Causal Structure and Quantum Field Theory in Curved Spacetimes

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I: A Quick Tour of Classical Relativity

Or, really important things about the local and global structure of spacetime that you may not have seen in your general relativity class because of all the time you spent on learning about coordinate transformations, covariant derivatives and Riemann curvature

Note in my lectures $c = G = 1$ and the Lorentz metric signature is $- + + \cdots +$

Spacetimes

A spacetime is a pair (M^n, g_{ab}) where M^n is a smooth n-manifold where g_{ab} is a symmetric, non-degenerate tensor with signature $-+ +\cdots+$.

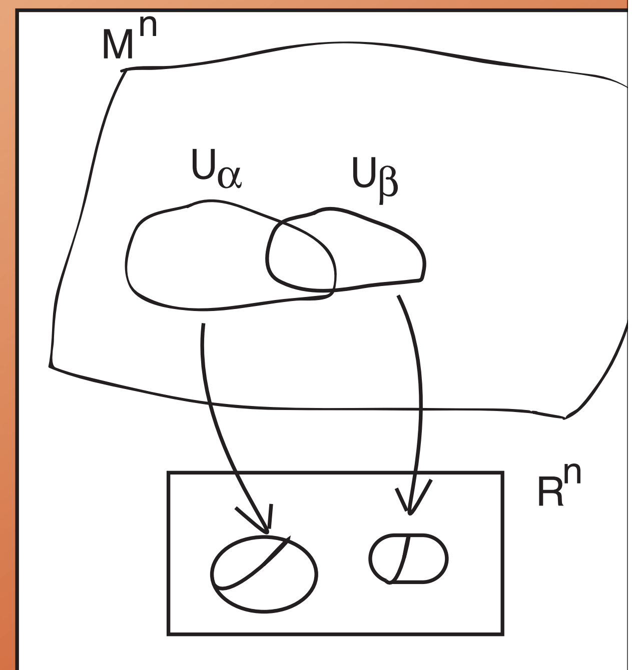
Definition: A topological space M^n is a C^k n-manifold if it satisfies the following conditions:

1. Every point has a neighborhood U_α which is homeomorphic to an open subset of R^n via a mapping $\phi_\alpha : U_\alpha \to R^n$.

2. Given any two of the U's with nonempty intersection, then the mapping

$$\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a C^k mapping between R^n and itself.



Why do we worry about smooth manifolds?

- In order to do field theory, global structure is important.
- Local knowledge of the metric does not determine the global structure of the spacetime.
- Global structure includes differentiable structure not just topology.

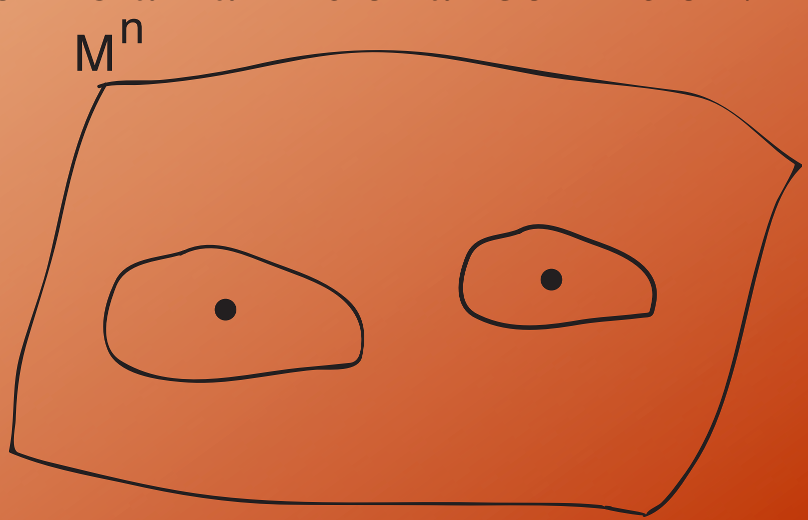
Extra topological conditions

- One example of a physical requirement is that one needs to define integration on a manifold.

  If the space is compact, our definition is enough. But in general, one needs the space to be *paracompact*.

  Theorem: Any differentiable n-manifold which admits a Lorentzian metric is paracompact.

- The Hausdorff condition is an additional condition.



Theorem: The following statements are equivalent for a differentiable manifold:

- It is Hausdorff and paracompact.
- Its topology can be induced by a metric.
- It admits a Riemannian metric.
- It is Hausdorff and admits a Lorentzian metric.

All manifolds will be taken to Hausdorff and paracompact, unless mentioned otherwise.

Example

First, define an open cover of

$$S^n = \{x \in R^{n+1} | x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

by

$$U_{2j-1} = \{x \in S^n | x_j < 0\} \; and \; U_{2j} = \{x \in S^n | x_j > 0\},$$

where $j = 1, \ldots, n+1$. Next, define the homeomorphisms of the chart by the maps

$$\square_i : U_i \to R^n$$

where

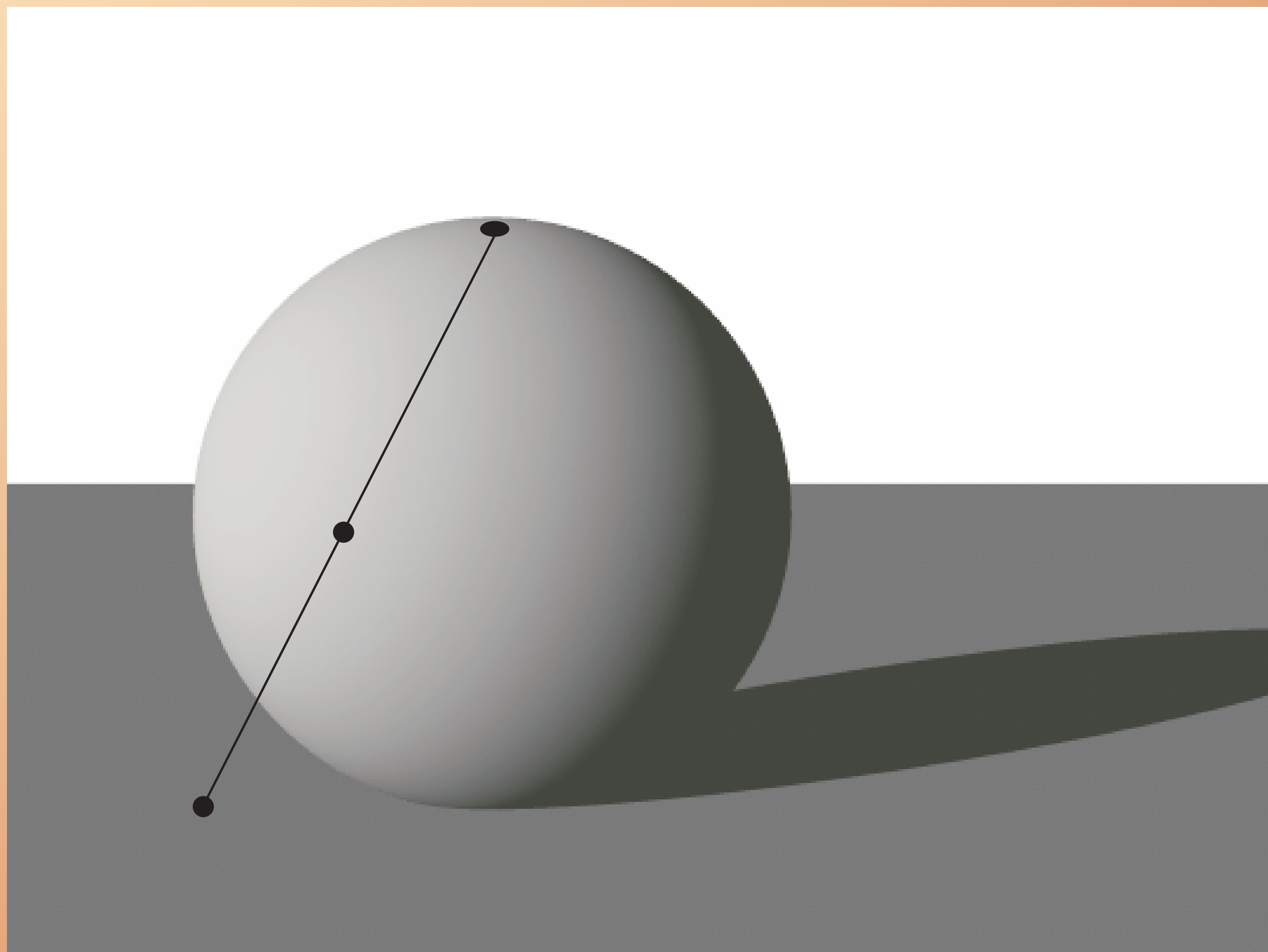
$$\square_i(x) = (x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}) \quad for \; i = 2j-1 \; or \; i = 2j,$$

the hat over the x_j means delete the j^{th} coördinate. The map $\square_i$ maps U_i onto the open ball B in R^n, i.e. the set of points with norm less than one. The inverse is given by

$$\square_i^{-1}(y) = (y_1, \ldots, y_{j-1}, (\pm 1)^i \sqrt{1 - |y|^2}, y_j, \ldots, y_n)$$

where y is a point in $B = \{y \in R^n | \, |y| < 1\}$.

Example of another chart on the n-sphere



Equivalence of Differentiable structures

Definition: A (continuous) map $f: M^m \to N^n$ is smooth if for each point in M^m there is chart (ϕ, U) containing it and a chart (ψ, V) in N^n with $f(U) \subset V$ such that:

$$\psi f \phi^{-1} : \phi(U) \to \psi(V)$$

is a smooth map from R^m to R^n.

Definition: A smooth homeomorphism with a smooth inverse is called a diffeomorphism. Furthermore, two smooth manifolds are equivalent if there is a diffeomorphism between them; they are said to be diffeomorphic.

Example on the line

Let $M = R$ with M the only open set in the atlas and the homeomorphism $\square\colon M \to R$, $\square(x) = x^{\frac{1}{3}}$. Therefore, this defines a smooth structure on M. By definition a function $f\colon M \to R$ is smooth if and only if

$$f\square^{-1}\colon M \to R$$

is smooth in the usual sense.

An example of a smooth function on M is $f(x) = x^{\frac{2}{3}}$ because $f\square^{-1}(x) = x^2$ is smooth in the usual sense! Now, M is diffeomorphic to R because $\square$ is a diffeomorphism. Observe that one can easily find many differential structures like the above on R and other manifolds.

Homework Problem 1

- Prove or disprove the following: Given a connected n-manifold M and a point p in M, then M minus p is not diffeomorphic to M.

- Example showing that the above is true in one case: An n-sphere minus a point is R^n. Clearly, R^n is not diffeomorphic to S^n.

