Phase Space and Path Integral Methods in Seismic Wave Propagation and Imaging

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Goals of Lectures

1) Overview of wave equation seismic imaging
2) Application of modern mathematical physics methods to acoustic wave scattering and propagation
   a) Incorporation of well-posed, one-way methods into inherently two-way, global formulations
   b) Exploitation of correspondences between classical wave propagation, quantum mechanics, and modern mathematical asymptotics
   c) Extension of Fourier analysis to inhomogeneous environments
3) Possible improvements to seismic imaging algorithms

Part 1

Seismic Imaging – The Seismic Way

Overview of wave equation seismic imaging

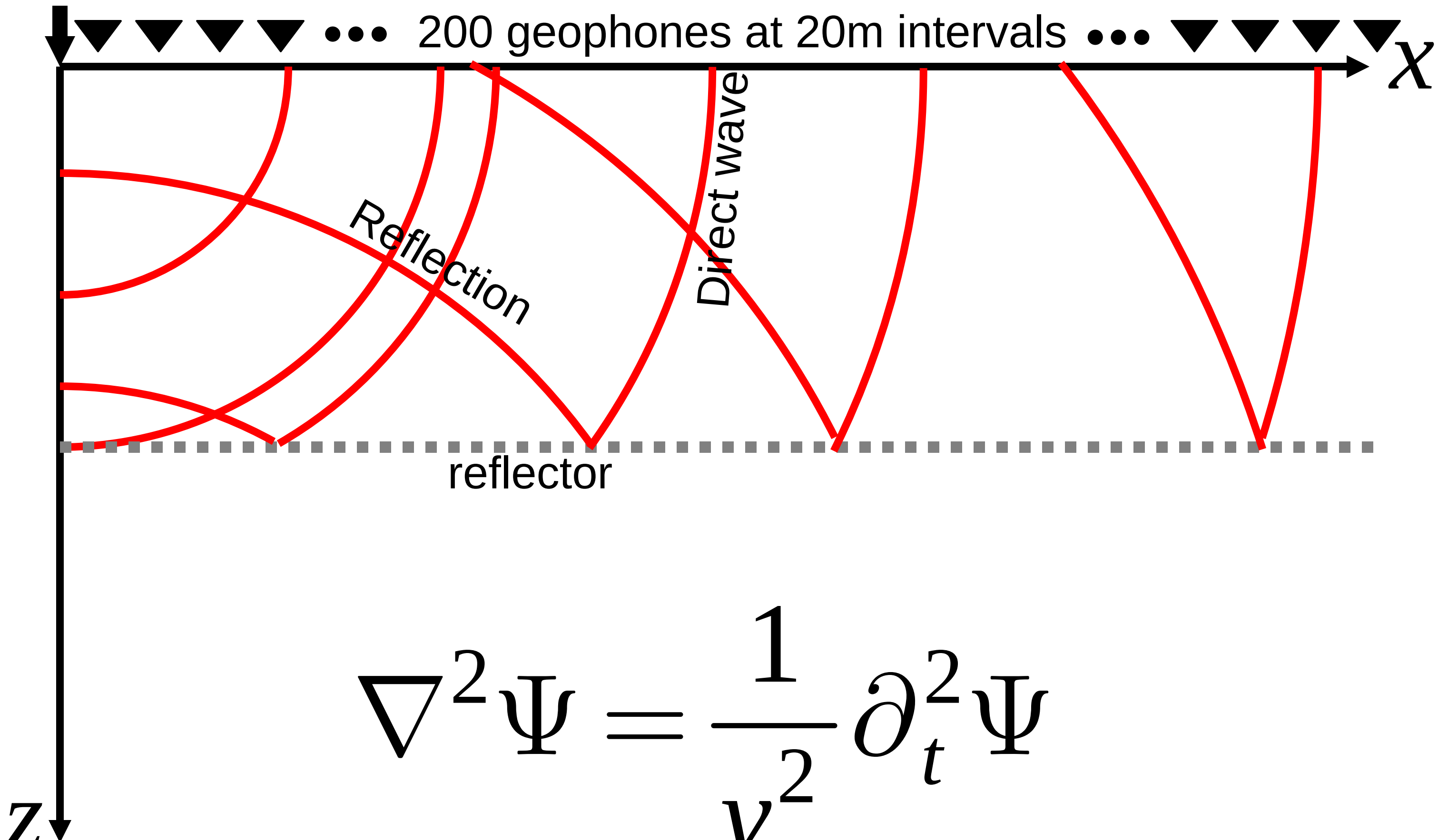
Homogeneous medium wavefield extrapolation

Locally homogeneous medium wavefield extrapolation

Common misconceptions in seismic imaging about locally homogeneous medium wavefield extrapolation

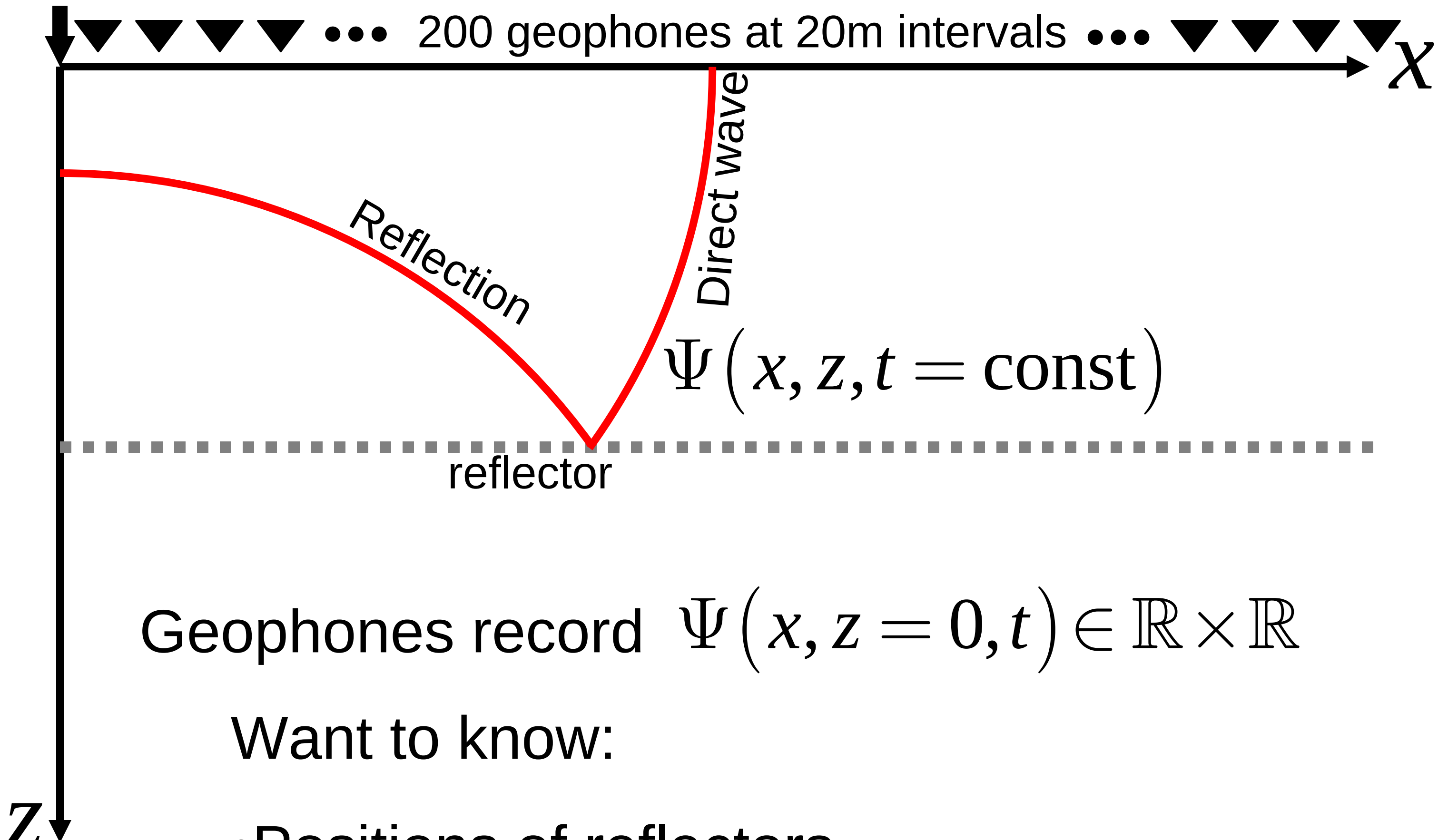
The shape of things to come

Seismic Shot Record



$$\nabla^2 \Psi = \frac{1}{v^2} \partial_t^2 \Psi$$

Seismic Shot Record



Geophones record $\Psi\left(x, z = 0, t\right) \in \mathbb{R} \times \mathbb{R}$

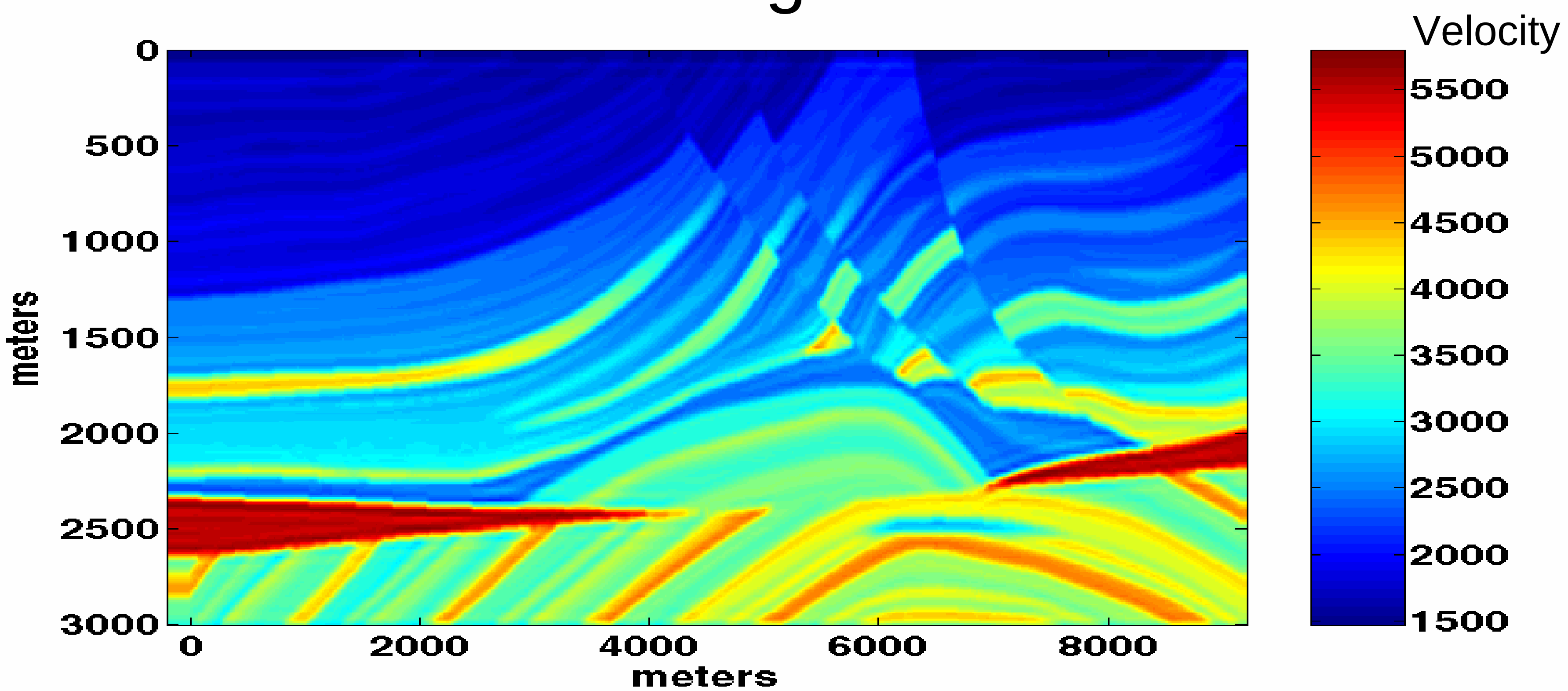
Want to know:

