Phase Space and Path Integral Methods in Seismic Wave Propagation and Imaging

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POTSI
PSEUDODIFFERENTIAL OPERATOR THEORY & SEISMIC IMAGING

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Goals of Lectures

1) Overview of wave equation seismic imaging

2) Application of modern mathematical physics methods to acoustic wave scattering and propagation

    a) Incorporation of well-posed, one-way methods into inherently two-way, global formulations

    b) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics

    c) Extension of Fourier analysis to inhomogeneous environments

3) Possible improvements to seismic imaging algorithms

Lecture 2

Phase Space and Path Integral Methods
Part 1

Approaches to Helmholtz equation modeling

Exact, well-posed, one-way reformulation

Connection to previous seismic work

Homogeneous medium constructions

Classical waves, quantum physics, and modern mathematical asymptotics

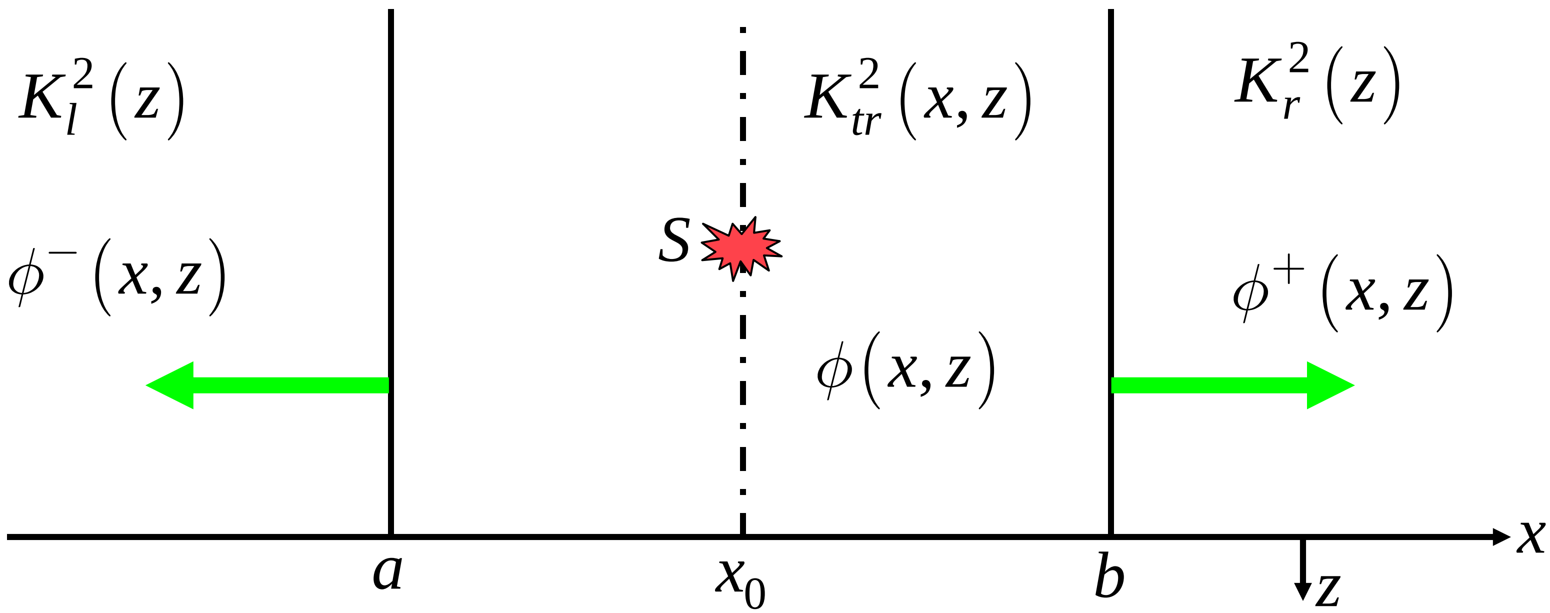
Explicit formulation in terms of phase space and path integral methods

Mathematical Illustration

Scalar Helmholtz Equation

$$\left(\partial_x^2 + \underbrace{\partial_z^2 + \bar{k}^2 K^2(x,z)}_{\mathbf{B}^2}\right)\phi(x,z) = -\delta(x - x_s)\,\delta(z - z_s)$$

General Radiation Formulation



Classical Wave Theory

Classical Wave Propagation Approaches - Traditional

1) Direct Wave Field Approximations

   (a) Perturbation theory (Born, Rytov)

   (b) Asymptotic Ray Theory

   (c) Normal-mode (spectral) analysis

   (d) Gaussian beams

2) Approximate Wave Equations

Classical Wave Theory

Classical Wave Propagation Approaches - Traditional

3) Discrete Numerical Computations

(a) Finite differences

(b) Finite elements

(c) Spectral methods

(d) Fast field programs (FFP)

Classical Wave Theory

Classical Wave Propagation Approaches – Modern Methods

1) Phase Space Analysis

   (a) Pseudodifferential operators (ψDO)

   (b) Fourier integral operators (FIO)

   (c) Microlocal analysis

   (d) Formal constructions from quantum physics

Think – modern mathematical formulation of “asymptotics”

Classical Wave Theory

Classical Wave Propagation Approaches – Modern Methods

2) Functional (Path) Integration

(a) Explicit wavefield representations

(b) Wiener (Brownian motion)

(c) Feynman (quantum mechanics)

Think – approximate ray theory methods as "sum over paths"

3) Phase Space + Path Integral Methods

Classical Wave Theory

Modern Approaches – Principal Themes

1) Incorporation of well-posed, one-way methods into inherently two-way, global formulations

2) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics

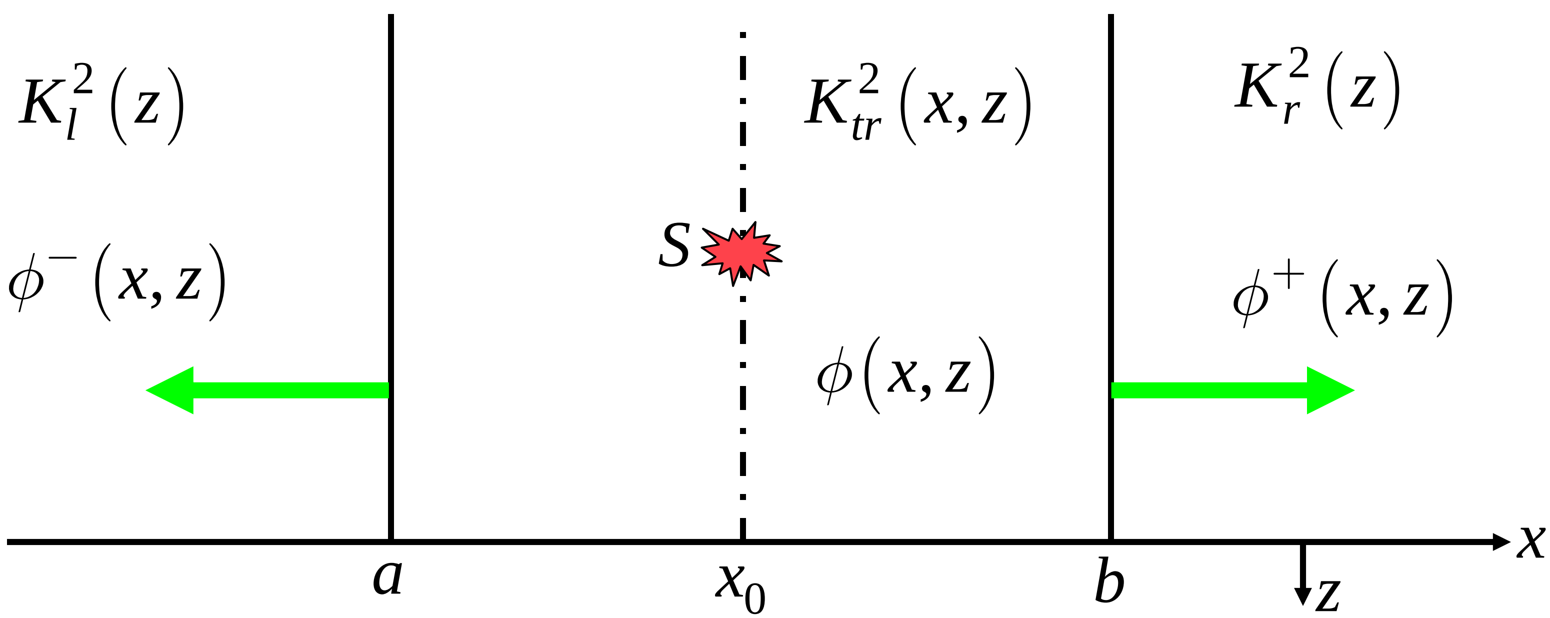
3) Extension of Fourier analysis to inhomogeneous environments

Mathematical Illustration

Scalar Helmholtz Equation

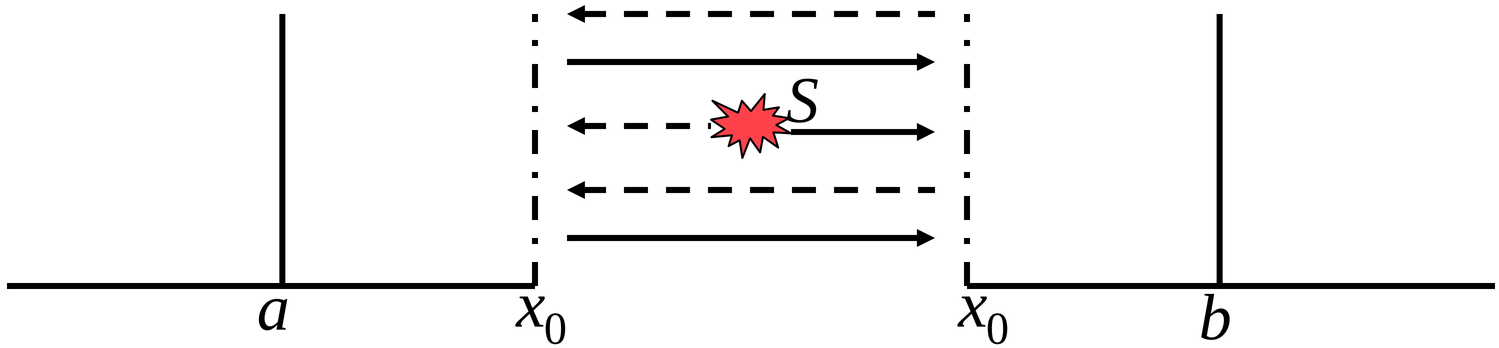
$$\left(\partial_x^2 + \underbrace{\partial_z^2 + \bar{k}^2 K^2(x,z)}_{\mathbf{B}^2} \right) \phi(x,z) = -\delta(x - x_s)\,\delta(z - z_s)$$

General Radiation Formulation



Basic Scattering Picture

(1) Scattering Block Decomposition



(a) Individual block scattering problems

(b) “Glue” solutions together (block multiple scattering)

Basic Scattering Picture

(2) Fundamental Scattering Problem

Basic Scattering Picture

(2) Fundamental Scattering Problem

Global nature of problem

a) Must compute wavefield everywhere

b) Very large-scale problems (in ocean acoustics, in FD formulation, 10^{11} unknowns)

c) Wavelets/multigrid do not produce significant acceleration in frequency-domain problem

d) Fundamental problem is small-scale, oscillatory character of wavefield solution

Basic Scattering Picture

(2) Fundamental Scattering Problem

Desire for one-way methods – simplest one-way marching scheme

$$\partial_x \begin{pmatrix} \phi \\ \partial_x \phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(\partial_z^2 + \bar{k}^2 K^2(x,z)\right) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \partial_x \phi \end{pmatrix}$$