Exotic Differentiable Structures

- In 3 or fewer dimensions, the differentiable structure is unique and all manifolds admit a differentiable structure
- In 4 or more dimensions, the differentiable structures are not unique. Furthermore, there are examples of topological manifolds which do not admit a smooth structure.
- In 8 or more dimensions, there are piecewise linear manifolds which are not smooth.

Physically, differentiable structures are important

- Recently solutions have been found which solve the Einstein equations for different differentiable structures. (K. Schleich and D. Witt Class. Quant. Gravity 1999, and preprint 2003)
- The piecewise linear case is interesting for discrete versions of gravity. (K. Schleich and D. Witt, Nucl. Phys. B 1993)
- Sum over topologies is incomplete in 4 or more dimensions unless a sum over differentiable structures is also included.

Geometry on a spacetime

- Define the covariant derivative ∇_a in terms of the metric tensor.
- Define the curvature tensor $\nabla_{[a}\nabla_{b]}\omega_c = \frac{1}{2}R_{abcd}\omega^d$
- Geodesics are given by curves

  which satisfy $u^b\nabla_b u^a = 0$

  All of the above concepts hold for metrics of any signature.
- Useful tip: Use Cartan Calculus for curvature calculations!

3 types of geodesics in Lorentzian Geometry

- Null geodesics $u_a u^a = 0$
- Timelike geodesics $u_a u^a < 0$
- Spacelike geodesics $u_a u^a > 0$

Note: Riemannian geometry only has one type.

Diffeomorphisms, Isometries, and Conformal Isometries

Definition of Physical Equivalence:

The spacetime (M^n, g_{ab}) is equivalent to $(N^n, \bar{g}_{ab})$ iff there is diffeomorphism $f : M^n \to N^n$ such that $f^* \bar{g}_{ab} = g_{ab}$

Definition of Isometry:

A diffeomorphism of (M^n, g_{ab}) to itself is an isometry iff $f^* g_{ab} = g_{ab}$.

conformal isometry iff $f^* g_{ab} = g_{ab}$.

Definition of Conformal Isometry:

A diffeomorphism of (M^n, g_{ab}) to itself is a conformal isometry iff $f^* g_{ab} = \square g_{ab}$ where $\square > 0$.

Killing and Conformal Killing Vectors

- These are infinitesimal versions of isometries and conformal isometries.
- A conformal Killing vector is a vector which satisfies

$$\nabla_a \xi_b + \nabla_b \xi_a = \frac{2}{n} \nabla_c \xi^c g_{ab}$$

- A Killing vector is a vector which satisfies

$$\nabla_a \xi_b + \nabla_b \xi_a = 0$$

- Set of all linearly independent Killing vectors forms a Lie algebra

How to find Conformal Killing and Killing vectors

- Any two conformally equivalent metrics have the same conformal Killing vectors. Try simplifying metric via conformal isometries.
- Find Killing vectors by finding all conformal Killing vectors first, then finding linear combination of these that are divergence free.
- In any space obtained from identifications, the Killing vectors can be found using basic group theory, namely,

$$\text{If } M^n = \bar{M}^n/G, \text{ then } Geo(M^n) = N_{Geo(\bar{M}^n)}(G)/G.$$

(see D. Witt J. Math Phys. 1986)

Weyl Curvature, C

- The Riemann curvature can be written as

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) - \frac{2}{(n-1)(n-2)}R(g_{a[c}g_{d]b})$$

where $R_{ab} = R_{acb}{}^{c}$ is the Ricci curvature and $R = R^{a}{}_{a}$ the Ricci scalar.

- C is conformally invariant and C = 0 iff the metric is locally conformally flat.

Homework 2

- Find the Killing vectors of the following metrics:
- Rindler
- Milne
- deSitter

Einstein's Equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \square g_{ab} = 8 \square T_{ab}$$

- The right side usually obeys energy conditions such as ANEC, weak, strong, or dominant.
- Notice that no sources imply only that the Ricci curvature vanishes; Weyl can be nonvanishing.
- Solutions do not need to obey any sort of evolution equations.

Evolution Equations and Constraints

- A Cauchy slice or surface is a spacelike hypersurface such that each causal curve, i.e. non-spacelike curve, intersects it once and only once.
- A globally hyperbolic spacetime is a spacetime with a Cauchy slice.
- A ***partial*** Cauchy slice is a spacelike hypersurface such that each causal curve intersects it ***at most*** once.
- The future domain of dependence, $D^+(S)$, is the set of all points in the spacetime such that every past directed past-inextendible casual curve intersects S.

Domain of Dependence

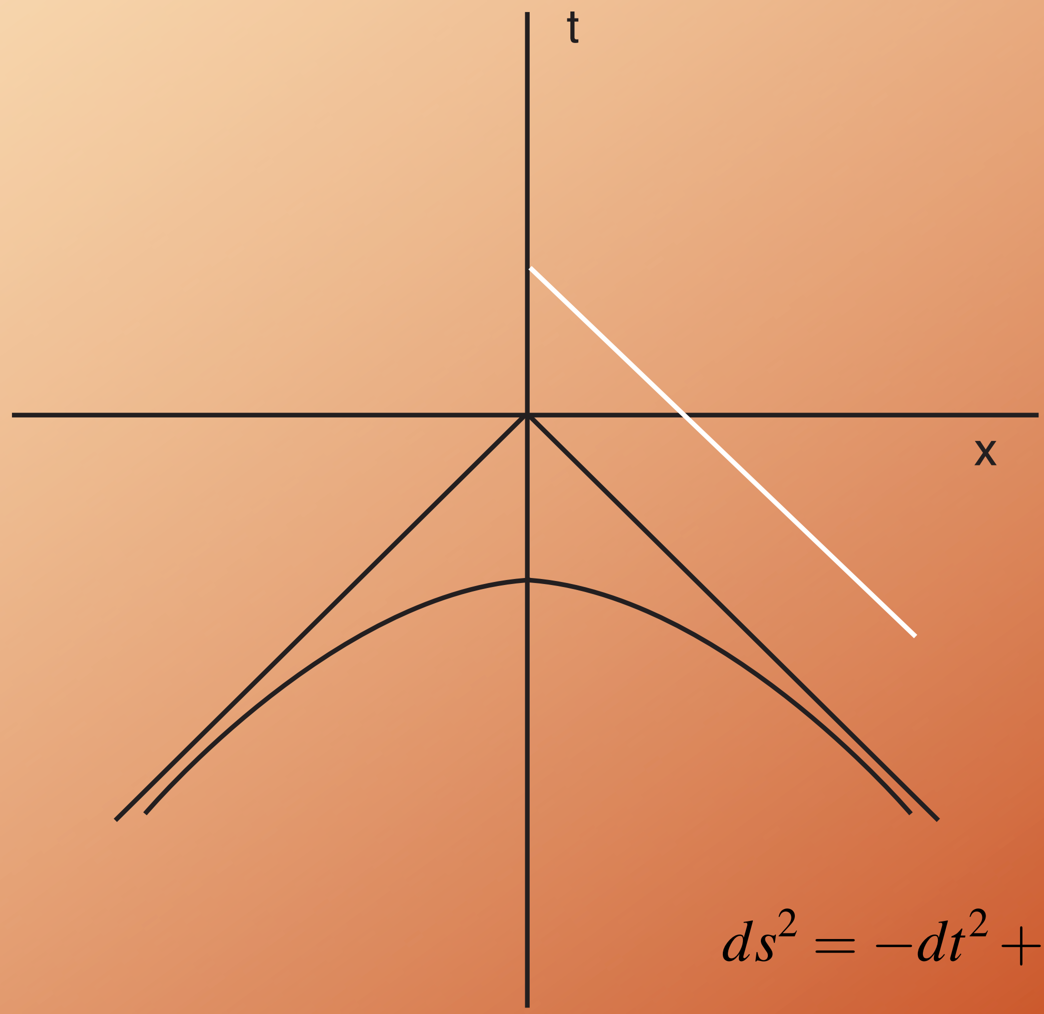
- The past domain of depence, $D^{-}(S)$, is the set of all points in the spacetime such that every future directed future-inextendible casual curve intersects S.
- The domain of dependence, $D(S) = D^{+}(S) \cup D^{-}(S)$.
- Theorem: A spacetime is globally hyperbolic iff $M=D(S)$.

Examples

- Minkowski space $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ is globally hyperbolic with Cauchy slice t=0 surface.

- Minkowski space with t periodically indentified is not globally hyperbolic.
- Example: Milne Universe - Take a mass hyperboloid in Minkowski space

Example (Milne Universe)



$$ds^2 = -dt^2 + a^2 t^2 (dH_{n-1}^2)$$

Constraints

The contraint equations on the Cauchy slice are

$$\bar{R} - p_{ab}p^{ab} + p^2 = 16\pi\rho,$$

and

$$D_b(p^{ab} - ph^{ab}) = 8\pi J^a\,.$$

Given any (n-1)-manifold with riemannian metric h and tensor p satisfying the constraints, one can evolve to obtain a spacetime satisfying Einstein's equations.

Examples of some useful results on the structure of the Cauchy surfaces.

- Theorem: Every smooth (n-1)-manifold is the Cauchy slice of a physically reasonable spacetime. (D. Witt PRL 1986)

- Theorem: A generic class of smooth (n-1)-manifolds are the Cauchy slices for locally deSitter spacetimes. A class of these is not constructible from identifying deSitter. These give counter examples to the cosmological principle. (J. Morrow-Jones & D. Witt)

Causal Structure

- The chronological future of point is

$$I^+(p) = \{q \in M | there\ is\ a\ future\ directed\ timelike\ curve$$
$$\gamma\ such\ that\ \gamma(1) = q\ and\ \gamma(0) = p\}$$

- The chronological future is an open set.

- The chronological future of a set A is

$$I^+(A) = \bigcup_{A \in M} I^+(p)$$

Causal structure

- The chronological past is defined in a similar way.
- The causal future $J^+(A)$ and past $J^-(A)$ are defined in a similar way using causal curves in place of timelike ones.

- Theorem:

  If $q \in J^+(p) - I^+(p)$, then any causal curve connecting p to q must be a null geodesic.