•Positions of reflectors

•Numerical estimates of earth properties

Classical Wave Theory

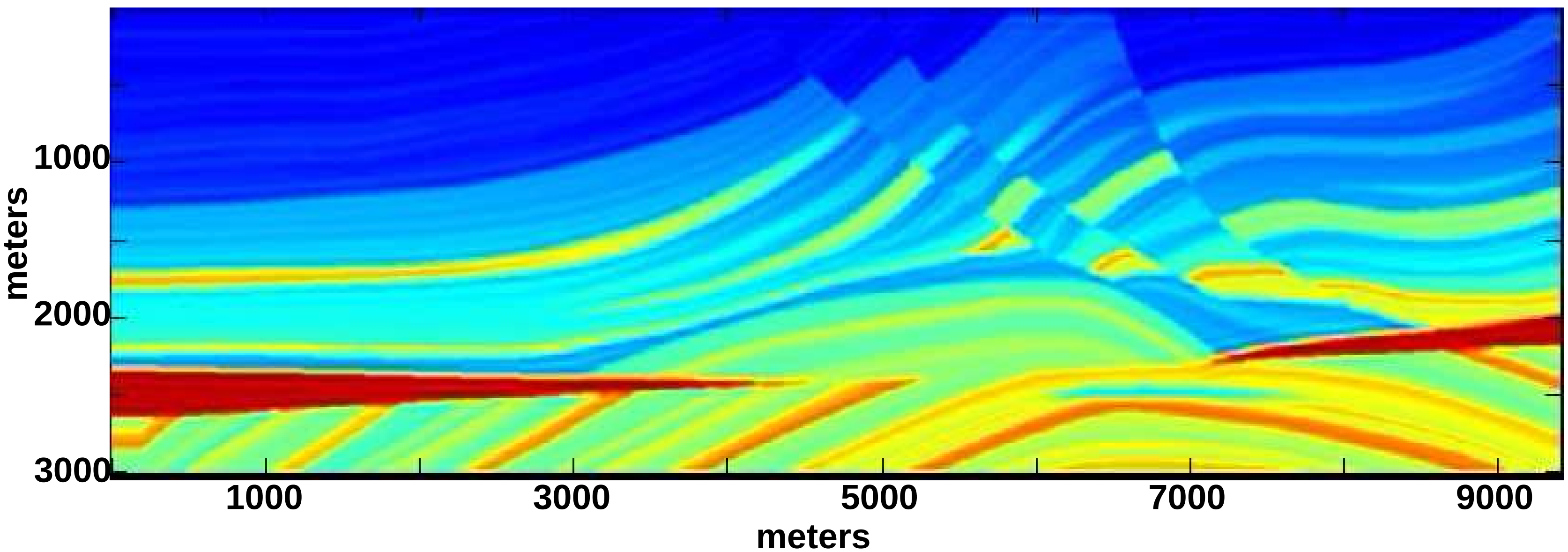
Environmental Modeling: Marmousi



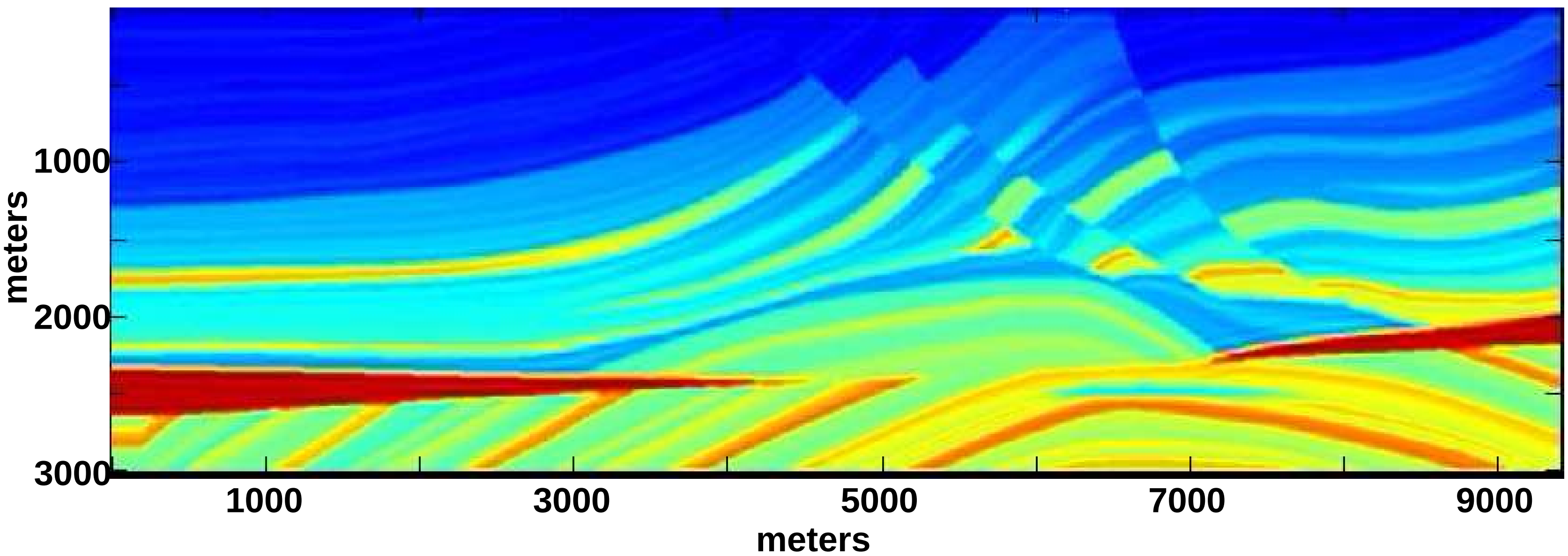
Environmental Difficulties

1) Complex, layered environments
2) Multidimensional environments
3) Inhomogeneous background
4) Large scale (many wavelengths)
5) Strongly inhomogeneous environments
6) Focusing and defocusing regimes

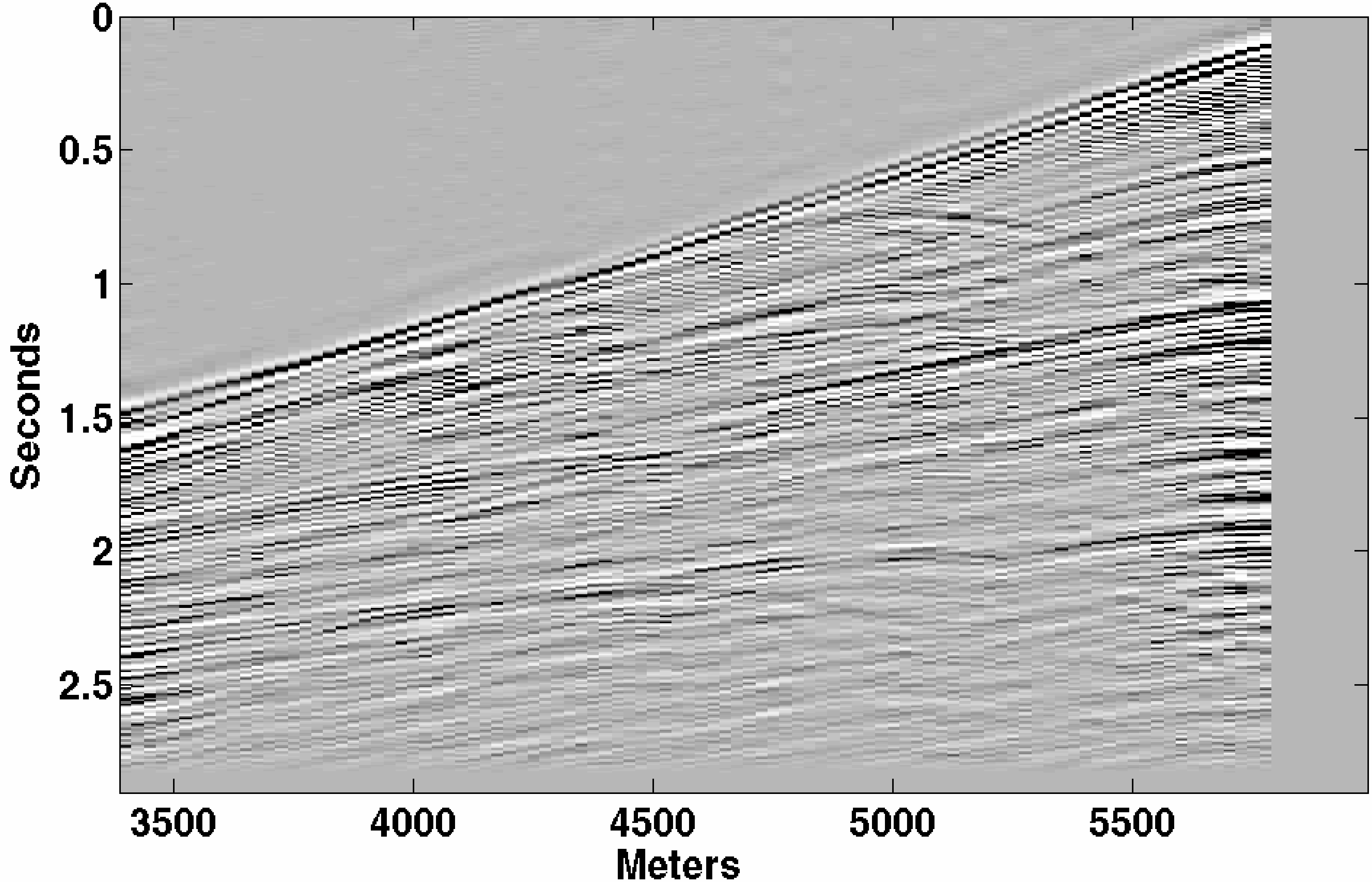
Marmousi Movie
Homogeneous Medium



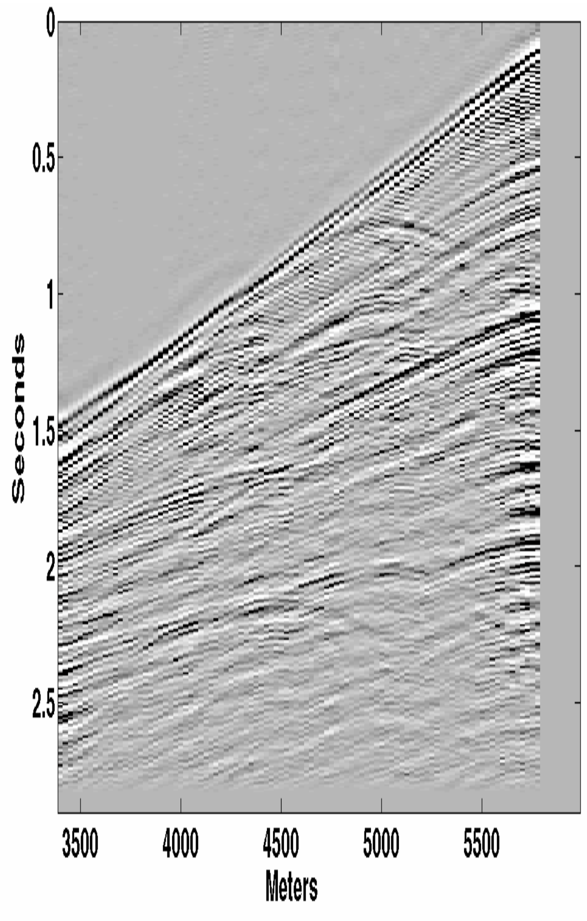
Marmousi Movie
Real Story



Marmousi Data



Marmousi Data



240 shots

96 receivers/shot

726 samples/receiver

8 bytes/samples

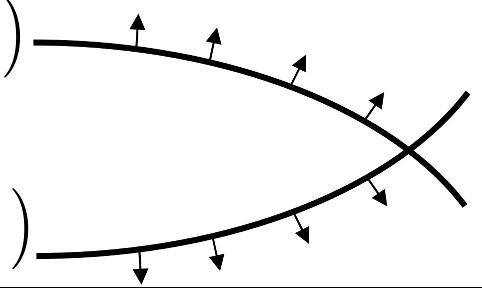
Dataset size= 240*96*726*8 ~ 134 Mbytes

Real datasets have 1000’s of shots, 1000’s of receivers/shot, and 1000’s of samples/receiver.