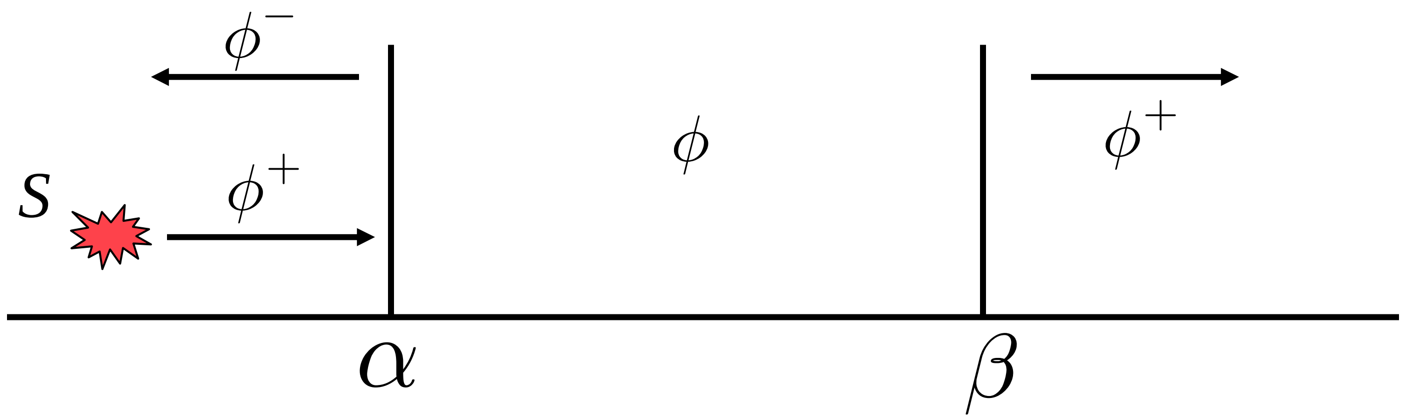
Ill-posed for simultaneous marching of wavefield and normal derivative in range direction

Non-independent initial data

Relationship between wavefield and normal derivative is key to well-posed marching method

Basic Scattering Picture

(3) Correct Scattering Kinematics (Geometry)



$$\phi^{\pm}(x,z) = \frac{1}{2}\left[\phi(x,z) \mp \left(i/\bar{k}\right)\mathbf{B}^{-1}\partial_x \phi(x,z)\right]$$

$$\mathbf{B} = \left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

Basic Scattering Picture

(3) Correct Scattering Kinematics (Geometry)

Equivalent Helmholtz formulation

$$\left(i/k \right) \partial_x \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{B} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \left(i/2k \right) \left(\partial_x \mathbf{B}^{-1} \right) \mathbf{B} \right\} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix}$$

Basic Scattering Picture

(3) Correct Scattering Kinematics (Geometry)

Equivalent Helmholtz formulation

a) Kinematically correct scattering picture

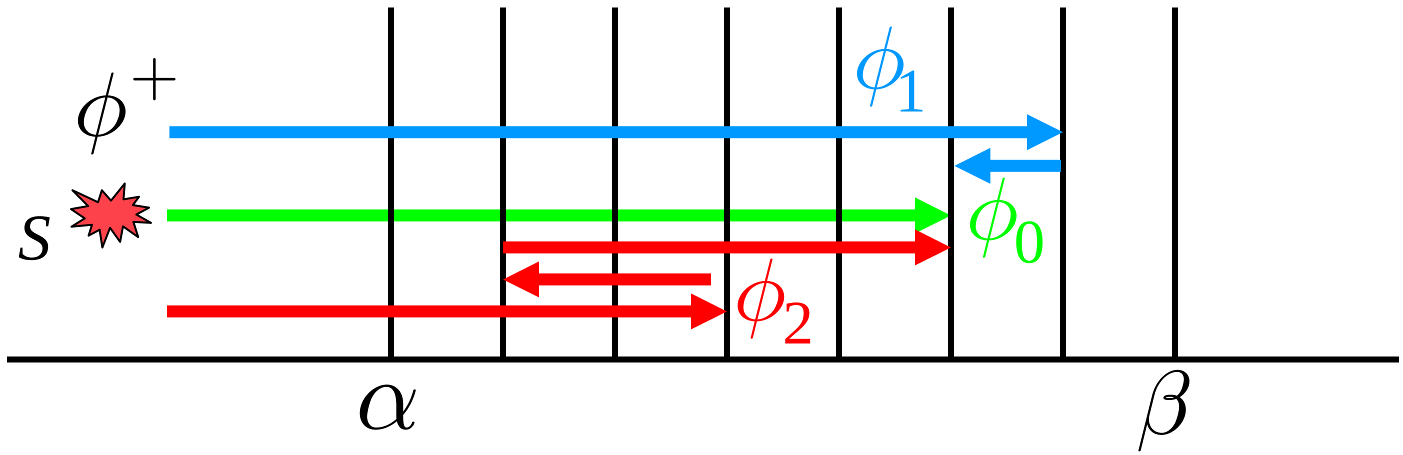
b) Decoupled, physical wavefield components in range-independent limit (diagonal)

c) Ill-posed for marching

d) Non-independent initial data

Well-Posed, One-Way Methods

(1) Generalized Bremmer Coupling Series



$$\phi = \sum_{\substack{\#\text{of} \\ \text{reflections} \\ n}} \phi_n$$

Well-Posed, One-Way Methods

(1) Generalized Bremmer Coupling Series

Fundamental Equation

$$\begin{pmatrix} \left(i/\bar{k}\right)\partial_x + \mathbf{B} & 0 \\ 0 & \left(i/\bar{k}\right)\partial_x - \mathbf{B} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} =$$

$$\begin{pmatrix} \mathbf{T} & \mathbf{R} \\ \mathbf{R} & \mathbf{T} \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \left(1/2\bar{k}^2\right) \mathbf{B}^{-1} \delta\left(\underline{x} - \underline{x}_s\right)$$