Cauchy Horizons

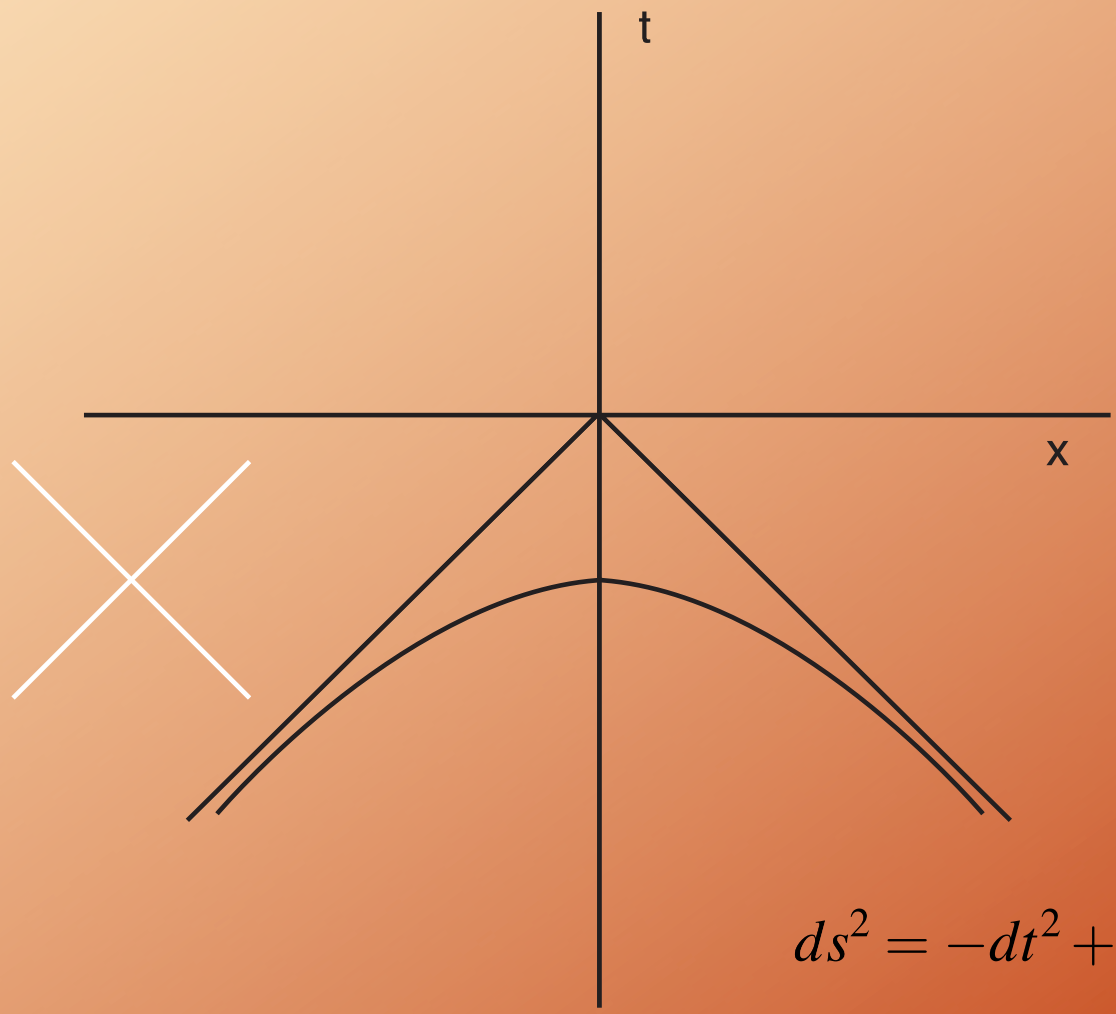
Definition: The future Cauchy Horizon of a achronal set is S is

$$H^+(S) = \overline{D^+(S)} - I^-[D^+(S)]$$

.

The definition for $H^-(S)$ is similar and $H(S) = H^+(S) \bigcup H^-(S)$.

Example (Milne Universe) Cauchy Horizon



$$ds^2 = -dt^2 + a^2 t^2 (dH_{n-1}^2)$$

The metric of Minkowski spacetime in spherical coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

r, t have infinite ranges - no way to draw where infinity is.

This is not a problem with studying causal structure; rather it is a problem with the choice of coordinates. Define t' and r'

$$2t = \tan(\frac{t'+r'}{2}) + \tan(\frac{t'-r'}{2}) \qquad 2r = \tan(\frac{t'+r'}{2}) - \tan(\frac{t'-r'}{2})$$

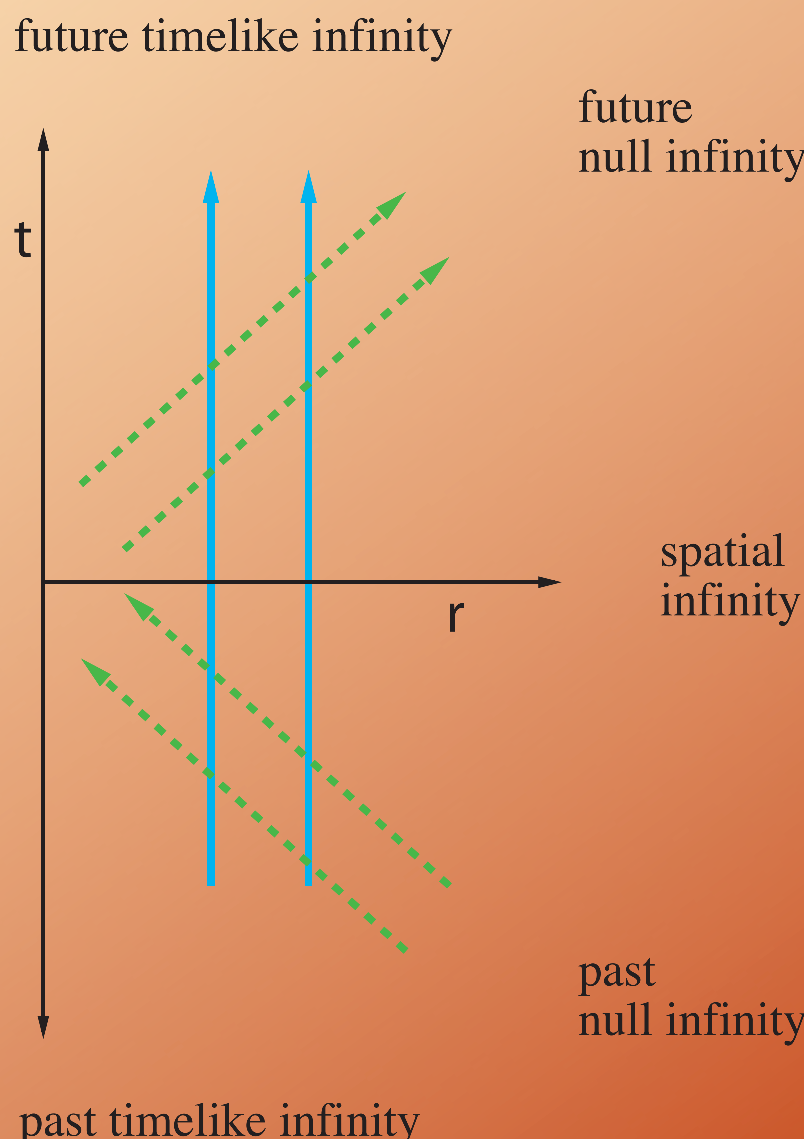
the Minkowski metric becomes

$$ds^2 = \Omega^2(t', r')\left(-dt'^2 + dr'^2 + r'^2(d\theta + \sin^2\theta d\phi^2)\right)$$

$$\Omega(t', r') = \frac{1}{2}\sec(\frac{t'+r'}{2})\sec(\frac{t'-r'}{2})$$

r', t' have finite ranges, $r' \geq 0$, $-\pi < t'+r' < \pi$, $-\pi < t'-r' < \pi$.

Minkowski spacetime in spherical coordinates



Two spacetimes related by a conformal transformation have the same causal structure. Thus $\square^2$ can be suppressed in a concrete illustration of the causal structure.

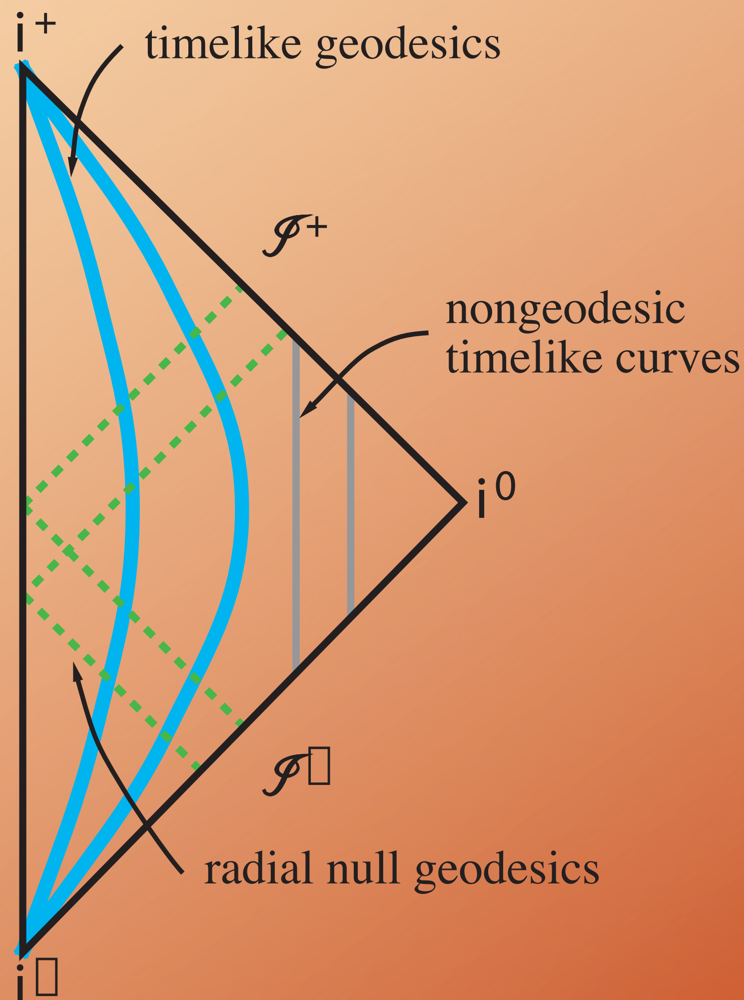
The metric

$$d\bar{s}^2 = \left(-dt'^2 + dr'^2 + r'^2(d\square + \sin^2\square d\square^2)\right)$$

with the given coordinate ranges is simply a region of flat spacetime (or the Einstein static universe).

The line segments $t' + r' = \square$, $t' - r' = -\square$ are not part of the original spacetime, but one can add these as a boundary, $\mathscr{I}$.

Penrose diagram of Minkowski spacetime



II: A Causal Structure and Field Theory

Some the ideas mentioned in the first lecture are now applied to example and the basics of field theory in curved spacetimes are considered.