Seismic Imaging Paradigm

A common seismic imaging methodology is derivable from first-order inverse Born scattering

$$\Psi_{refl}\left(\vec{x},t_{inc}\right)=r\left(\vec{x}\right)\Psi_{inc}\left(\vec{x},t_{inc}\right)$$



$$\Psi_{\mathrm{inc}}\left(\vec{x},t_{inc}\right)$$

reflector

$$\frac{\Psi_{refl}\left(\vec{x},t_{inc}\right)}{\Psi_{inc}\left(\vec{x},t_{inc}\right)}=r\left(\vec{x}\right)$$ A reflectivity estimate.

Seismic Imaging Paradigm

Seismic imaging typically is done in the frequency domain and uses depth steps not time steps, so a more common imaging condition is:

$$r(x, y, \Delta z) = \sum_{k} \frac{\psi_{refl}(x, y, z = \Delta z, k\Delta\omega)}{\psi_{inc}(x, y, z = \Delta z, k\Delta\omega)}$$

Seismic Imaging Paradigm

So for each depth, we must calculate two fields:

$$\psi_{refl}\left(x, y, n\Delta z, \omega\right)$$

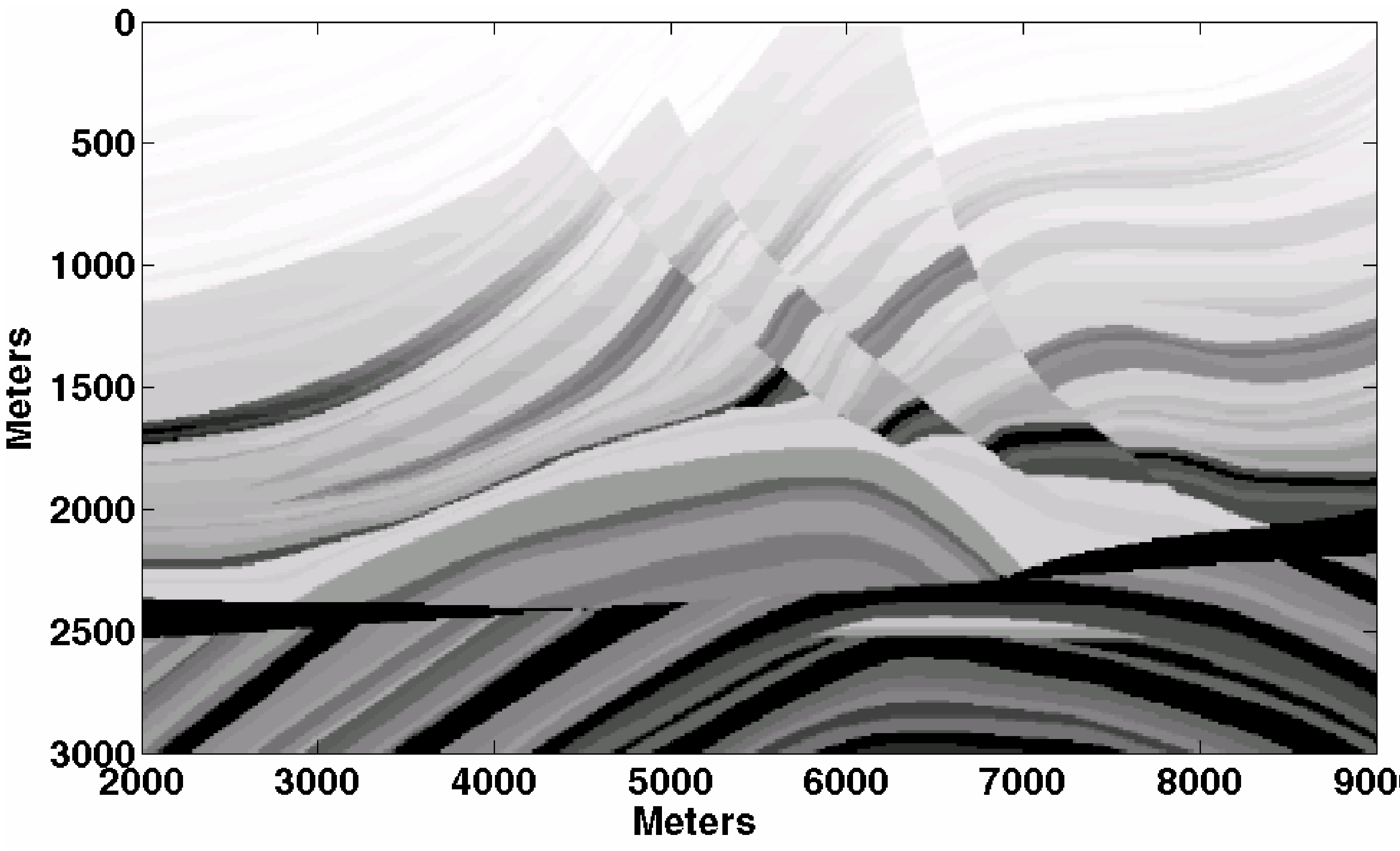
The reflected field comes from mathematically marching the recorded data down into the earth.

$$\psi_{inc}\left(x, y, n\Delta z, \omega\right)$$

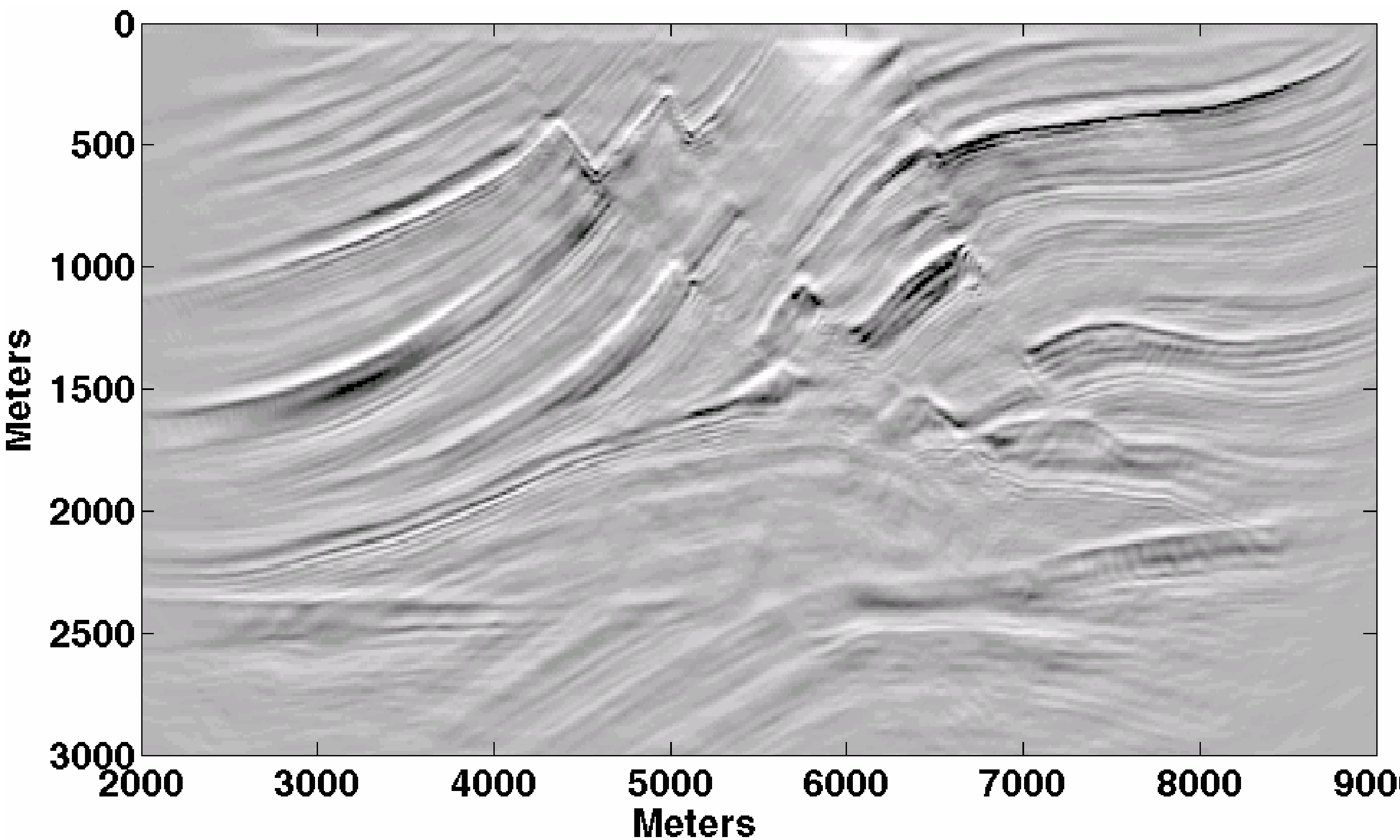
The incident field comes from a mathematical model of the source wavefield that is also marched down.

In both cases, the wavefield marching is done through a "background" velocity field that is presumed known.

