pi)^{1/2} \int_0^\infty dt \exp\left(-\left(Yt + \mathrm{X}\tanh t\right)\right)$$

$$\bullet t^{-1/2} \sec ht \left(Y + \mathrm{X} \sec h^2 t + \tanh t\right)$$

Propagation of Singularities

where

$$\mathrm{X}=\frac{1}{\varepsilon}\left(\omega^2 q^2+p^2\right), Y=K_0^2/\varepsilon, \varepsilon=\omega/\bar{k}$$

High-frequency $(\varepsilon \to 0)$ asymptotics

$$\Omega_{\mathbf{B}}(p,q) \sim \left(K_0^2+\omega^2 q^2+p^2\right)^{1/2}+\frac{\varepsilon^2 K_0^2}{8\left(K_0^2+\omega^2 q^2+p^2\right)^{5/2}}+\ldots$$

$+O$(exponentially decreasing)

Nonsingular algebraic terms; decaying exponential terms

Uniform algebraic approximation

Uniform High-Frequency Wave Theory

Path Integral + UHF Standard Operator Symbol

(1) Distinct from high-frequency approximations made directly on the wave field, including the globally uniform, high-frequency constructions of Maslov (Fourier transform) and Klauder (coherent-state transform)

(2) Incorporates wave (diffraction) effects via “sum over paths” in phase space and uniform approximation of phase functional (operator symbol) in path integral

(3) Since there is functional stationary phase approximation, there are no caustic-related phenomena

(4) Essentially the difference of classical, high-frequency asymptotic wavefield as (1) global solution and (2) local solution repeatedly composed to produce a global solution

Uniform High-Frequency Wave Theory

Path Integral + UHF Standard Operator Symbol

(5) Correct incorporation of both high-propagating-angle and post-critical wave phenomena

(6) Since uniform asymptotic construction, approximation is independent of the reference sound speed

(7) More accurate incorporation of energy flux conservation

(8) Since approximations are made at the “infinitesimal” (operator symbol) level, far greater range of computational validity

(9) Algorithm runs just like GPSPI computational algorithm

(10) Reformulation of path integral in terms of coherent states (Gabor basis) should result in faster computational algorithms

Lecture 3

Phase Space and Path Integral Methods
Part 2

The focus on solving the composition equations for the operator symbols

Can the desired solutions be taken directly from the quantum mechanical and microlocal literatures?

Nonuniform operator symbol constructions

Exact operator symbol constructions

Uniform asymptotic operator symbol constructions

Uniform high-frequency wavefield extrapolator