Note in my lectures $c = G = 1$ and the Lorentz metric signature is $- + + \cdots +$

The infinite future and past are now clearly described; future infinities of null geodesics and timelike geodesics are distinct.

Observe that not all timelike curves end at i^+; curves corresponding to accelerated timelike observers can reach $\mathscr{I}^+$.

Radially directed photons travel along paths at 45 degree angles; any timelike or null curve leaving a point in this spacetime has tangent lying between the inward and outward directed radial null geodesics.

The Schwarzschild metric is

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{(1-\frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$r = 2M$ is a coordinate singularity; $r = 0$ is a curvature singularity.
The maximal extension (Kruskal (1960)):

$$v = \sqrt{r-2M}\exp\frac{(t+r)}{4M} \quad w = -\sqrt{r-2M}\exp-\frac{(t-r)}{4M}$$

The metric becomes

$$ds^2 = F^2(v,w)\,dv\,dw + r^2(d\square^2 + \sin^2\square\, d\square^2)$$

where

$$vw = -(r-2M)\exp(r/2M) \qquad F^2 = \frac{16M^2}{r} exp(-r/2M)$$

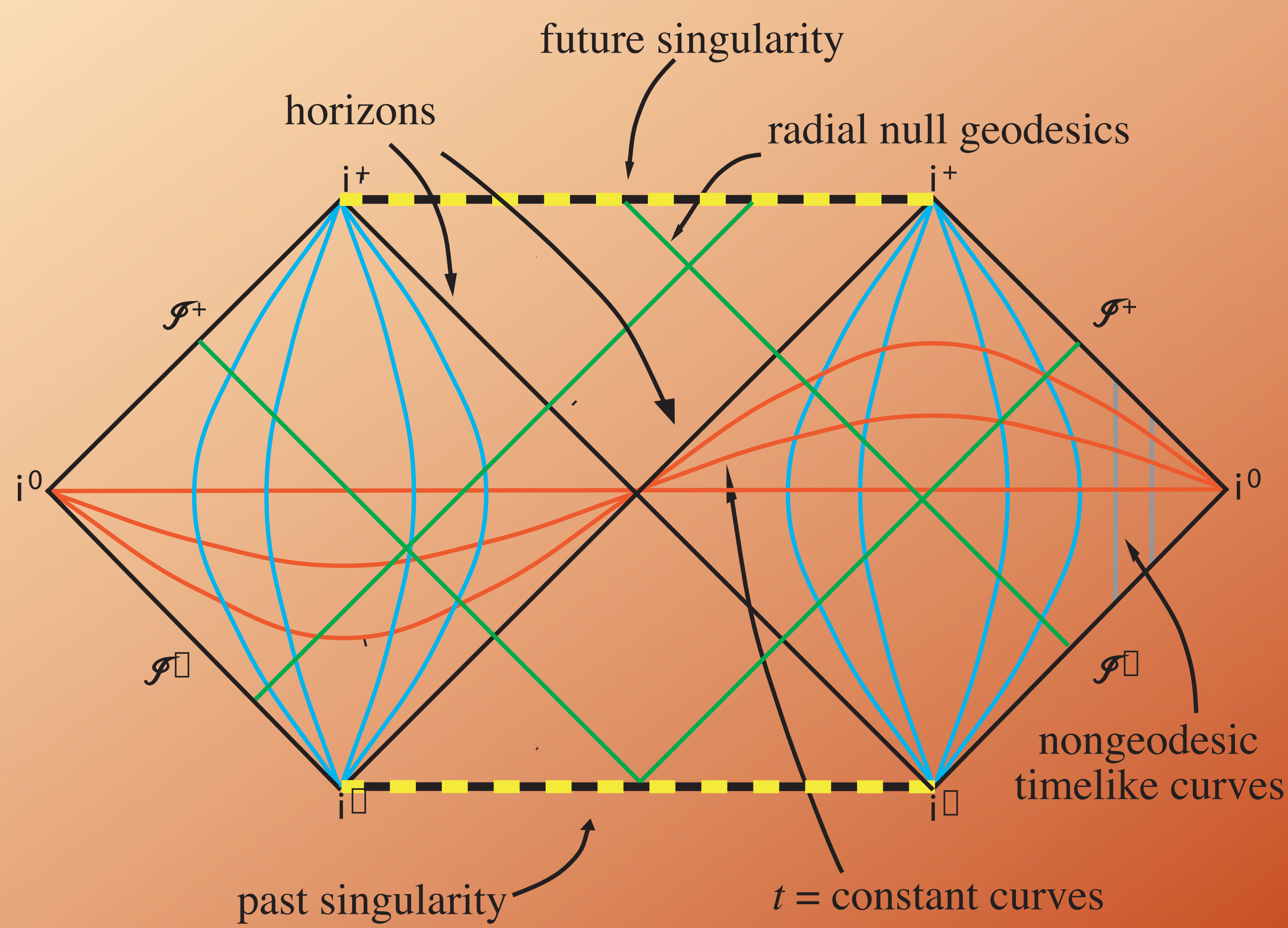
The metric is not singular at $w = 0$ or $v = 0$ ($r = 2M$) and the range of coordinates can be extended to v, w such that $vw < 2M$.

One can construct the Penrose diagram by transforming to coordinates

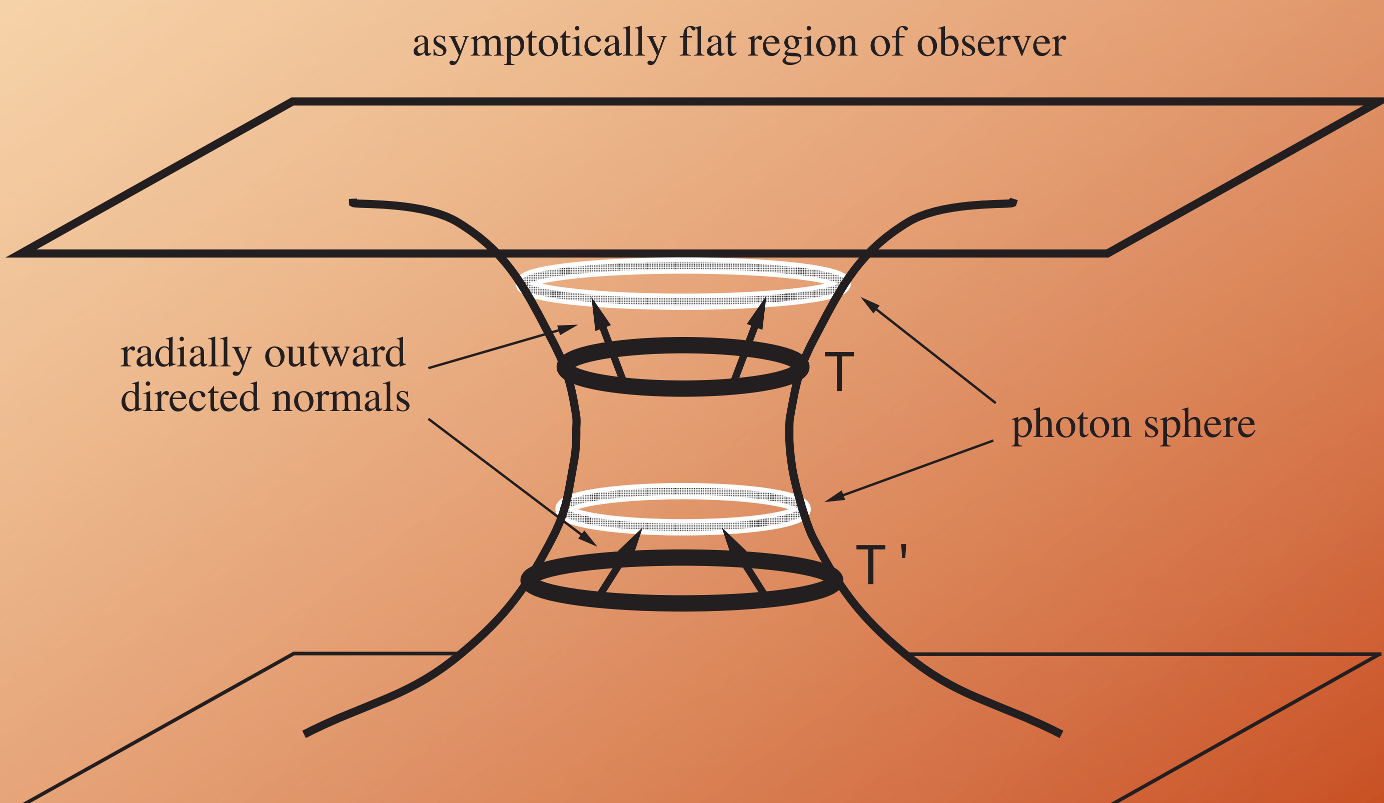
$$v' = \arctan\frac{v}{\sqrt{2M}} \qquad w' = \arctan\frac{w}{\sqrt{2M}}$$

with ranges $-\square < v' + w' < \square$, $-\square/2 < v' < \square/2$, $-\square/2 < w' < \square/2$

Schwarzschild spacetime



An outer trapped surface

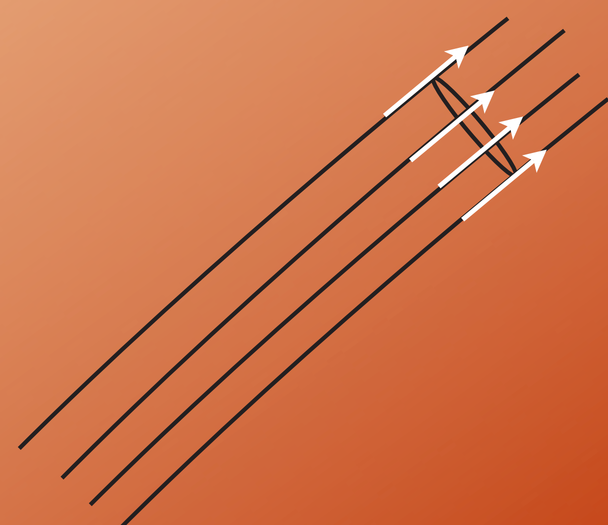


Expansion: θ

Let $B_{ab} = \nabla_a k_b$ where k_a is tangent to a congruence of null geodesics. Decompose B in the following form

$$\hat{B}_{ab} = \frac{1}{n-2}\theta\hat{h}_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab}$$

where the ˆ denote the restriction onto the $(n-2)$ subspace given by tensors orthogonal to k_a.