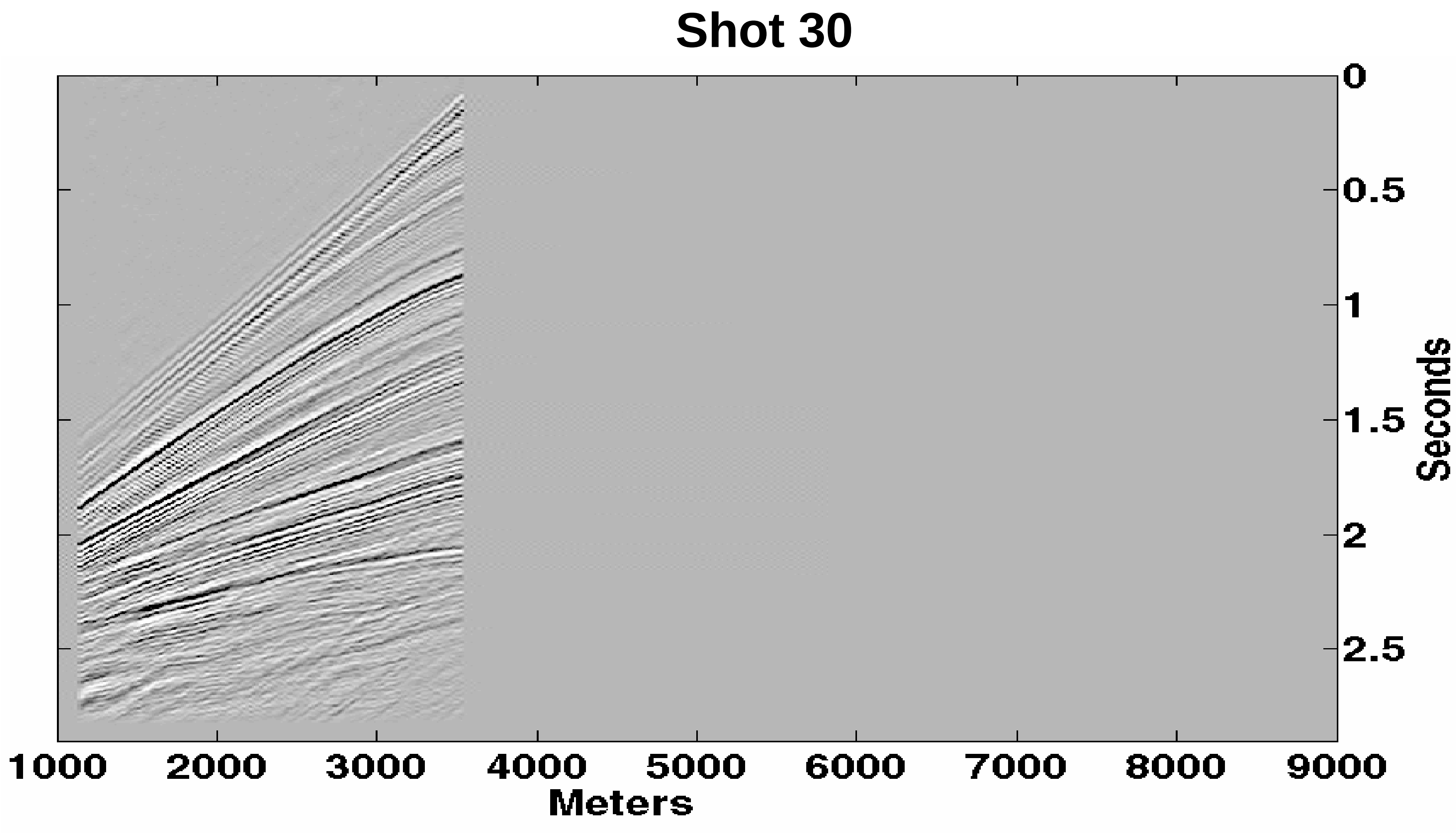
Marmousi Velocity Model



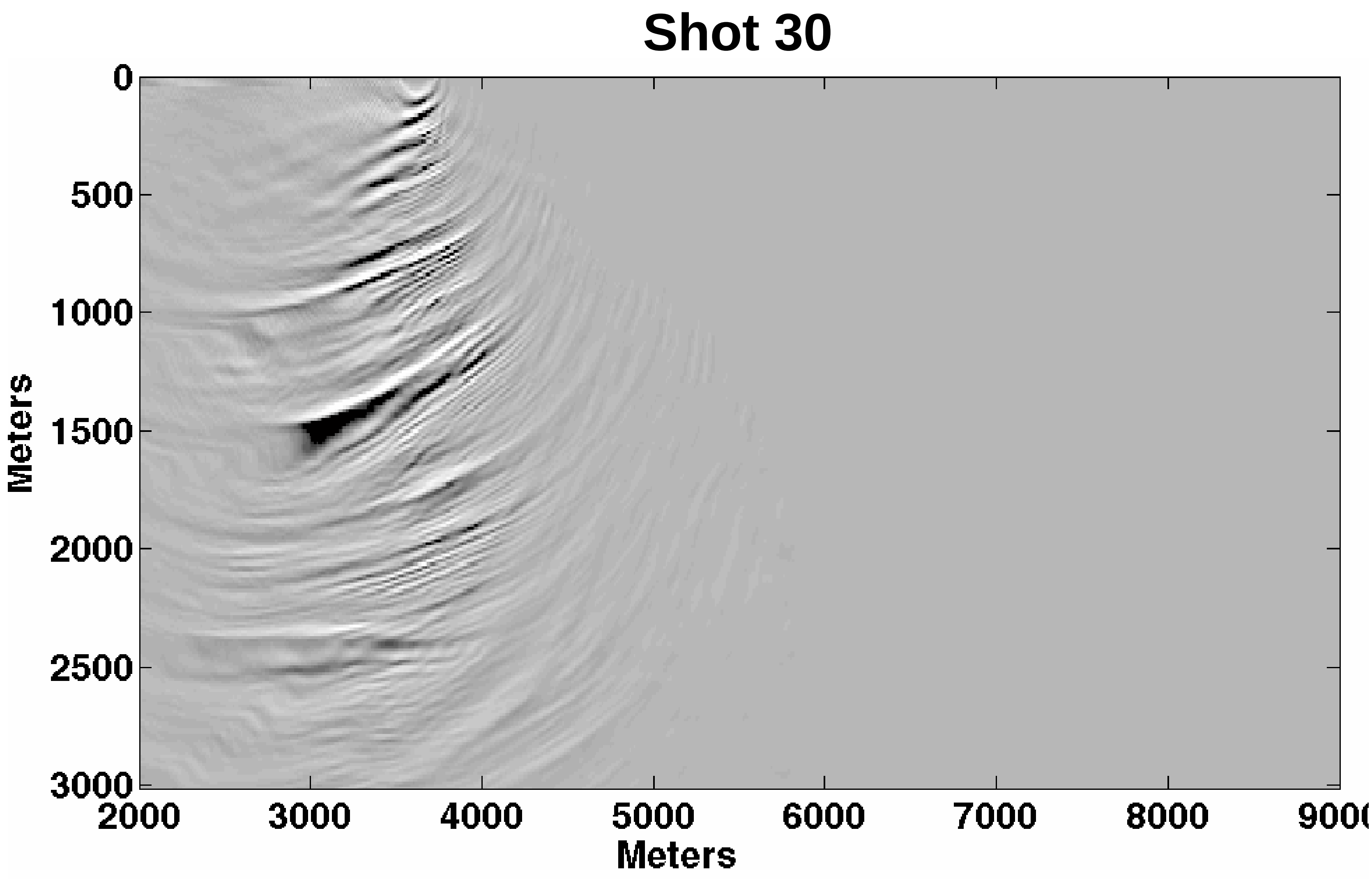
FOCI Pre-Stack Migration



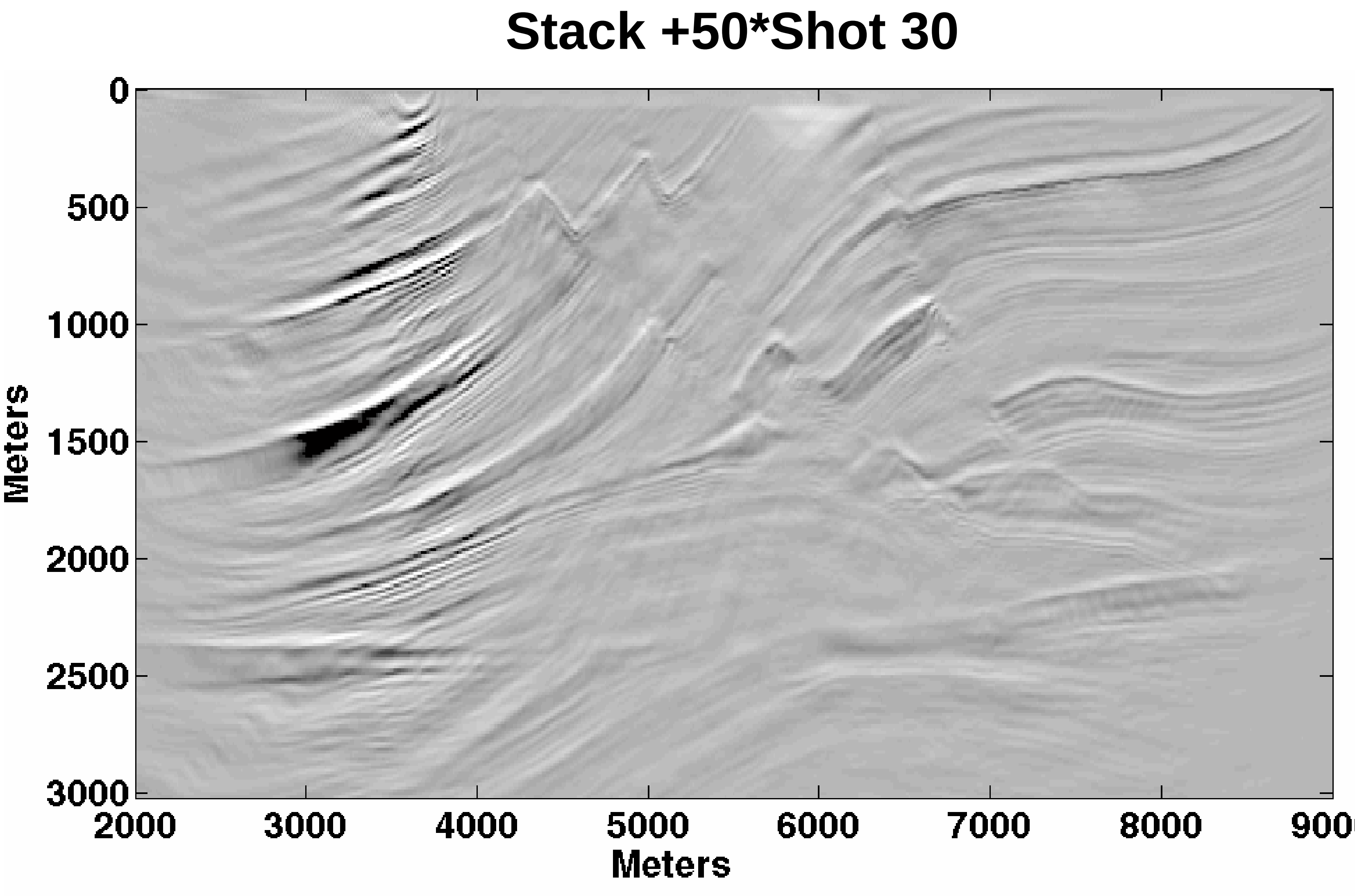
FOCI Pre-Stack Migration



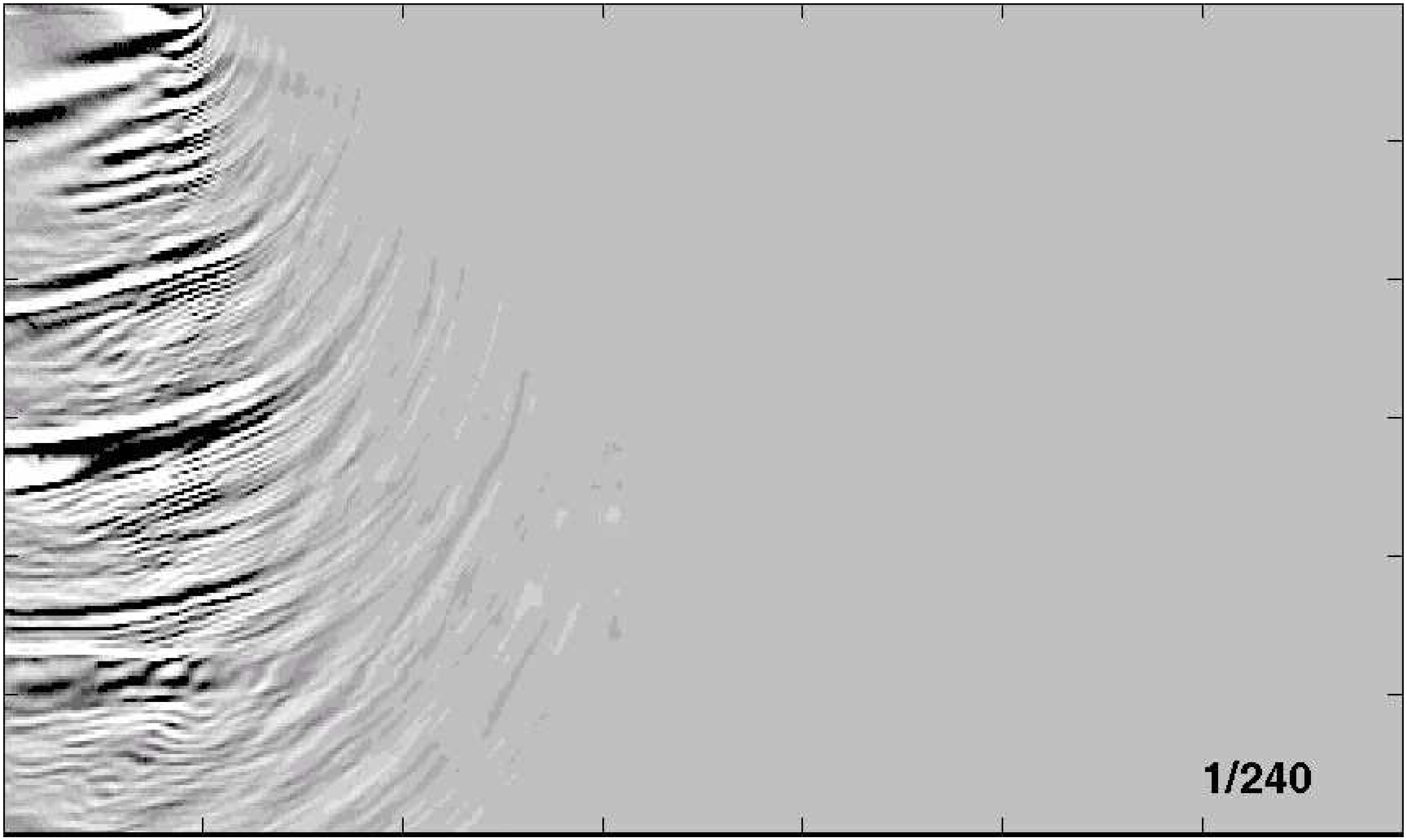
FOCI Pre-Stack Migration



FOCI Pre-Stack Migration



Depth Migration Movie



Frequency Domain

While the seismic experiment is conducted in the time domain, the data processing is often done in the frequency domain using the Fourier-transform of the time-domain wave equation, the Helmholtz equation.