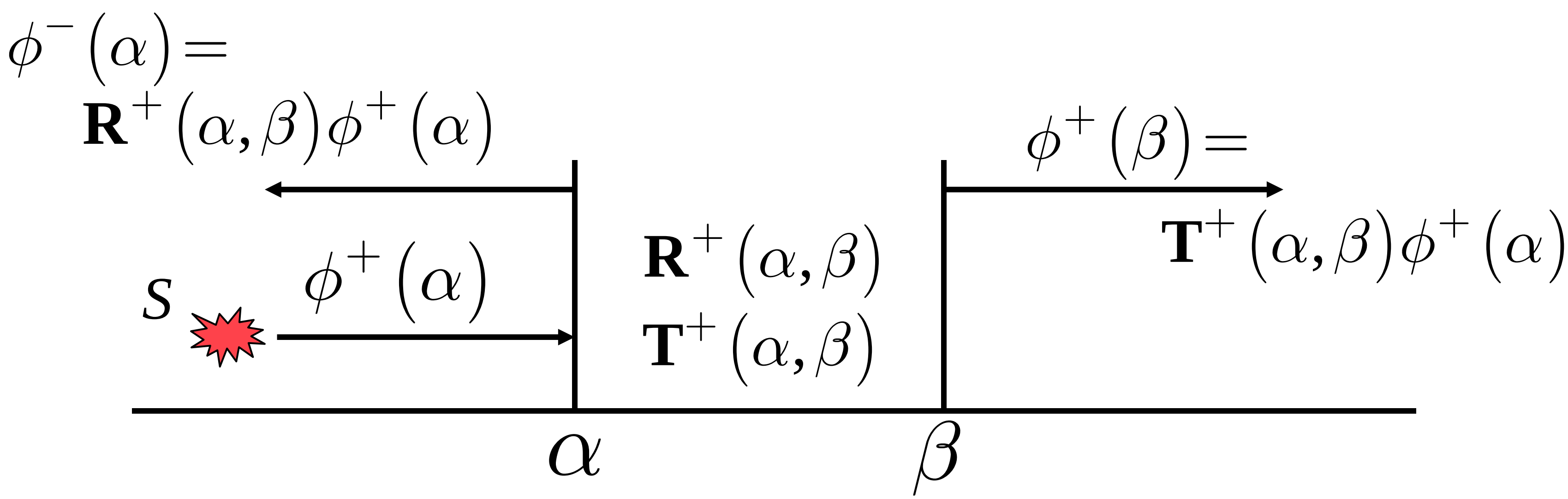
Born-Neumann series solution

Basis for “wave tracing” – following multiples

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

Scattering Picture



Scattering operators associated with global block.

All internal multiple scattering incorporated into scattering operators.

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

Scattering Picture

Invariant Imbedding

(1) Derive first-order equations on scattering operators

(2) Perturb the scattering block boundaries

(3) How do the scattering operators depend upon the location of the boundaries?

(4) The resulting system of equations:

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

$$\left(i/\bar{k}\right)\partial_x \mathbf{R}^+(x,b) = \boldsymbol{\gamma}(x) + \boldsymbol{\varepsilon}(x)\mathbf{R}^+(x,b) - \mathbf{R}^+(x,b)\boldsymbol{\alpha}(x)$$
$$+\mathbf{R}^+(x,b)\boldsymbol{\beta}(x)\mathbf{R}^+(x,b), \mathbf{R}^+(b,b) = 0$$

$$\left(i/\bar{k}\right)\partial_x \mathbf{T}^+(x,b) = -\mathbf{T}^+(x,b)\boldsymbol{\alpha}(x)$$
$$+\mathbf{T}^+(x,b)\boldsymbol{\beta}(x)\mathbf{R}^+(x,b), \mathbf{T}^+(b,b) = \mathbf{I}$$

with

$$\boldsymbol{\alpha}(x) = \left(\left(i/2\bar{k}\right)\partial_x \mathbf{B}^{-1} - 1\right)\mathbf{B}, \boldsymbol{\varepsilon}(x) = \left(\left(i/2\bar{k}\right)\partial_x \mathbf{B}^{-1} + 1\right)\mathbf{B}$$

$$\boldsymbol{\beta}(x) = -\boldsymbol{\gamma}(x) = \left(i/2\bar{k}\right)\left(\partial_x \mathbf{B}^{-1}\right)\mathbf{B}$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

(1) Operator Riccati equations

(2) First-order, quadratically nonlinear, nonlocal equations

Boundary-Value (DtN) Picture

(1)Transform from scattering picture

$$\left\{\phi^{+},\phi^{-}\right\}$$

to boundary-value picture

$$\left\{\phi,\partial_{x}\phi\right\}$$

(2) “Glue” block decomposition solutions together

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

Given $\phi\left(x_0, z\right)$, then propagation from x_0 is given by

$$\left(\left(1/\bar{k}\right)\partial_x + \mathbf{\Lambda}^{+}\left(x,b\right)\right)\phi\left(x,z\right) = 0$$

$$\left(\left(1/\bar{k}\right)\partial_x - \mathbf{\Lambda}^{-}\left(a,x\right)\right)\phi\left(x,z\right) = 0$$

where

$$\left(1/\bar{k}\right)\partial_x \mathbf{\Lambda}^{+}\left(x,b\right) = \left(\mathbf{\Lambda}^{+}\left(x,b\right)\right)^2 + \mathbf{B}^2\left(x\right)$$

with the initial condition

$$\mathbf{\Lambda}^{+}\left(b,b\right) = -i\mathbf{B}\left(b\right)$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

and

$$-\left(1/\bar{k}\right)\partial_x \mathbf{\Lambda}^-(a,x) = \left(\mathbf{\Lambda}^-(a,x)\right)^2 + \mathbf{B}^2(x)$$

with the initial condition

$$\mathbf{\Lambda}^-(a,a) = -i\mathbf{B}(a)$$

with

$$\mathbf{B}(x) = \left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

and where the initial wavefield is given by

$$\phi(x_0,z) = \left(1/\bar{k}\right)\left(\mathbf{\Lambda}^+(x_0,b) + \mathbf{\Lambda}^-(a,x_0)\right)^{-1}\delta(z - z_s)$$