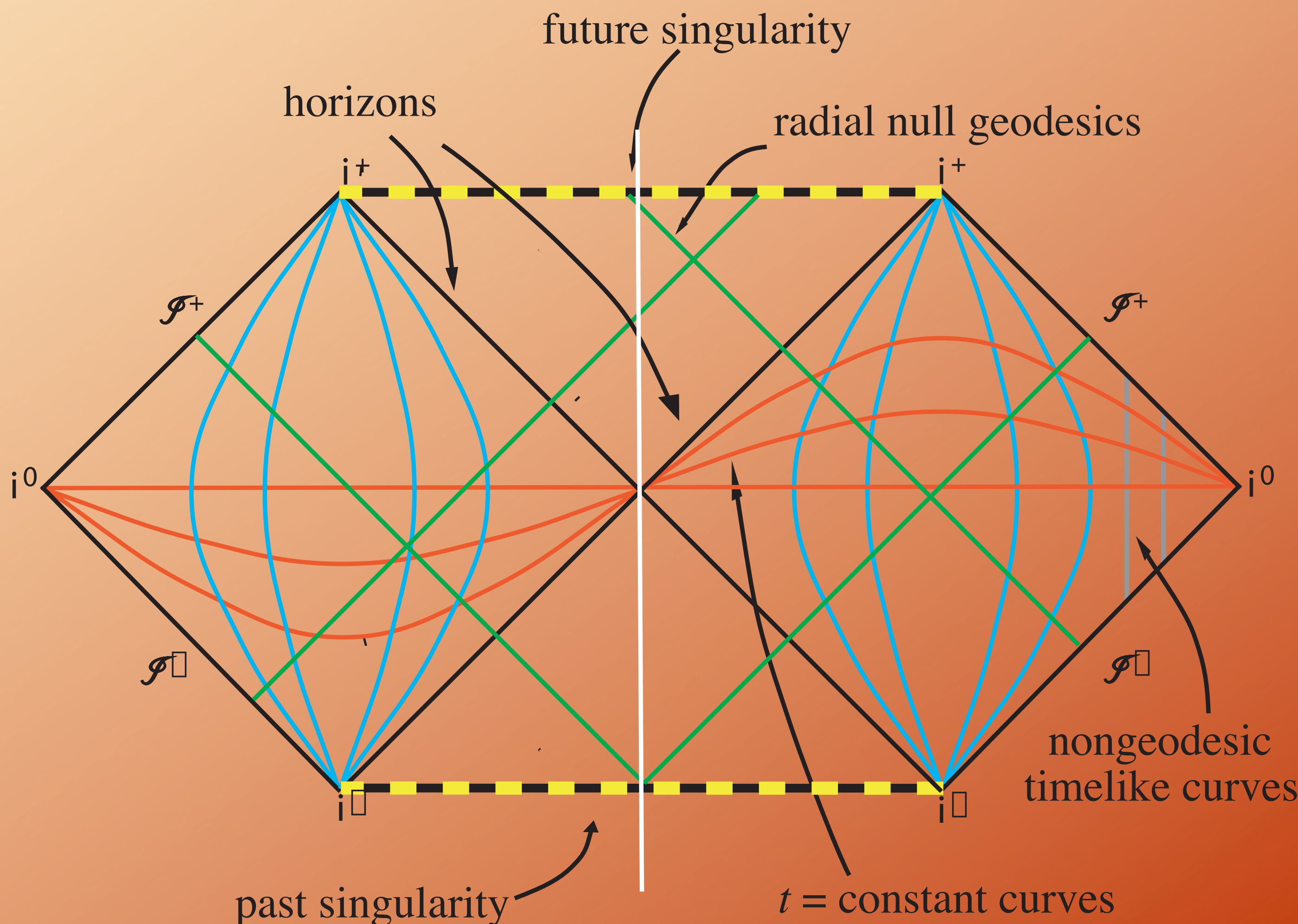


$$\theta = \hat{h}^{ab}\hat{B}_{ab}$$

$$\hat{\sigma}_{ab} = \hat{B}_{(ab)} - \frac{1}{n-2}\theta\hat{h}_{ab}$$

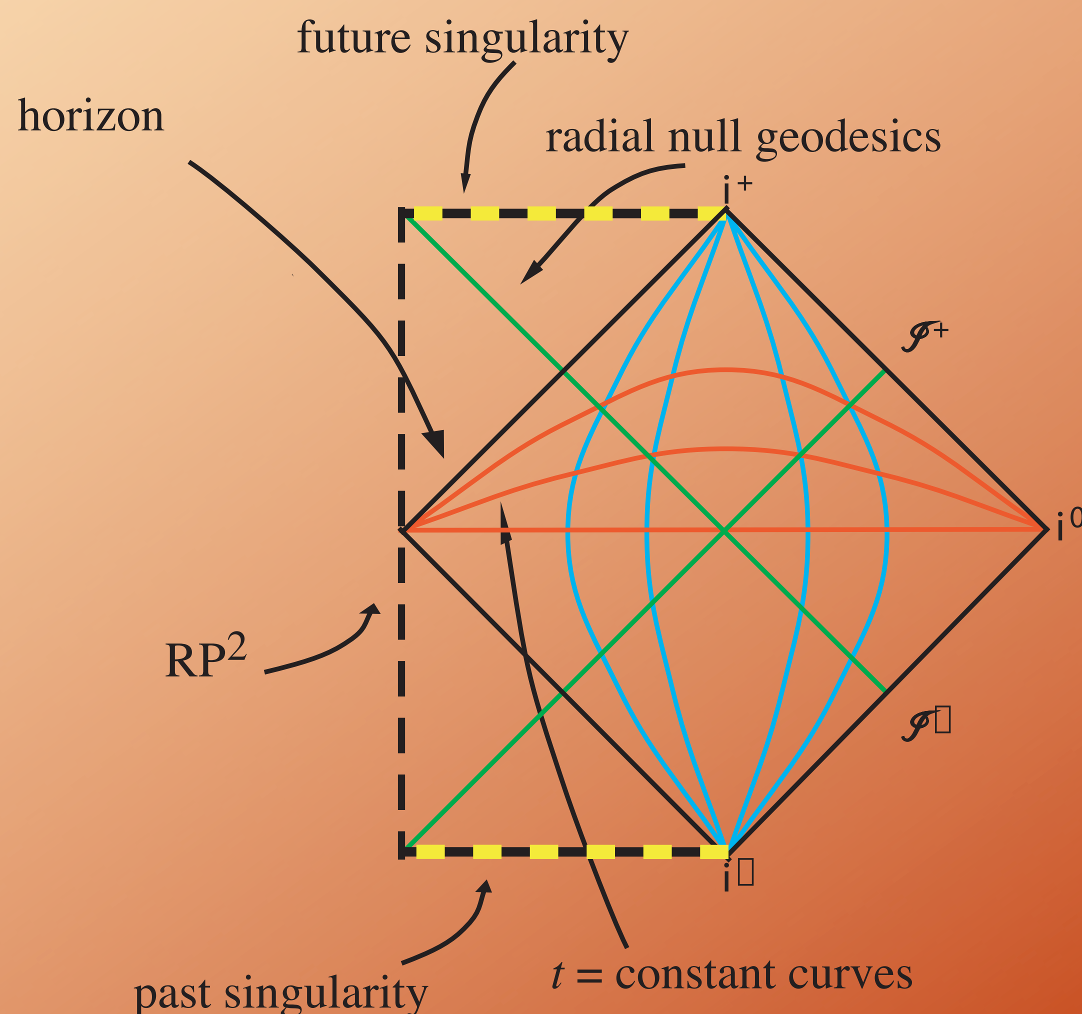
$$\hat{\omega}_{ab} = \hat{B}_{[ab]}$$

RP^3 geon



future singularity
horizons
radial null geodesics
i+
i+
i0
i0
nongeodesic
timelike curves
past singularity
t = constant curves

RP^3 Schwarzschild spacetime



Raychaudhuri's Equation

$$\frac{d\theta}{d\lambda} = \frac{-1}{n-2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b$$

A few important results:

- Singularity Theorems (1965 R. Penrose and S. Hawking)
- Area theorem for Blackholes (1971 S. Hawking)
- Topological Censorhip Conjecture and Theorem (1993 J. Friedman, K. Schleich, and D. Witt)

The metric for anti - de Sitter spacetime can be written

$$ds^2 = -\square^2 \cosh^2 \frac{r}{\square} dt^2 + dr^2 + \square^2 \sinh^2 \frac{r}{\square} (d\square^2 + \sin^2 \square d\square^2)$$

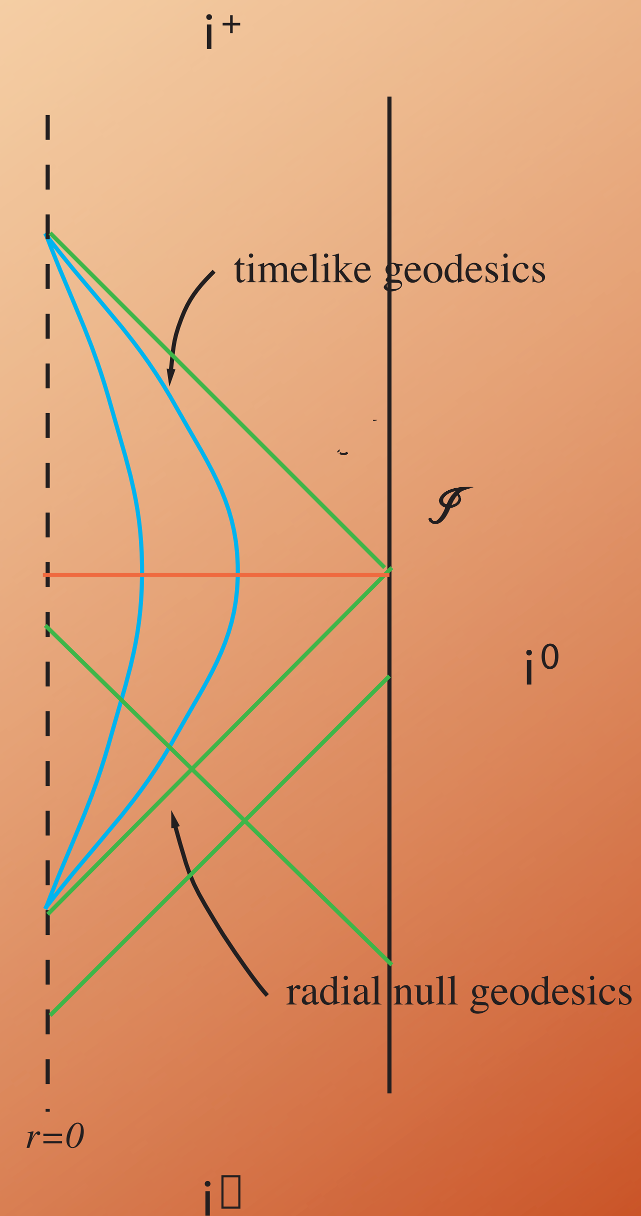
Constant curvature spacetime with $R < 0$. Note $\square^2 = -3/\square$ where $R_{ab} = \square g_{ab}$.