$$\left(\nabla^2 + \frac{\omega^2}{v^2} \right) \psi = 0$$

Homogeneous Medium Wavefield Extrapolation
(the phase-shift method)

$$\psi^{+}(x,z,\omega)=\mathbf{F}\left(\Psi^{+}(x,z,\bullet)\right)(\omega)$$

Δz

homogeneous

$$\psi^{+}(x,z+\Delta z,\omega)$$

$$\psi^{+}(x,z+\Delta z)=\int_{\mathbb{R}} dk_x e^{i\Delta z\sqrt{\frac{\omega^2}{v^2}-k_x^2}}\, e^{ik_x x}\,\hat{\psi}^{+}(k_x,z)$$

This is exact for a constant velocity

Equivalent One-Way Wave Equation

The homogeneous medium wavefield extrapolator solves

$$i\partial_z \psi^+(x,z) + \frac{1}{2\pi}\int_{\mathbb{R}^2} dk_x dx' \left(\frac{\omega^2}{v^2} - k_x^2\right)^{1/2} \mathrm{e}^{ik_x(x-x')}\, \psi^+(x',z) = 0$$

Exact one-way wave equation for a homogeneous medium

Inherently nonlocal wave equation

If the square-root term is approximated by a polynomial, then the one-way wave equation becomes a (local) pde.

Locally Homogeneous Medium Wavefield Extrapolation

(the GPSPI method)

$$\psi^{+}\left(x, z+\Delta z, \omega\right) \approx \int_{\mathbb{R}} dk_x e^{i\Delta z\sqrt{\frac{\omega^2}{v(x)^2}-k_x^2}} e^{ik_x x}\,\hat{\psi}^{+}\left(k_x, z, \omega\right)$$

In the limit of an infinitesimal step, the corresponding one-way wave equation is

$$i\partial_z\psi^{+}\left(x,z\right)+\frac{1}{2\pi}\int_{\mathbb{R}^2} dk_x dx'\left(\frac{\omega^2}{v\left(x\right)^2}-k_x^2\right)^{1/2} \mathrm{e}^{ik_x\left(x-x'\right)}\,\psi^{+}\left(x',z\right)=0$$

Locally Homogeneous Medium Wavefield Extrapolation
(physics formulation)

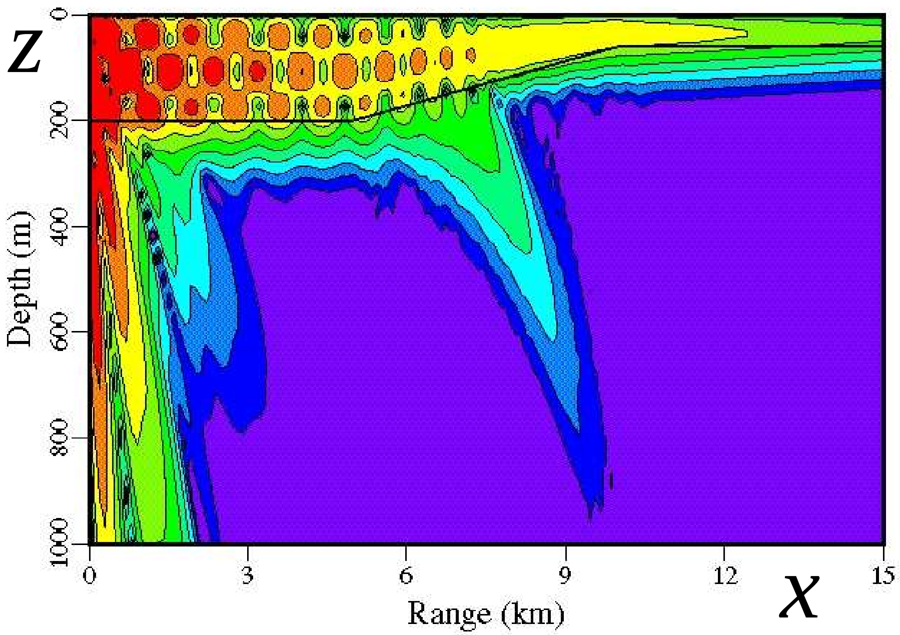
$$\phi^{+}(x+\Delta x, z) \approx \int_{\mathbb{R}} dp \exp\left(i\bar{k}pz\right)\left[\exp\left(i\bar{k}\Delta x\left(K^{2}(z)-p^{2}\right)^{1/2}\right)\hat{\phi}^{+}(x,p)\right]$$

$$\left(i/\bar{k}\right)\partial_{x}\phi^{+}(x,z)+\frac{\bar{k}}{2\pi}\int_{\mathbb{R}^{2}} dp dz'\left(K^{2}(z)-p^{2}\right)^{1/2}$$
$$\bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^{+}\left(x,z'\right)=0$$

$$K(z)=\frac{c_{0}}{c(z)} \qquad \bar{k}=\frac{\omega}{c_{0}} \qquad c(z)=v(z)$$

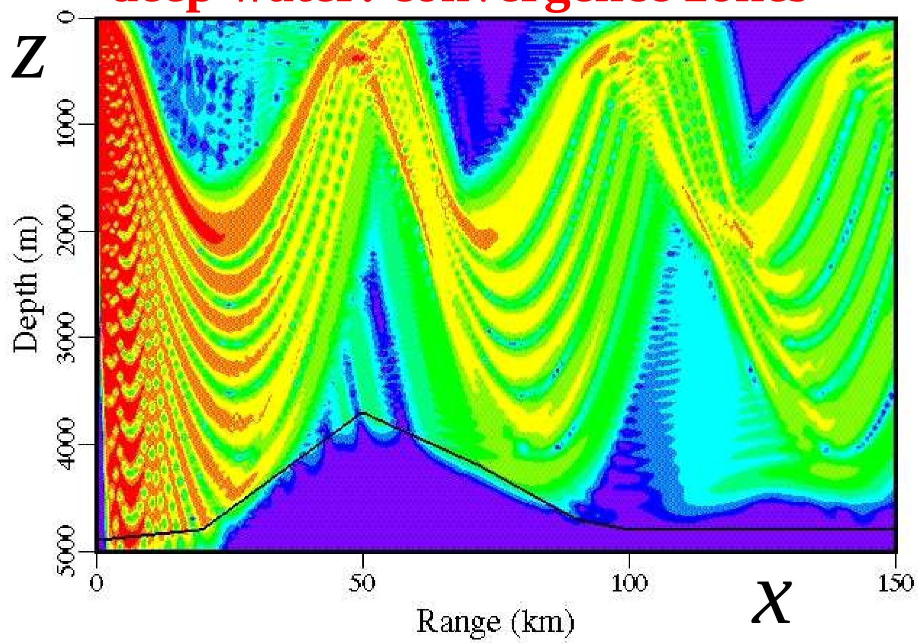
Range-Dependent Problems in Ocean Acoustics

shallow water: modes



[Jensen and Kuperman, J. Acoust. Soc. Am., 1980]

deep water: convergence zones



Three Common Misconceptions About GPSPI

1) The one-way wave equation corresponding to the limiting form of the GPSPI algorithm

$$\left(i/\bar{k}\right)\partial_x \phi^+(x,z) + \frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2} dp\,dz' \overbrace{\left(K^2(z) - p^2\right)^{1/2}}^{\text{symbol}} \bullet \exp\left(i\bar{k}p\left(z-z'\right)\right)\phi^+\left(x,z'\right) = 0$$

is believed exact for a range-independent medium. The square-root function is believed to be the correct function (symbol) for infinitesimal wavefield extrapolation.

Three Common Misconceptions About GPSPI

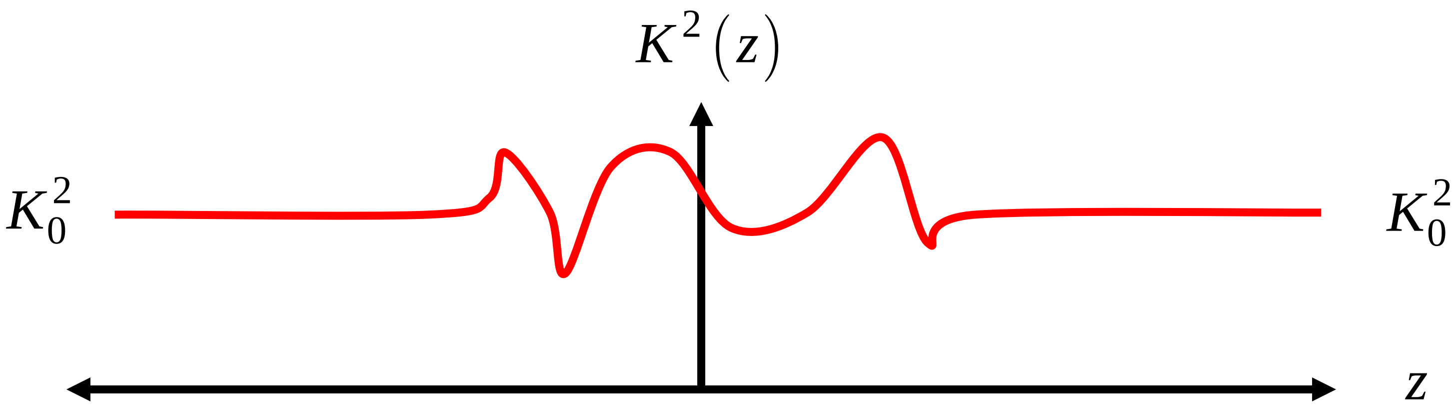
2) The wavefield growth problems, which can develop for finite, range step-size, are believed to vanish in the limit of zero, range step-size, since, in that limit, the theory is believed to be exact. The amplitude problems are assumed a result of the numerical discretization, and are not viewed as fundamental in nature.

3) Since, in the typical derivation of GPSPI, the up- and down-going wavefields are assumed to be independent, it is thought to be impossible to extrapolate a full, two-way wavefield by well-posed, one-way marching methods.

Misconception 1

Physical arguments (and rigorous mathematical arguments) establish that the symbol must be frequency dependent.

Consider the following environment:



in the (1) high- and (2) low-frequency limits.