Well-Posed, One-Way Methods

(2) Exact, Well-Posed, One-Way Reformulation

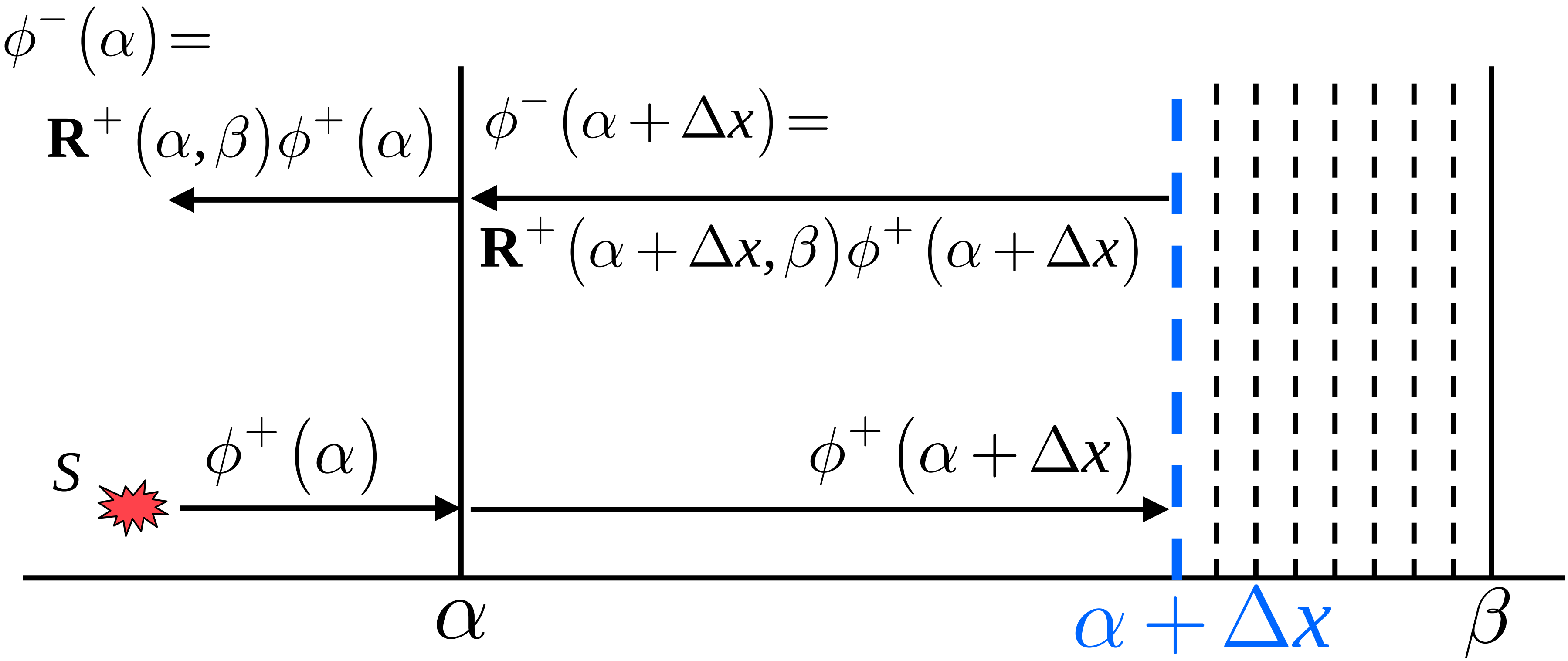
(1) Now have two, well-posed marching problems, in opposite directions, done in secession

(2) Must cover both directions for the two-way "elliptic wave propagation" problem: one direction to get the DtN operator and the other direction to propagate the total wavefield with the DtN operator

(3) The idea is not to do the first marching procedure (DtN operator construction) computationally, but, rather, to solve the problem asymptotically. Thus, there will be only one, one-way marching computational procedure.

(4) The initial field calculation will also be done asymptotically.

One-Way Marching of a Total Wavefield



Bremmer Series – multiple-sweep algorithm

DtN Marching – single-sweep algorithm

Two Complementary Approaches

Generalized Bremmer Coupling Series

(1) Propagation operator:

$$\mathbf{B}(x,z)$$

(2) One-way wave equation:

$$\left(\left(i/\bar{k}\right)\partial_x+\mathbf{B}(x,z)\right)\phi^+(x,z)=0$$

(3) Fundamental solution (propagator):

$$\mathbf{G}^+(x,0)=\exp\left(i\bar{k}x\mathbf{B}(x,z)\right)$$

(4) Multiple-sweep algorithm (“wave tracing”)

Two Complementary Approaches

DtN Operator One-Way Reformulation

(1) Propagation operator:

$$\mathbf{\Lambda}^{+}\left(x,b\right)$$

(2) One-way wave equation:

$$\left(\left(1/\bar{k}\right)\partial_{x}+\mathbf{\Lambda}^{+}\left(x,b\right)\right)\phi\left(x,z\right)=0$$

(3) Fundamental solution (propagator):

$$\mathbf{G}^{+}\left(x,x_{0}\right)=\lim_{\delta\to 0}\left[\exp\left(-\bar{k}\delta\mathbf{\Lambda}^{+}\left(x_{N},b\right)\right)\ldots\exp\left(-\bar{k}\delta\mathbf{\Lambda}^{+}\left(x_{1},b\right)\right)\right]$$

(4) Single-sweep algorithm on total wavefield

Seismo-Acoustic Operator Constructions

Propagation Operators

(1) Formal operator series/rational approximations on square-root operator (Claerbout, C. J. Thomson, Tappert, etc.)

$$\mathbf{B} = \left(\mathbf{I} + \mathbf{L}\right)^{1/2} \approx \mathbf{I} + \left(1/2\right)\mathbf{L} + \ldots$$

$$\mathbf{B} \approx \mathbf{B}^{\mathbf{ara}} = \mathbf{I} + \sum_{j=1}^{N} c_{j,N}\left(\mathbf{I} - \left(\mathbf{I} + b_{j,N}\mathbf{L}\right)^{-1}\right)$$

$$\mathbf{L} = \mathbf{B}^2 - \mathbf{I} = \left(\left(K^2\left(x,z\right) - 1\right)\mathbf{I} + \left(1/\bar{k}^2\right)\partial_z^2\right)$$