The metric for de Sitter spacetime can be written

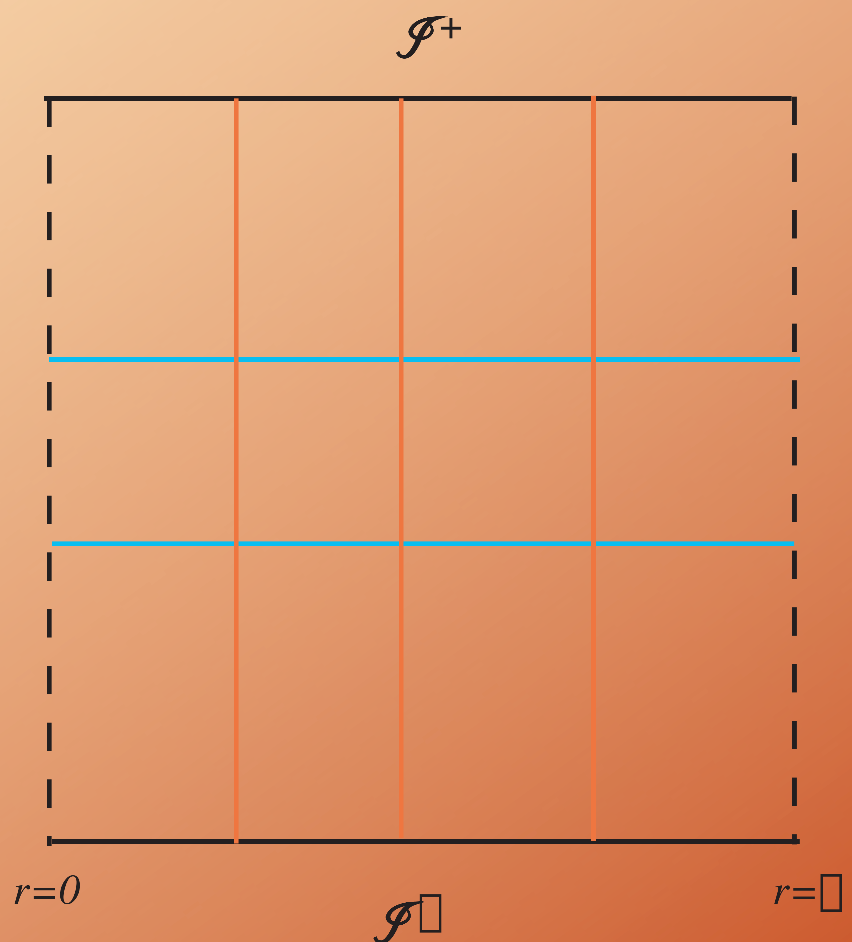
$$ds^2 = -dt^2 + \square^2 \cosh^2 \frac{t}{\square} (dr^2 + \sin^2 r(d\square^2 + \sin^2 \square d\square^2))$$

Constant curvature spacetime with $R > 0$. Note $\square^2 = 3/\square$ where $R_{ab} = \square g_{ab}$.

Anti - de Sitter Spacetime



de Sitter Spacetime



Note: there are other coordinatizations of (parts of) all of these spacetimes

Example: de Sitter spacetime

$$ds^2 = -dt^2 + \square^2 \exp\frac{2t}{\square}\,(dr^2 + r^2(d\square^2 + \sin^2\square\, d\square^2))$$

- flat $t =$ constant surfaces

$$ds^2 = -dt^2 + \square^2 \sinh^2\frac{t}{\square}\,(dr^2 + \sinh^2 r(d\square^2 + \sin^2\square\, d\square^2))$$

- hyperbolic $t =$ constant surfaces

$$ds^2 = -dt^2 + \square^2 \sinh^2\frac{t}{\square}\,dr^2 + \cosh^2\frac{t}{\square}\,(d\square^2 + \sin^2\square\, d\square^2)$$

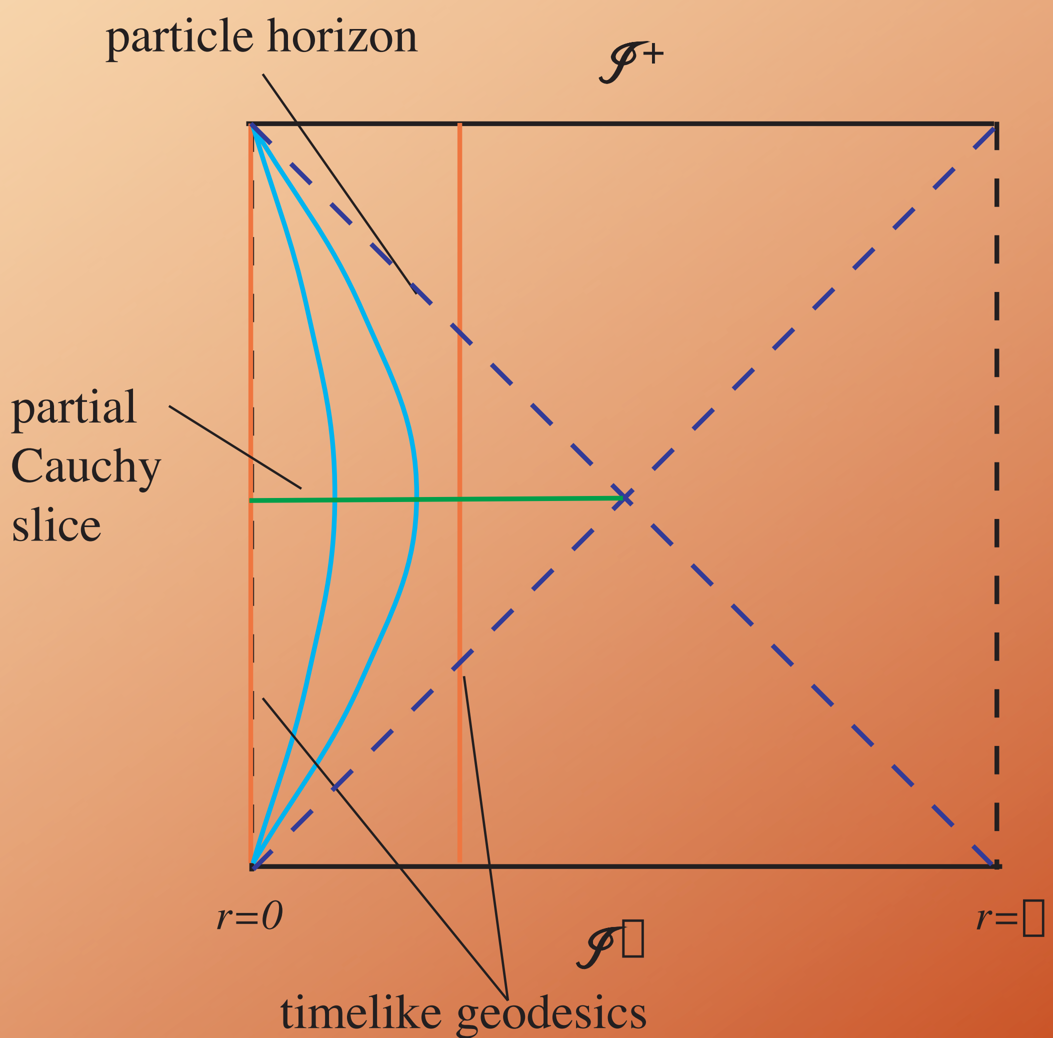
- $S^2 \times R$ $t =$ constant surfaces

$$ds^2 = -(1 - \square r^2)dt^2 + \frac{1}{(1 - \square r^2)}dr^2 + r^2(d\square^2 + \sin^2\square\, d\square^2)$$

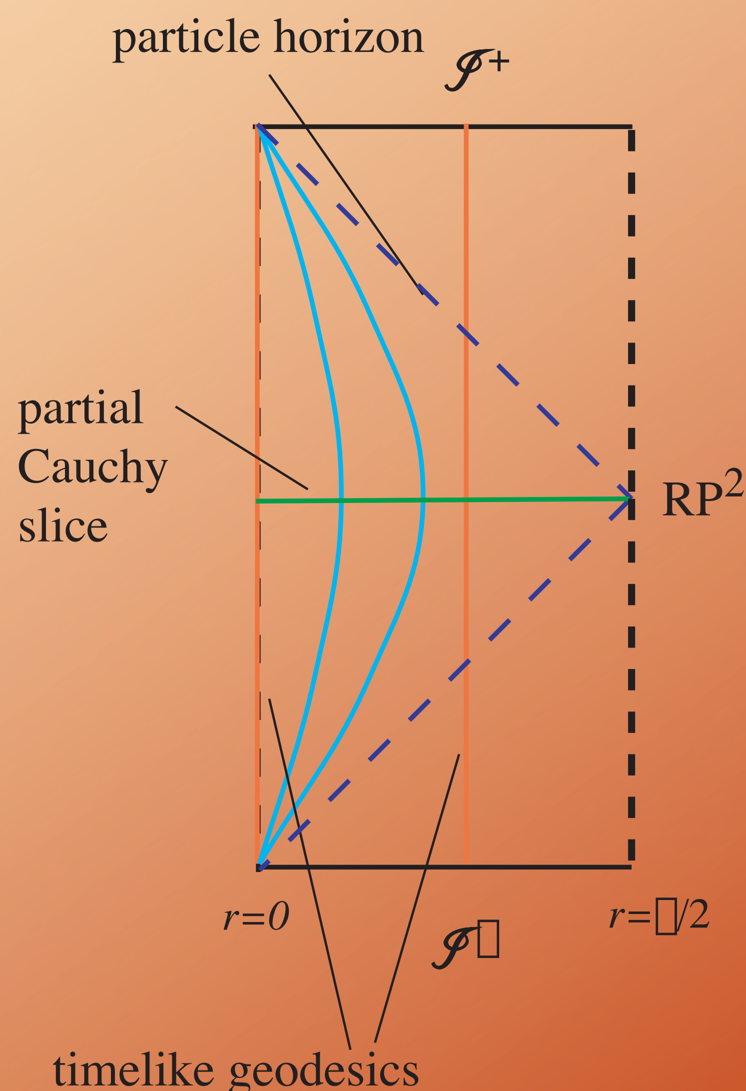
- static coordinates

Why? All these coordinate systems only cover part of the de Sitter spacetime

Static de Sitter coordinates



RP^3 de Sitter Spacetime



Minkowski spacetime

$$ds^2 = -dt^2 + t^2(dr^2 + \sinh^2 r(d\square^2 + \sin^2\square d\square^2))$$

- hyperbolic $t =$ constant slices; coordinate singularity at $t = 0$. This coordinate singularity implies geodesic incompleteness if points on the hyperboloid are identified under a discrete isometry.

$$ds^2 = -x^2dt^2 + dx^2 + dy^2 + dz^2$$

- also known as Rindler spacetime. coordinate singularity at $x = 0$.