Misconception 1

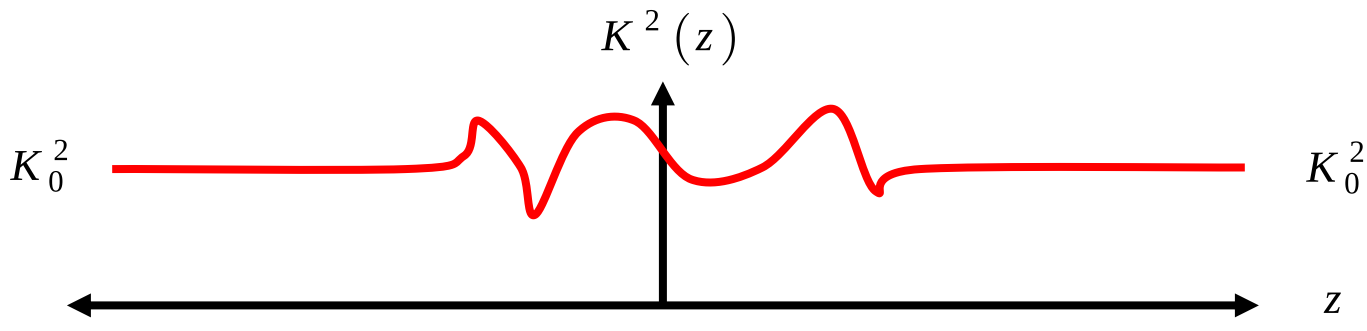
(1) In the high-frequency limit, the environment appears locally homogeneous, and the exact symbol approaches

$$\left(K^2(z) - p^2\right)^{1/2}$$

(2) In the low-frequency limit, the environment appears globally homogeneous, and the exact symbol approaches

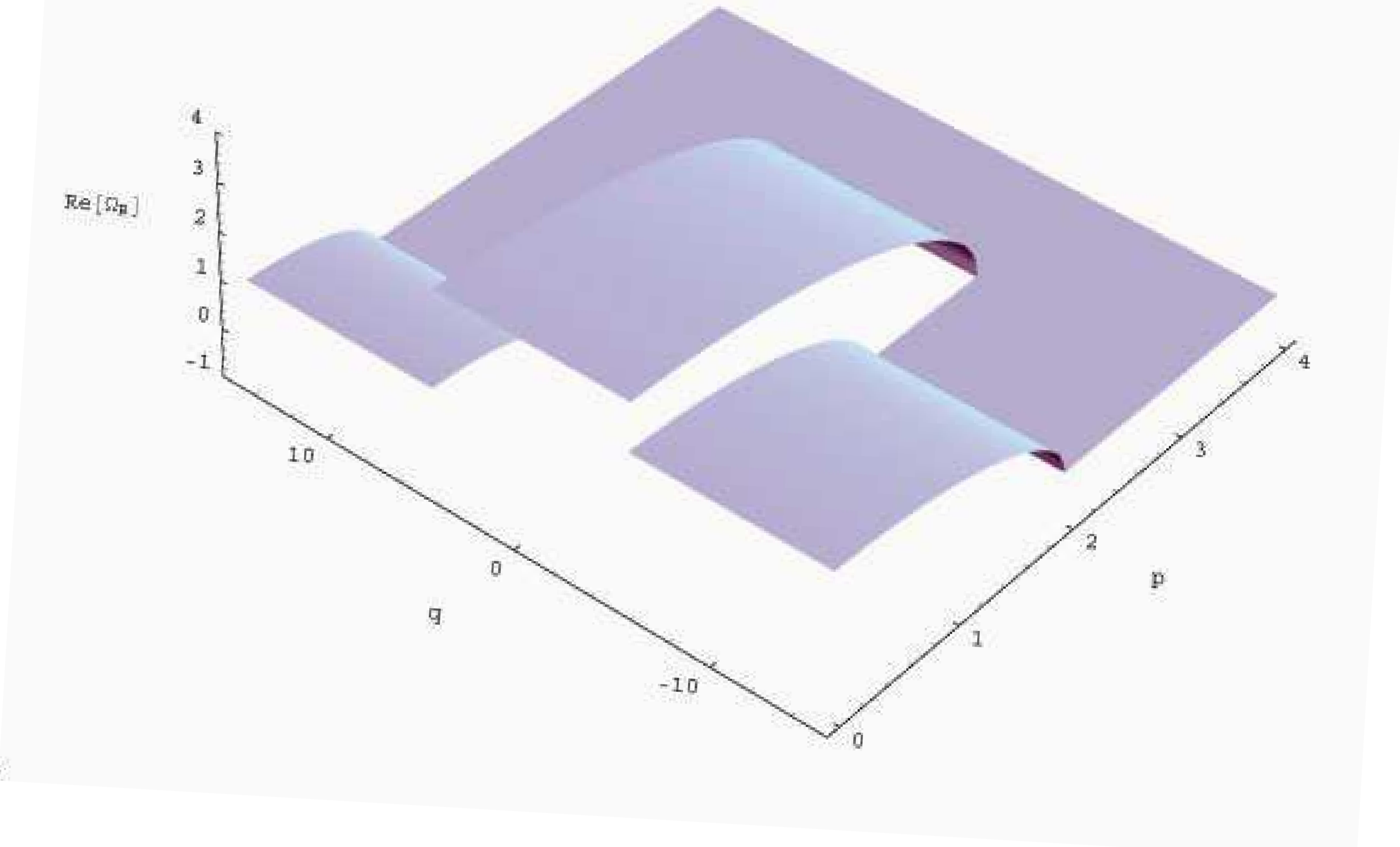
$$\left(K_0^2 - p^2\right)^{1/2}$$

Thus, the exact symbol must be frequency dependent, and cannot be the locally homogeneous symbol for all frequencies.



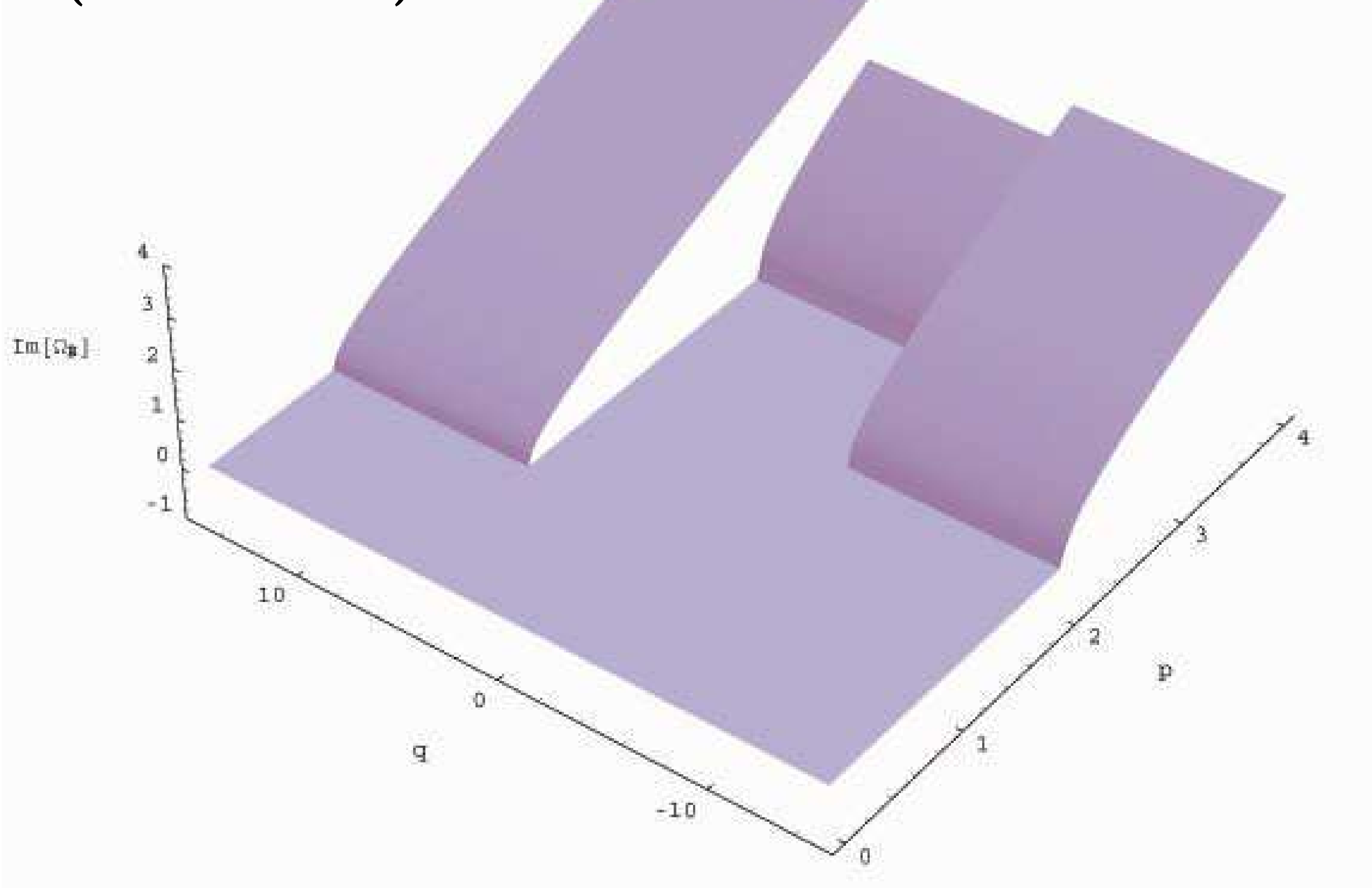
Locally Homogeneous Approximation
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



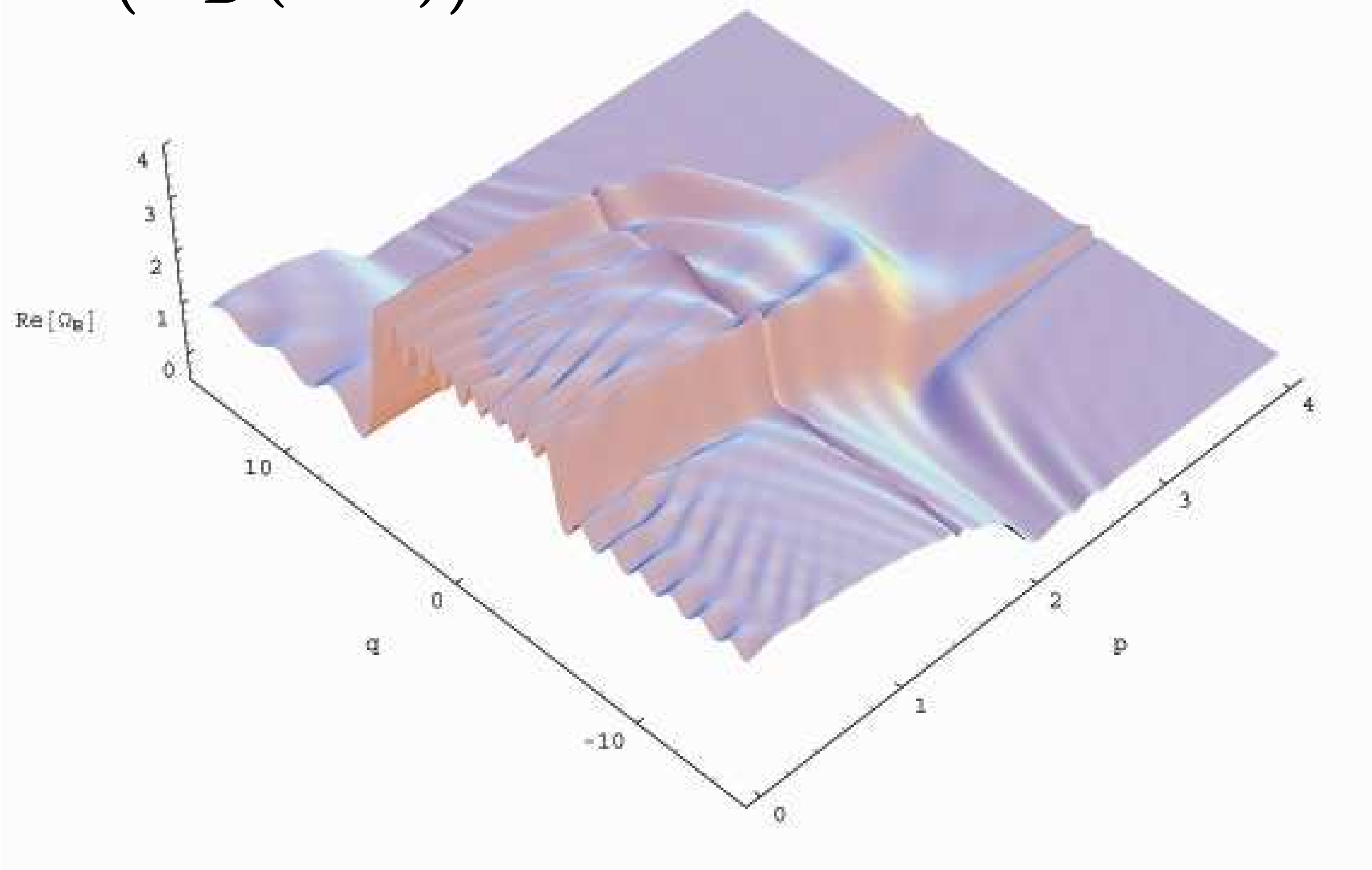
Locally Homogeneous Approximation
3-Layer Profile