(2) Formal operator series on DtN operator (Bleistein, Zhang, and Zhang; Chapman)

Seismo-Acoustic Operator Constructions

Fundamental Solutions (Propagators)

(1) Operator rational approximations for the propagator (Collins)

$$\mathbf{G}^{+}(x,0) = \exp\left(i\bar{k}x\mathbf{B}\right)$$

(2) Path integral representations: time-domain wave equation (Schlottmann); fixed-frequency, one-way, anisotropic, elastic wave equation (C. J. Thomson)

Critical Comments

(1) $\left(K^2(x,z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$ does not define the operator

(2) Formal operator Taylor series do not define the operator

(3) Operator series expansions are nonuniform and singular

Seismo-Acoustic Operator Constructions

Critical Comments

(4) Operator rational approximations can provide uniform approximation

(5) Operator rational approximations are fits – both strength and weakness

(6) Schlottmann path integral construction is, in general, approximate, being exact only for a homogeneous medium; at level of scalar Helmholtz equation, it is essentially the approximate Feynman/Garrod representation from the 60’s

(7) Thomson path integral is purely formal

Mathematical Framework

Homogeneous Half-Space $K^2(z) = K_0^2$

(1) Wave Equation

$$\left((i/\bar{k})\partial_x + \left(K_0^2 + (1/\bar{k}^2)\partial_z^2 \right)^{1/2} \right)\phi^+(x,z) = 0$$

$$(i/\bar{k})\partial_x\phi^+(x,z) + \frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2} dp\,dz' \left(K_0^2 - p^2 \right)^{1/2} \bullet \exp\left(i\bar{k}p(z-z') \right)\phi^+(x,z') = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x, z \middle| 0, z'\right) = \lim_{N\to\infty} \int_{\mathbb{R}^{2N-1}} \prod_{j=1}^{N-1} dz_j \prod_{j=1}^{N} \left(\frac{\bar{k}}{2\pi}\right) dp_j$$

$$\bullet \exp\left[i\bar{k} \sum_{j=1}^{N} \left(p_j \left(z_j - z_{j-1}\right) + \left(\frac{x}{N}\right)\left(K_0^2 - p_j^2\right)^{1/2} \right) \right]$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi^{+}\left(x+\Delta x, z\right)=$$

$$\int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\Delta x\left(K_0^2-p^2\right)^{1/2}\right)\hat{\phi}^{+}\left(x,p\right)\right]$$

$$=F^{-1}\left[\exp\left(i\bar{k}\Delta x\left(K_0^2-p^2\right)^{1/2}\right)F\left[\phi^{+}\left(x,z'\right),p\right],z\right]$$

Phase Space and Path Integral Methods

General Problem:

At the simplest level, explicitly construct operator functions of the type:

$$\mathbf{B} = \left(K^2(z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}$$

and the corresponding fundamental solution,

$$\exp\left(i\bar{k}\,x\,\mathbf{B}\right)$$

History:

(1) Development of Quantum Mechanics

(2) Development of Modern Mathematical Asymptotics

Quantum Mechanical Correspondence

(1) Start from classical function and develop general phase-space mappings to produce a corresponding operator

$$h(p,q) \rightarrow \mathbf{H}(\mathbf{P},\mathbf{Q}), \quad \mathbf{P} = -i\hbar\partial_q, \quad \mathbf{Q} = q$$

(2) Classical mechanics → quantum mechanics via Schrödinger equation,

$$\left(i\hbar\partial_t - \mathbf{H}(t,z,-i\hbar\partial_z)\right)\phi(t,z) = 0$$

Quantum Mechanical Correspondence

(3) In classical wave problems, we have formal operators and want to construct them in terms of functions.

(4) Fundamental solution to Schrödinger equation is then expressed via path integral construction:

$$\exp\left(\frac{-i}{\hbar} t\mathbf{H}\left(t, z, -i\hbar\partial_z\right)\right) = \int_{\substack{all\ paths \\ z(\sigma),\, p(\sigma)}} D\left(z(\sigma)\right) D\left(p(\sigma)\right) F\left(z(\sigma), p(\sigma), t\right)$$

Quantum Mechanical Correspondence

Schrödinger equation (non-relativistic QM)

$$\left(i\hbar\partial_t + \left(\hbar^2/2m\right)\nabla^2 - V\left(\underline{x}\right)\right)G\left(\underline{x},t\middle|\underline{x}_s,0\right) = 0$$

$$G\left(\underline{x},0\middle|\underline{x}_s,0\right) = \delta\left(\underline{x} - \underline{x}_s\right)$$