Penrose Compactification

A spacetime, M, is AF if it can be conformally included in $M' = M \cup \mathsf{I}$. Furthermore, there is a $\Omega > 0$ such that $g' = \Omega^2\, g$ on M and $\Omega = 0$ and $d\Omega$ is pointwise non-vanishing on I.

Blackholes

- A blackhole is given by

$$B = M - J^{-}(I^{+})$$

- The event horizon is the boundary of B
- Whiteholes are similarly defined.

Field Theory

Given a field □ it is quantized by treating it as an operator obeying the the equal time commutation relations

$$[\square(t,x),\square(t,x')] = 0$$

$$[\square(t,x),\square(t,x')] = 0$$

$$[\square(t,x),\square(t,x')] = i\square^{n-1}(x-x')$$

The fields are quantized by expanding the fields

$$\square(t,x) = \square_{k}\left[a_k u(t,x) + a_k^{\dagger} u^{*}(t,x)\right]$$

The equal time commutation relations for □ and □ are equivalent to the following relations

$$[a_k, a_{k'}] = 0$$

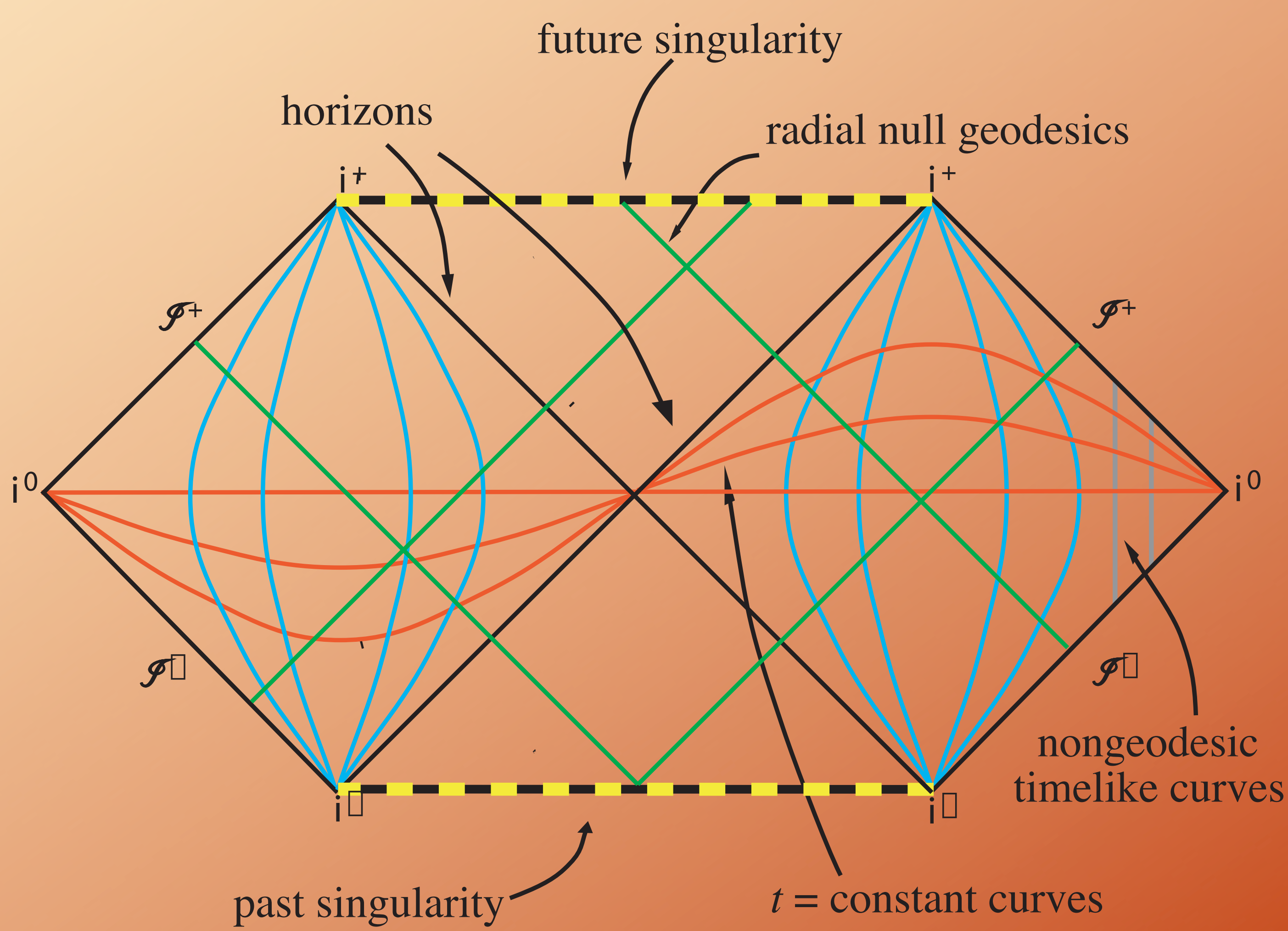
$$[a_k^\dagger, a_{k'}^\dagger] = 0$$

$$[a_k, a_{k'}^\dagger] = □_{kk'}$$

$$a_k|0\rangle = 0 \ \ \forall k$$

The state $|0\rangle = 0$ is the vacuum. The operator $a_k^\dagger$ acting on the vacuum gives the particle state $|1_k\rangle = a_k^\dagger|0\rangle$

Field Theory in Curved Spacetime



Curved Spacetime modes

Defining the operators in Minkowski space used the mode expansion

$$\phi(t,x) = \sum_k \left[a_k u(t,x) + a_k^\dagger u^*(t,x)\right] .$$

The modes are not unique in curved spacetime. This should not be a surprise because this true even in quantum mechanics!

Pick a new set of modes

$$\phi(t,x) = \sum_k \left[\bar{a}_k \bar{u}(t,x) + \bar{a}_k^\dagger \bar{u}^*(t,x)\right] .$$

Bogolubov Transformations

A new vacuum is given by

$$\bar{a}_k|\bar{0}\rangle = 0 \;\; \forall k \; .$$

The new modes can be written in terms of the old and the old in terms of the new

$$\bar{u}_j = \underset{i}{\square} \left[\square_{ji} u_i + \square_{ji} u_i^* \right] ,$$

and

$$u_i = \underset{j}{\square} \left[\square_{ji}^* \bar{u}_i - \square_{ji} \bar{u}_i^* \right] .$$

Expressions for operators

The operators can be written in terms of the Bogolubov transformations

$$\bar{a}_j = \sum_i \left[\square^*{}_{ji} a_i - \square^*{}_{ji} a_i^\dagger\right] ,$$

and

$$a_i = \sum_j \left[\square_{ji} \bar{a}_i + \square^*{}_{ji} \bar{a}_i^\dagger\right] .$$

Particle Creation

If $\square_{ij} \neq 0$ means that

$$a_i|\bar{0}\rangle = \square_j \square^*{}_{ji}|\bar{1}_j\rangle \neq 0 \ .$$

Furthermore,

$$\langle\bar{0}|N_i|\bar{0}\rangle = \square_j |\square_{ji}|^2$$

where $N_i = a^\dagger{}_i a_i$.

Timelike Killing vector implies unique vacuum

- If there is a timelike Killing ξ and u_i is set of positive frequency modes satisfying $L_\xi\ u_j = -i\ \omega\ u_j$ with positive frequency.
- Given another set of modes w_i which is the sum positive frequency u_i modes.
- Then both modes have a common vacuum!

More Homework

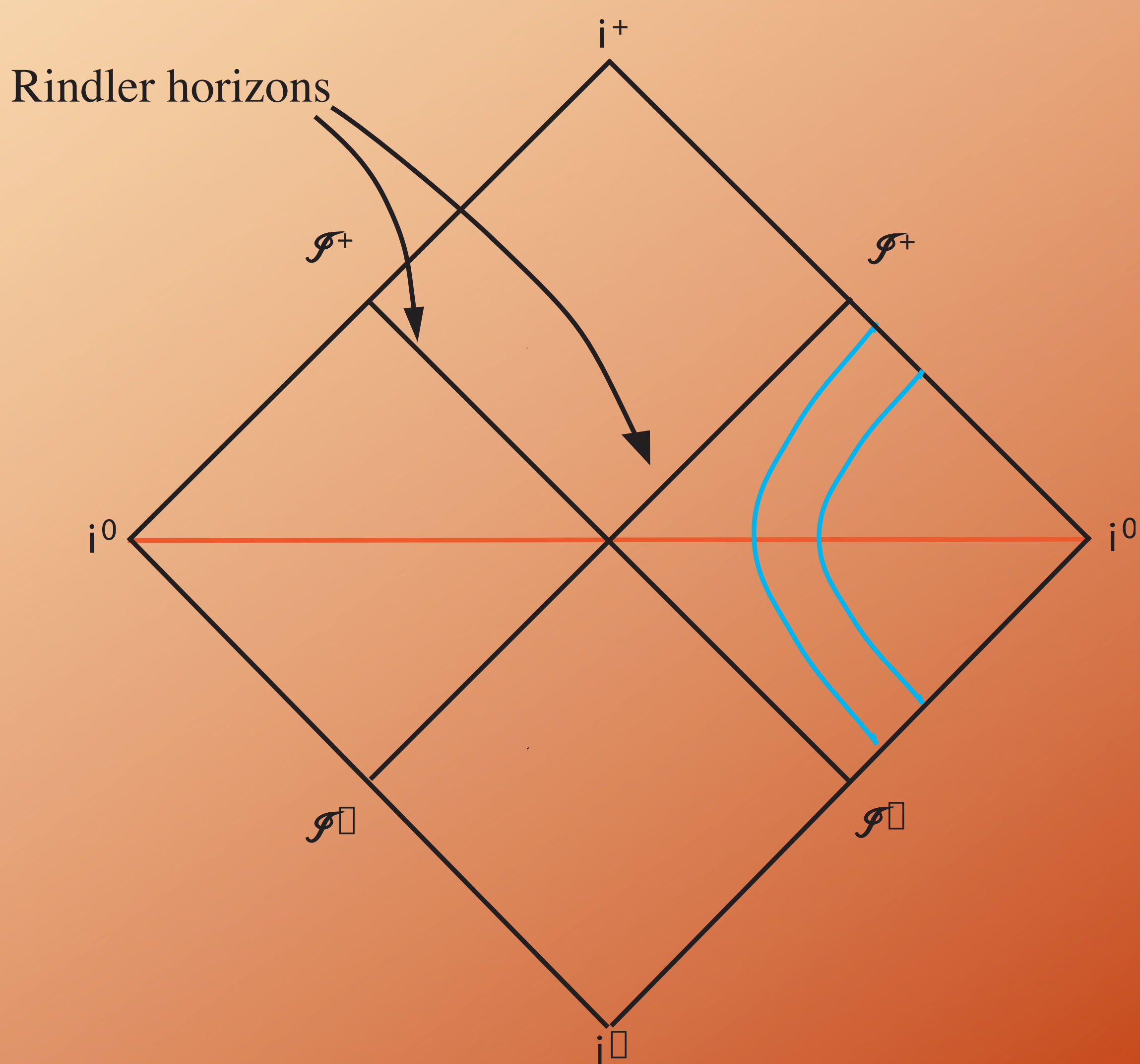
- Homework 3: Study the causal structure of the Rindler spacetime and work out the quantization of a scalar field.
- Homework 4: Does a point charge radiate in a uniform gravitational field?