$$\operatorname{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



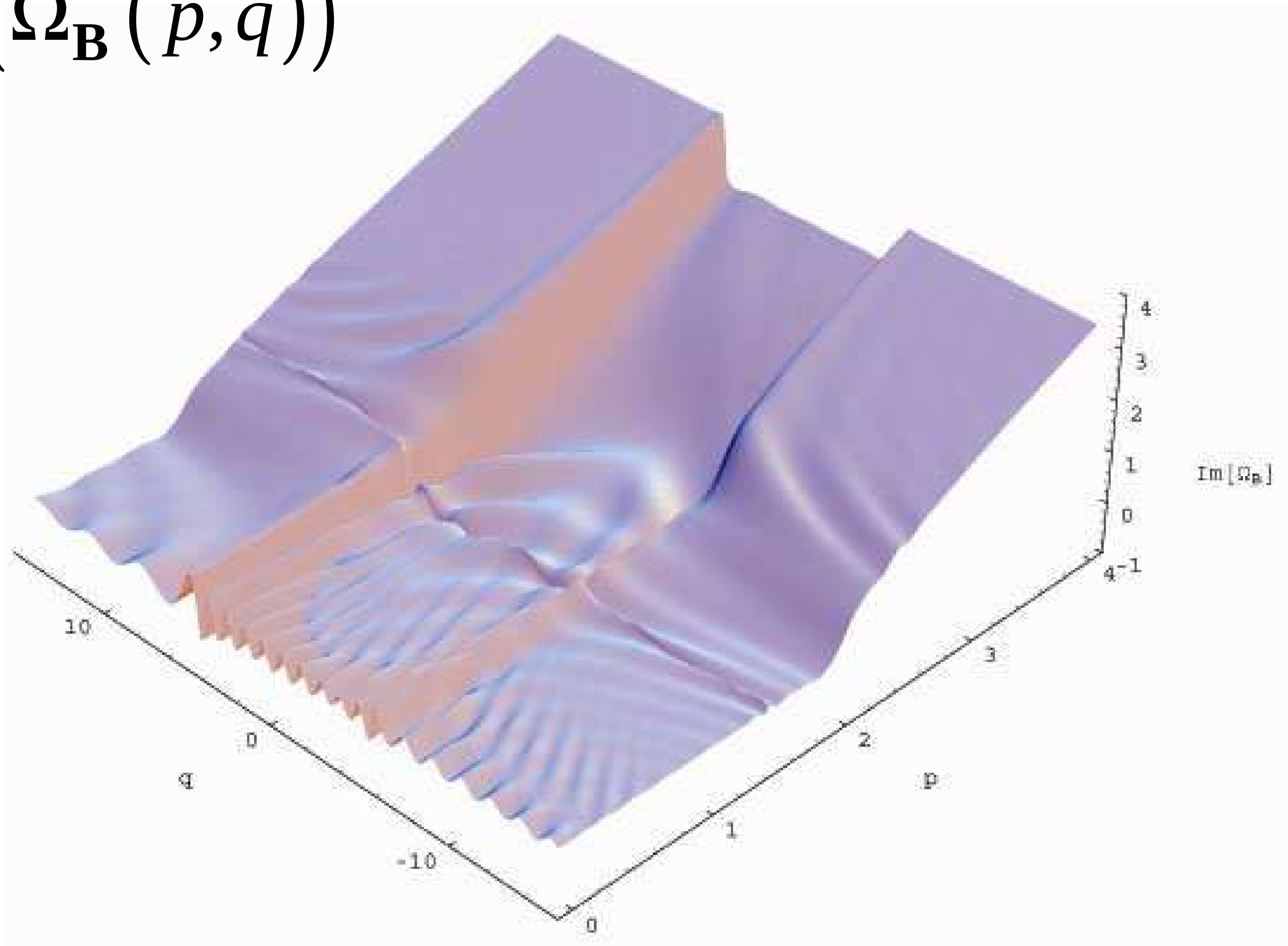
Exact Operator Symbol
3-Layer Profile

$$\mathrm{Re}\left(\mathbf{\Omega_B}\left(p,q\right)\right)$$



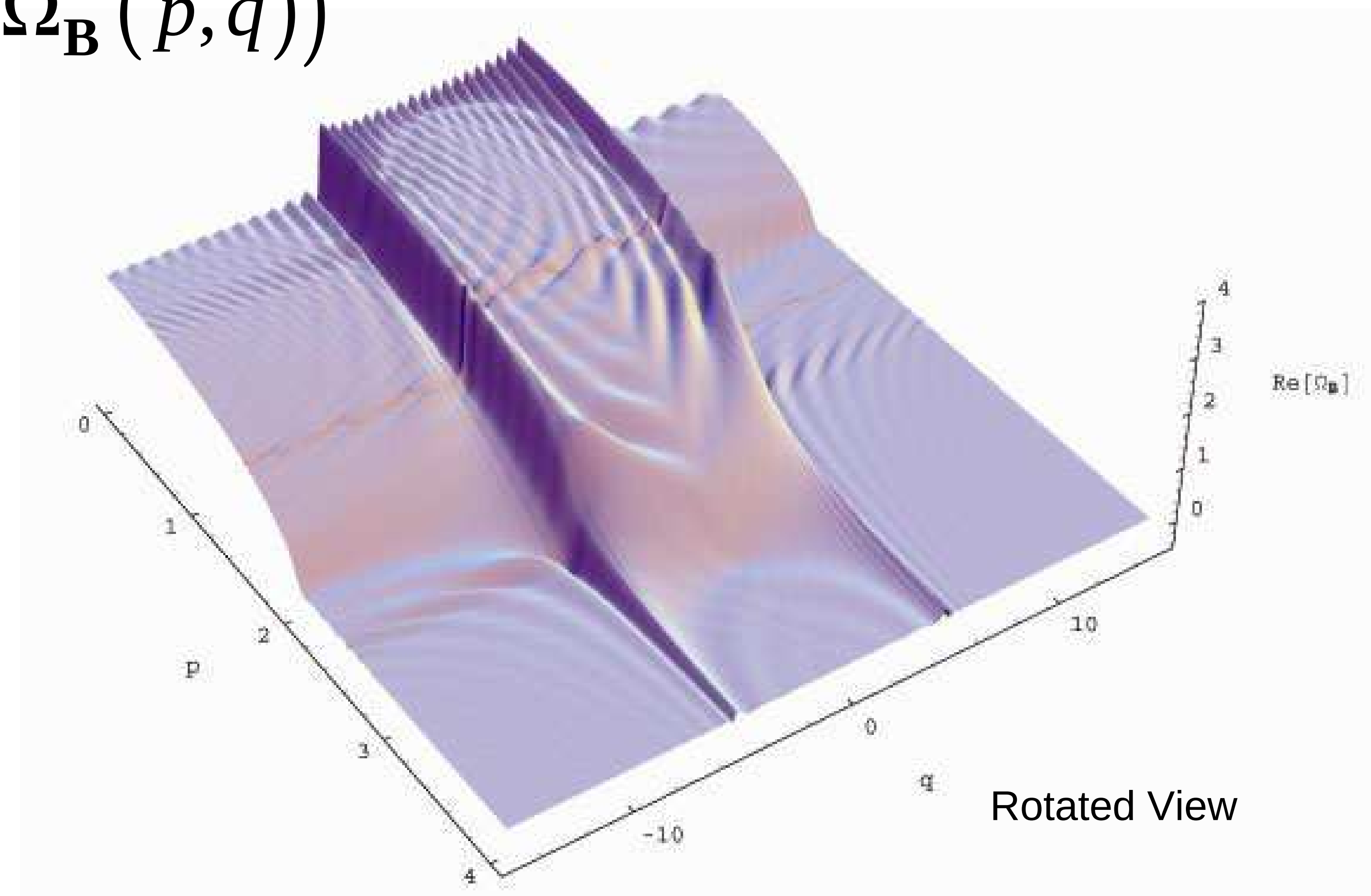
Exact Operator Symbol
3-Layer Profile

$$\mathrm{Im}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$$



Exact Operator Symbol
3-Layer Profile

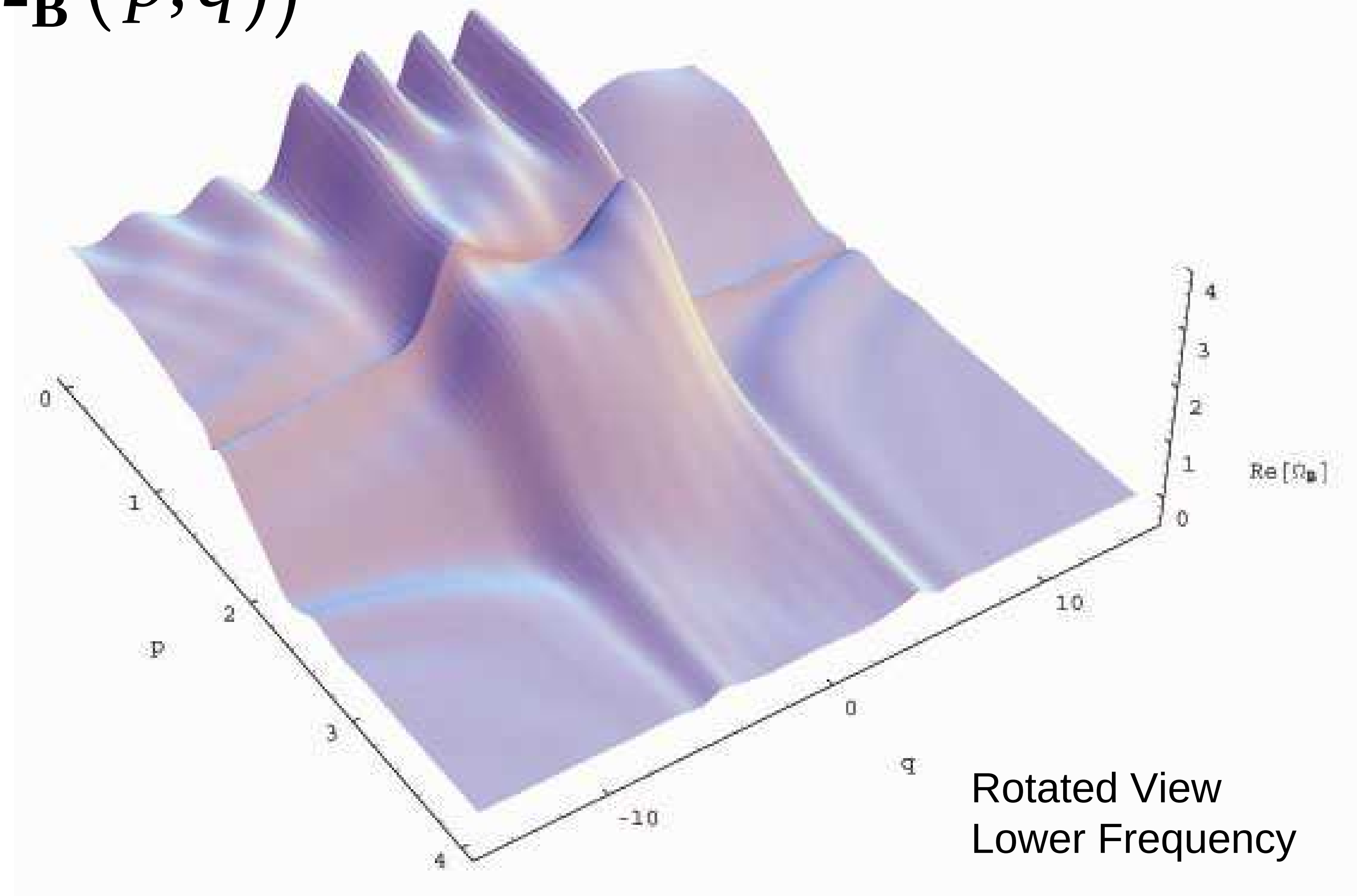
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View