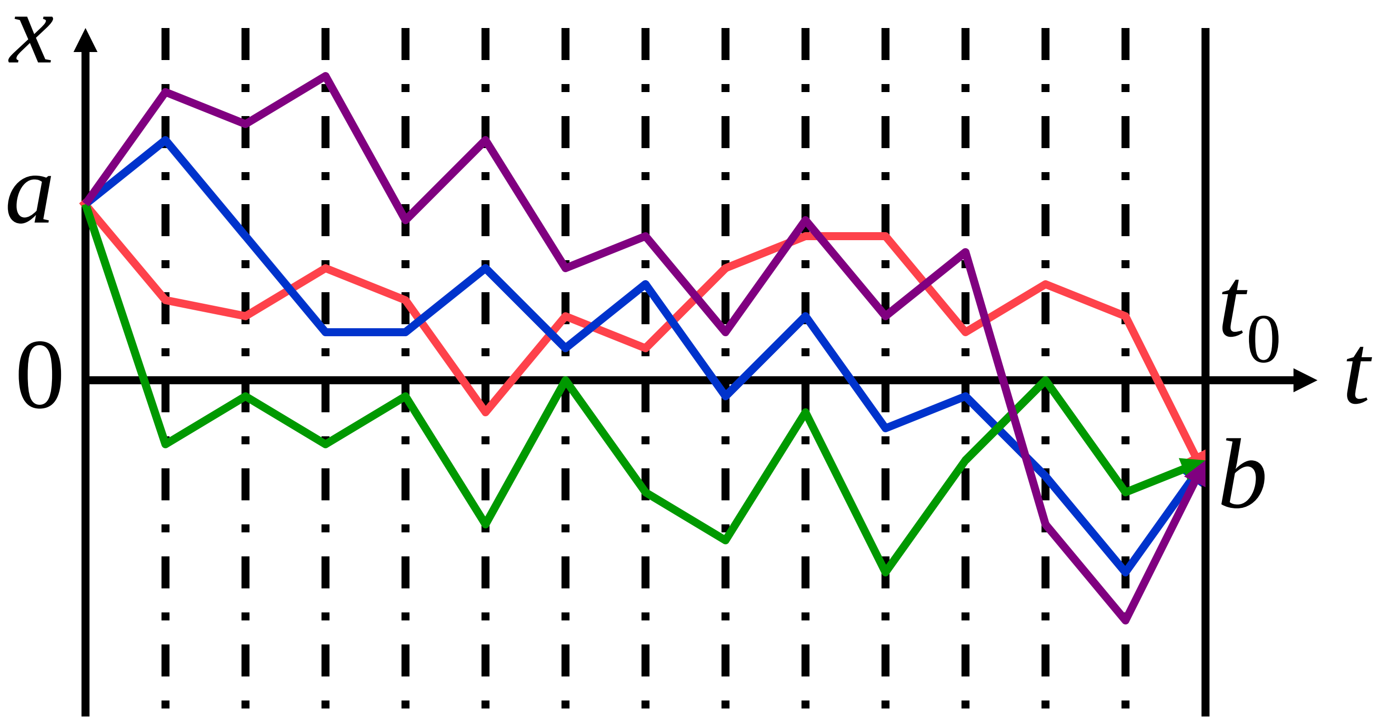
Feynman configuration space path integral (1948)

$$G\left(\underline{x},t\middle|\underline{x}_s,0\right) = \lim_{N\to\infty}\left\{\left(mN/2\pi i\hbar t\right)^{nN/2}\int_{\mathbb{R}^{n(N-1)}}\prod_{j=1}^{N-1}d\underline{x}_j\right.$$

$$\left.\bullet\exp\left[\left(i/\hbar\right)\sum_{j=1}^{N}\left(\left(m/2\right)\left(\underline{x}_j - \underline{x}_{j-1}\right)^2\Big/\left(t/N\right) - \left(t/N\right)V\left(\underline{x}_j\right)\right)\right]\right\}$$

Quantum Mechanical Correspondence

Path integration conceptually



A few of the infinitely many paths
between “a” and “b” in time “t_0”

Quantum Mechanical Correspondence

Feynman phase space path integral (1951)

$$G\left(\underline{x},t\,\middle|\,\underline{x}_s,0\right)=\lim_{N\to\infty}\int_{\mathbb{R}^{n(2N-1)}}\prod_{j=1}^{N-1}d\underline{x}_j\prod_{j=1}^{N}\left(2\pi\hbar\right)^{-n}d\underline{p}_j$$

$$\bullet\exp\left[\left(i/\hbar\right)\sum_{j=1}^{N}\left(\underline{p}_j\bullet\left(\underline{x}_j-\underline{x}_{j-1}\right)-\left(t/N\right)\left(\underline{p}_j^2/2m+V\left(\underline{x}_j\right)\right)\right)\right]$$

Phase space path integrals for Schrödinger equations with general pseudodifferential operator Hamiltonians have been constructed (Dowker and Mayes, Cohen)

$$\left(i\hbar\partial_t-\mathbf{H}\left(t,\underline{x},-i\hbar\nabla_{\underline{x}}\right)\right)\psi\left(\underline{x},t\right)=0$$

Microlocal Correspondence

(1) Extend homogeneous Fourier analysis to inhomogeneous environments

(2) Natural generalization of PDEs with nonconstant coefficients

(3) Focus is on operator symbol

(4) Example – quantum mechanics

(5) Operator symbol plays role analogous to Hamiltonian in quantization of classical dynamical systems

Microlocal Correspondence

Operator Symbols in Everyday Life

(1) “Generalized vertical slowness” in geophysics

(2) Multidimensional extension of scattering coefficients

(3) Framework for quantization

Mathematical Framework

Generally-Inhomogeneous Half-Space $K^2(x,z) = K^2(x,z)$

(1) Wave Equation

$$\left(\left(1/\bar{k}\right)\partial_x + \mathbf{\Lambda}^+(x,b)\right)\phi(x,z) = 0$$

$$\left(1/\bar{k}\right)\partial_x\phi(x,z) + \frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2} dp\,dz'\,\Omega_{\mathbf{\Lambda}^+}\left(x,b;p,\left(z+z'\right)/2\right)$$

$$\bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi\left(x,z'\right) = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x,z\middle|x_0,z'\right)=\lim_{N\to\infty}\int_{\mathbb{R}^{2N-1}}\prod_{j=1}^{N-1}dz_j\prod_{j=1}^{N}\left(\frac{\bar{k}}{2\pi}\right)dp_j$$

$$\bullet\exp\left[i\bar{k}\sum_{j=1}^{N}\left(p_j\left(z_j-z_{j-1}\right)+i\left(\frac{\Delta x}{N}\right)h^{s}_{\mathbf{\Lambda}^{+}}\left(x_j,b;p_j,z_j\right)\right)\right]$$

$$\Delta x=x-x_0$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi\left(x+\delta, z\right) \approx$$

$$\int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(-\bar{k}\,\delta h^{s}_{\mathbf{\Lambda}^{+}}\left(x, b; p, z\right)\right)\hat{\phi}\left(x, p\right)\right]$$

$$\delta = \Delta x \big/ N = \left(x - x_0\right) \big/ N$$

$$h^{s}_{\mathbf{\Lambda}^{+}}\left(x, b; p, q\right) = \left(\bar{k} \big/ \pi\right) \int_{\mathbb{R}^2} ds\,dt\,\Omega_{\mathbf{\Lambda}^{+}}\left(x, b; s, t\right) \exp\left(-2i\bar{k}\left(q - t\right)\left(p - s\right)\right)$$