Exact Operator Symbol
3-Layer Profile

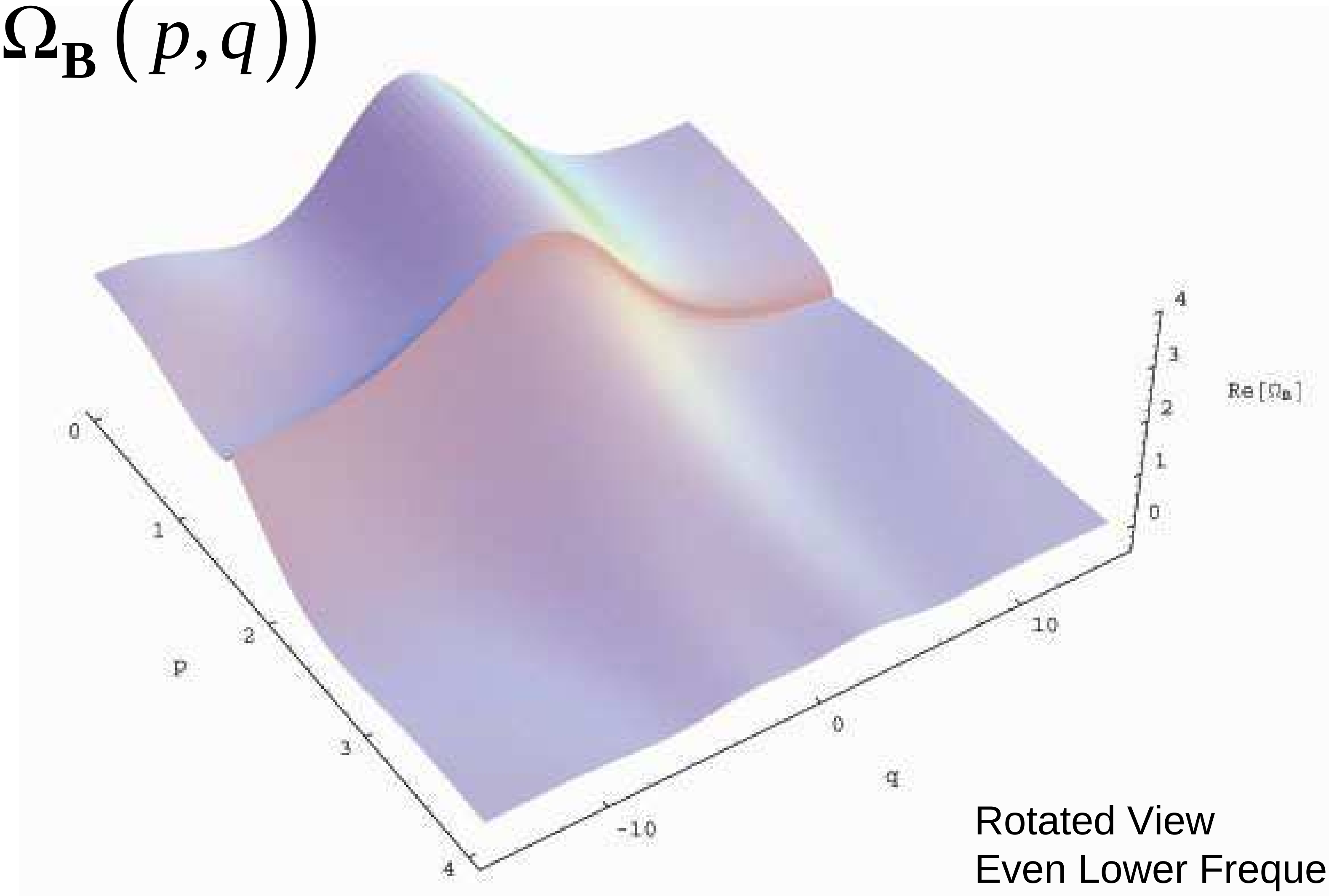
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View
Lower Frequency

Exact Operator Symbol
3-Layer Profile

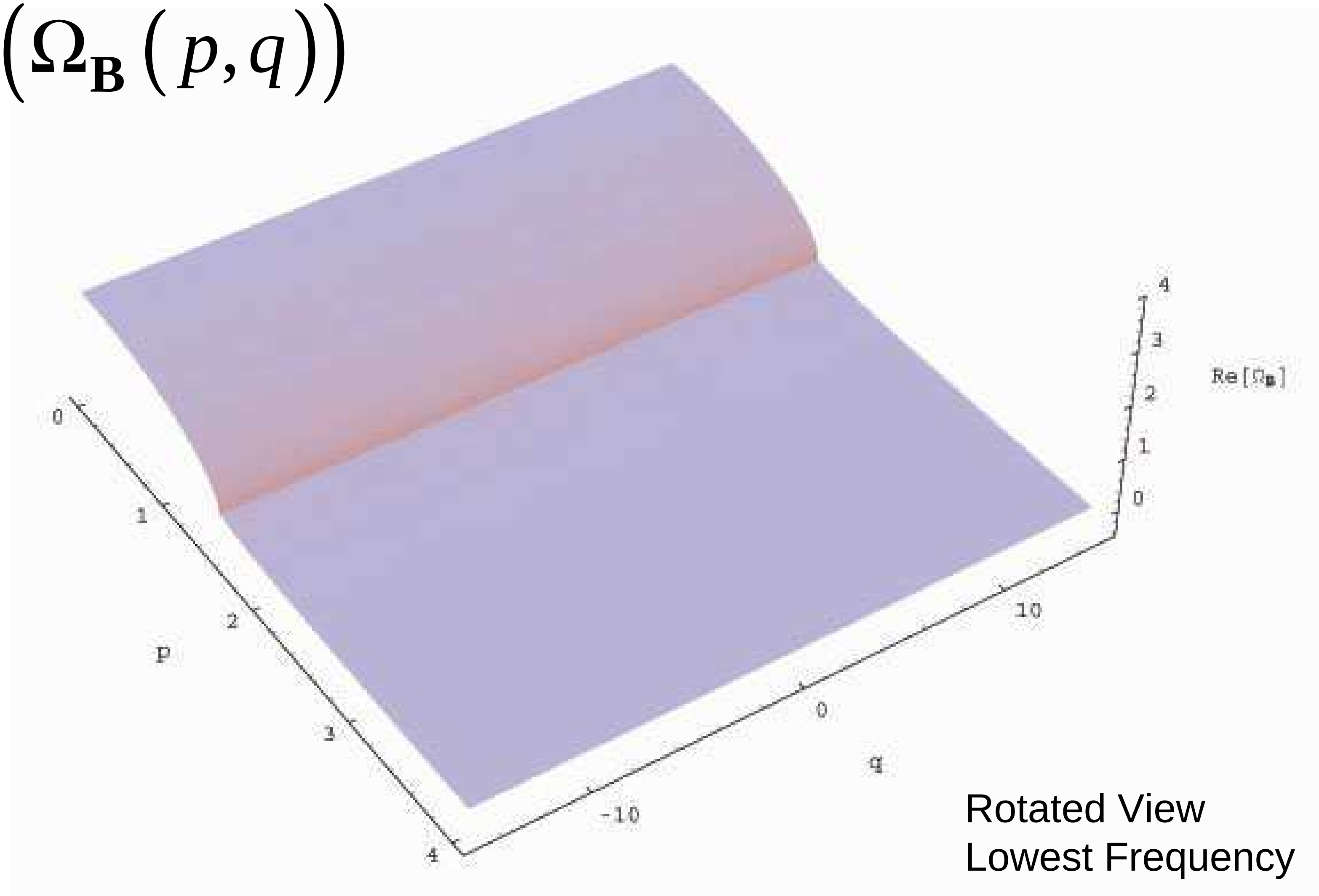
$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View
Even Lower Frequency

Exact Operator Symbol
3-Layer Profile

$\mathrm{Re}\left(\Omega_{\mathbf{B}}\left(p,q\right)\right)$



Rotated View
Lowest Frequency

Misconceptions 2 and 3

2) The locally homogeneous square-root function is just an approximation to the exact symbol. It does not inherently conserve the integrated energy flux. The wavefield growth problems experienced by GPSPI are fundamental, and not solely a discretization artifact. This is a reflection of the true nature of this APPROXIMATION.

3) We will, indeed, construct an exact, well-posed, one-way reformulation of the two-way Helmholtz equation.

The Shape of Things to Come

- The construction of an exact, well-posed, one-way reformulation of the Helmholtz equation in terms of complementary scattering and boundary-value pictures. This will allow for the one-way marching of the two-way wavefield. These will be mathematically formal operator equations.

- The construction of a mathematically explicit one-way wave equation, fundamental solution representation, and computational algorithm in terms of the appropriate symbols. Formal operator equations will be replaced by explicit equations in terms of functions (symbols).

Example

(1) Formal one-way wave equation

$$\left(\left(i/\bar{k}\right)\partial_x + \left(K_0^2 + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}\right)\phi^+(x,z) = 0$$

(2) Mathematically explicit one-way wave equation

$$\left(i/\bar{k}\right)\partial_x\phi^+(x,z) + \frac{\bar{k}}{2\pi}\int_{\mathbb{R}^2} dp\,dz'\left(K_0^2 - p^2\right)^{1/2}$$

$$\bullet\ \exp\left(i\bar{k}p\left(z - z'\right)\right)\phi^+\left(x,z'\right) = 0$$

Example

(3) Formal fundamental solution representation

$$\mathbf{G}^{+} = \exp\left[i\bar{k}x\left(K_0^2 + \left(1/\bar{k}^2\right)\partial_z^2\right)^{1/2}\right]$$

(4) Explicit fundamental solution representation

$$G^{+}\left(x, z \middle| 0, z'\right) = \lim_{N\to\infty}\int_{\mathbb{R}^{2N-1}} \prod_{j=1}^{N-1} dz_j \prod_{j=1}^{N}\left(\frac{\bar{k}}{2\pi}\right) dp_j$$

$$\bullet \exp\left[i\bar{k}\sum_{j=1}^{N}\left(p_j\left(z_j - z_{j-1}\right) + \left(\frac{x}{N}\right)\left(K_0^2 - p_j^2\right)^{1/2}\right)\right]$$

The Shape of Things to Come

The previous examples are well understood consequences of Fourier analysis (nontrivial functions of commuting operators).

These constructions will be extended to nontrivial functions of non-commuting operators, such as

$$\mathbf{B} = \left(K^2(z) + \left(1/\bar{k}^2 \right) \partial_z^2 \right)^{1/2}$$

The Shape of Things to Come

- The construction of exact equations explicitly defining the symbols, and allowing for the appropriate asymptotic analysis

- The resulting approximate wavefield extrapolators will be accurate over the entire spectral range (0 – 90 degrees in the propagating regime and throughout the evanescent regime) for several different parameter regimes.

- The computational algorithms will run exactly as GPSPI. The locally homogeneous square-root function will just be replaced by the above asymptotic results.

The Shape of Things to Come

- The analysis and constructions will have essentially extended Fourier methods to inhomogeneous environments. The basic form of the one-way wave equation, fundamental solution representation, and computational algorithm, for a homogeneous medium, will all survive all the way through. Only the complexity of the symbols will change. This is the beauty of the approach.