Mathematical Framework

Transversely-Inhomogeneous Half-Space $K^2(x,z) = K^2(z)$

(1) Wave Equation

$$\left(\left(i/\bar{k}\right)\partial_x + \left(K^2(z) + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}\right)\phi^+(x,z) = 0$$

$$\left(i/\bar{k}\right)\partial_x\phi^+(x,z) + \frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2} dp\,dz'\,\Omega_{\mathbf{B}}\left(p,\left(z+z'\right)/2\right)$$

$$\bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^+\left(x,z'\right) = 0$$

Mathematical Framework

(2) Path Integral

$$G^{+}\left(x,z\middle|0,z'\right)=\lim_{N\to\infty}\int_{\mathbb{R}^{2N-1}}\prod_{j=1}^{N-1}dz_{j}\prod_{j=1}^{N}\left(\frac{\bar{k}}{2\pi}\right)dp_{j}$$

$$\bullet\exp\left[i\bar{k}\sum_{j=1}^{N}\left(p_{j}\left(z_{j}-z_{j-1}\right)+\left(\frac{x}{N}\right)h_{\mathbf{B}}^{s}\left(p_{j},z_{j}\right)\right)\right]$$

Mathematical Framework

(3) Marching Numerical Algorithm

$$\phi^{+}\left(x+\Delta x, z\right) \approx$$

$$\int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\Delta x h_{\mathbf{B}}^{s}\left(p,z\right)\right)\hat{\phi}^{+}\left(x,p\right)\right]$$

$$h_{\mathbf{B}}^{s}\left(p,q\right)=\left(\bar{k}/\pi\right)\int_{\mathbb{R}^2} ds dt \Omega_{\mathbf{B}}\left(s,t\right)\exp\left(-2i\bar{k}\left(q-t\right)\left(p-s\right)\right)$$

Mathematical Framework

Comparing the forms of the one-way wave equation, path integral representation, and marching computational algorithm, it is seen that the forms for the transversely- and generally-inhomogeneous cases are the same as the forms for the homogeneous medium case.

It is in this sense that Fourier analysis has been extended to the inhomogeneous case.

The differences are in the composition equations satisfied by the operator symbols. These contain the appropriate scattering physics.

Mathematical Framework

Homogeneous Composition Equation

$$\Omega_{\mathbf{B}^2}(p) = K_0^2 - p^2 = \Omega_{\mathbf{B}}^2(p)$$

Transversely-Inhomogeneous Composition Equation

$$\Omega_{\mathbf{B}^2}(p,q) = K^2(q) - p^2 =$$

$$\left(\bar{k}/\pi\right)^2 \int_{\mathbb{R}^4} dtdsdvdu\,\Omega_{\mathbf{B}}(t+p, s+q)$$

$$\bullet\, \Omega_{\mathbf{B}}(v+p, u+q) \exp\left(2i\bar{k}(sv - tu)\right)$$

supplemented with right-traveling-wave radiation condition

Mathematical Framework

Generally-Inhomogeneous Composition Equation

$$\left(1/\bar{k}\right)\partial_x \Omega_{\mathbf{\Lambda}^+}(x,b;p,q) =$$

$$\left(\bar{k}/\pi\right)^2 \int_{\mathbb{R}^4} dtdsdvdu\Omega_{\mathbf{\Lambda}^+}(x,b;t+p,s+q)$$

$$\bullet\,\Omega_{\mathbf{\Lambda}^+}(x,b;v+p,u+q)\exp\left(2i\bar{k}(sv-tu)\right)$$

$$+K^2(x,q)-p^2$$

Initial condition: $\Omega_{\mathbf{\Lambda}^+}(b,b;p,q) = -i\Omega_{\mathbf{B}}(b;p,q)$

Generalized (nonlocal) Riccati equation

Mathematical Framework

Arbitrary transverse inhomogeneity

Operator symbols determined by appropriate composition equations, e.g.,

$$\Omega_{\mathbf{B}^2}\left(p,q\right)=K^2\left(q\right)-p^2=$$

$$\left(\bar{k}/\pi\right)^2\int_{\mathbb{R}^4}dtdsdvdu\Omega_{\mathbf{B}}\left(t+p,s+q\right)$$

$$\bullet\,\Omega_{\mathbf{B}}\left(v+p,u+q\right)\exp\left(2i\bar{k}\left(sv-tu\right)\right)$$

supplemented with right-traveling-wave radiation condition

Mathematical Framework

Linear pde transformed into quadratically nonlinear, nonlocal, composition equation understood in terms of generalized functions

Why do this seemingly complicated transformation?

1) Approach allows for exploitation of correspondences between classical wave propagation, quantum physics, and modern mathematical asymptotics (microlocal analysis)
2) Operator symbol is natural physical quantity that encodes underlying profile and dynamical information in very convenient fashion (useful point for inverse formulations)
3) Operator symbol leads to explicit wavefield representation in terms of the path integral

Mathematical Framework

4) In addressing problem at level of infinitesimal generator, approximations made at level of operator symbol have a greater range of validity, in general, then corresponding approximations made at level of wavefield (important computational point)

   For example, high frequency on the operator symbol is a full-wave approximation valid in the moderate- to low-frequency wavefield regimes

5) Sharp, uniformly approximate solutions to composition equation can be constructed, making this entire discussion relevant in practice

Lecture 2 Summary

Phase Space and Path Integral Methods
Part 1

Approaches to Helmholtz equation modeling

Exact, well-posed, one-way reformulation

Connection to previous seismic work

Homogeneous medium constructions

Classical waves, quantum physics, and modern mathematical asymptotics

Explicit formulation in terms of phase space and path integral methods