III: Examples of Field Theory in Curved Spacetimes

Examples mention in lecture 2 are worked out.

Note in my lectures $c = G = 1$ and the Lorentz metric signature is $- + + \cdots +$

Rindler spacetime



Definitions

Scalar product

$$(\phi_1,\phi_2) = -i\int_{\Sigma}(\phi_1\,\partial_\mu\phi_2^* - \partial_\mu\phi_1\,\phi_2^*)\,d\Sigma^\mu$$

where $d\Sigma^\mu$ is the volume element of the Cauchy surface with normal n^μ.

Complete set of states: $u_k(x)$ satisfying

$$(-\nabla^2 + m^2 + \xi R(x))u_k(x) = 0$$

such that

$$(u_k,u_l) = \delta_{kl} \quad (u_k^*,u_l^*) = -\delta_{kl} \quad (u_k,u_l^*) = 0$$

Two dimensional Rindler Spacetime

The metric is

$$ds^2 = -x^2 dt^2 + dx^2 = \exp 2\square(-dt^2 + d\square^2)$$

where

$$\square = \int \frac{dx}{x} = \ln x$$

Note: the range in $(t, \square)$ coordinates is infinite, $- \quad < t < \quad$,
$- \quad < \square <$

However a complete set on Minkowski spacetime (different from the usual) is given by use of complete sets on both left and right wedges:

$$\square' = \sum_{k=-} {}^{R}b_k^{R}u_k + {}^{R}b_k^{\dagger R}u_k^* + {}^{L}b_k^{L}u_k + {}^{L}b_k^{\dagger L}u_k^*$$

where $\square'$ is a field on the full Minkowski spacetime

$$^{L}u_k = \frac{1}{\sqrt{4\square\square}}\exp(ik\square + i\square t)$$

and modes are extended to the whole Cauchy slice by taking them to vanish on the partner wedge.

The massless scalar field

$$-\square^2\square(\square,t) = \exp -2\square(\partial_t^2 - \partial_\square^2)\square(\square,t) = 0.$$

Conformally invariant in 2-d.

Complete set of states:

$${}^{R}u_k = \frac{1}{\sqrt{4\square\square}}\exp(ik\square - i\square t)$$

positive frequency with respect to timelike K.V. $t^\square = \partial_t$.

$$\square = \sum_{k=-} {}^{R}b_k^{R}u_k + {}^{R}b_k^{\dagger R}u_k^*$$

$${}^{R}b_k|0_R >= 0$$

vacuum state in the right Rindler wedge.

Complete set of states only on right wedge - incomplete on full Minkowski spacetime.

Also can expand in the usual complete set:

$$\square' = \sum_{k=-}^{} a_k \frac{a_k}{\sqrt{4\square\square}} \exp(ikx - i\square t) + a_k^\dagger \frac{a_k^\dagger}{\sqrt{4\square\square}} \exp(-ikx + i\square t)$$

$$a_k|0_M >= 0$$

Can express one set of operators in terms of the other by orthogonality and

$$^R b_k = (\square', ^R u_k) = \sum_{k'} (\square_{kk'} a_{k'} + \square_{kk'} a_{k'}^\dagger)$$

$$\square_{kk'} = (\frac{1}{\sqrt{4\square\square}} \exp(ik'x - i\square' t), ^R u_k)$$

$$\square_{kk'} = (\frac{1}{\sqrt{4\square\square}} \exp(-ik'x + i\square' t), ^R u_k)$$

Rindler observer travelling $\square =$ constant line in right wedge has number operator

$$N_k = {}^R b_k^{\dagger R} b_k$$

If travelling in the Minkowski vacuum,

$$< 0_M | N_k | 0_M > = \sum_{k'} | \square_{kk'} |^2 = \frac{1}{(\exp 2 \square\square - 1)}$$

Planck distribution for radiation at $T = 1/2\square$

Units - taking acceleration a, $T = a/2_B$

Example

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2$$

where $0 < t <$. Change coordinates to $t \rightarrow t' = \frac{1}{t}$ and our new metric is

$$ds^2 = -dt'^2 + dx^2$$

In the new coordinates, $t' =$ is the part in the original coordinates which seemed singular. The new coordinates still leave the spacetime singular!

Example (cont’ d)

Just extend spacetime to negative time.

Rindler in different approach

$$ds^2 = -x^2 dt^2 + dx^2$$

where $- \quad < t < \quad$ and $0 < x < \quad$. Now, take a null geodesic in the above spacetime

$$0 = g_{ab} k^a k^b = -x^2 \dot{t}^2 + \dot{x}^2$$

which implies

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2} \, .$$

Finally, integrate to obtain $t = \pm \ln x + C$ and use new coordinates given by

$$u = t - \ln x$$

and

$$v = t + \ln x$$

Rindler (cont' d)

$$ds^2 = -e^{v-u} du dv$$

where $u = t - \ln x$ and $v = t + \ln x$. The coordinate ranges are $- \quad < u < \quad$ and $- \quad < v < \quad$. This still the region for $x > 0$.

To extend to $x < 0$, we define new coordinates U and V. Do this by using the Killing vector t^a,

$$E = -g_{ab} k^a t^a = x^2 \frac{dt}{d\square}$$

is a constant of motion.

Global Coordinates

$$E = -g_{ab}k^a t^a = x^2 \frac{dt}{d\square}$$

is a constant of motion.

$$\square = \frac{1}{2E}\int e^{v-u}dv = C + \frac{e^{-u}}{2E}e^{v}$$

Now,

$$\square_{outgoing} = e^{v}$$

and

$$\square_{ingoing} = -e^{-u} \ .$$

So our space is singular again because the range of $\square$.

Finally, let

$$U = -e^{-u}$$

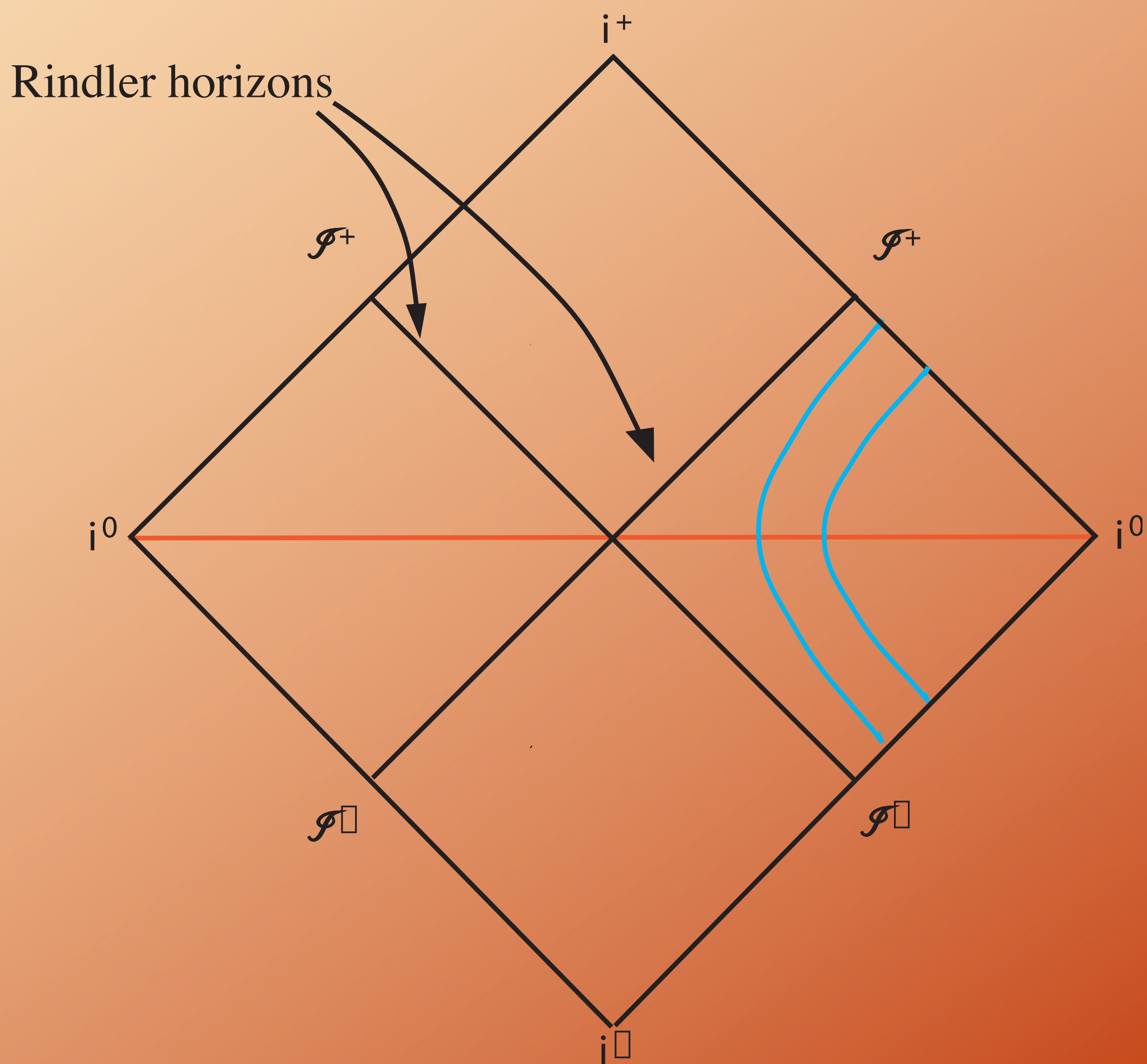
and

$$V = e^{v}$$

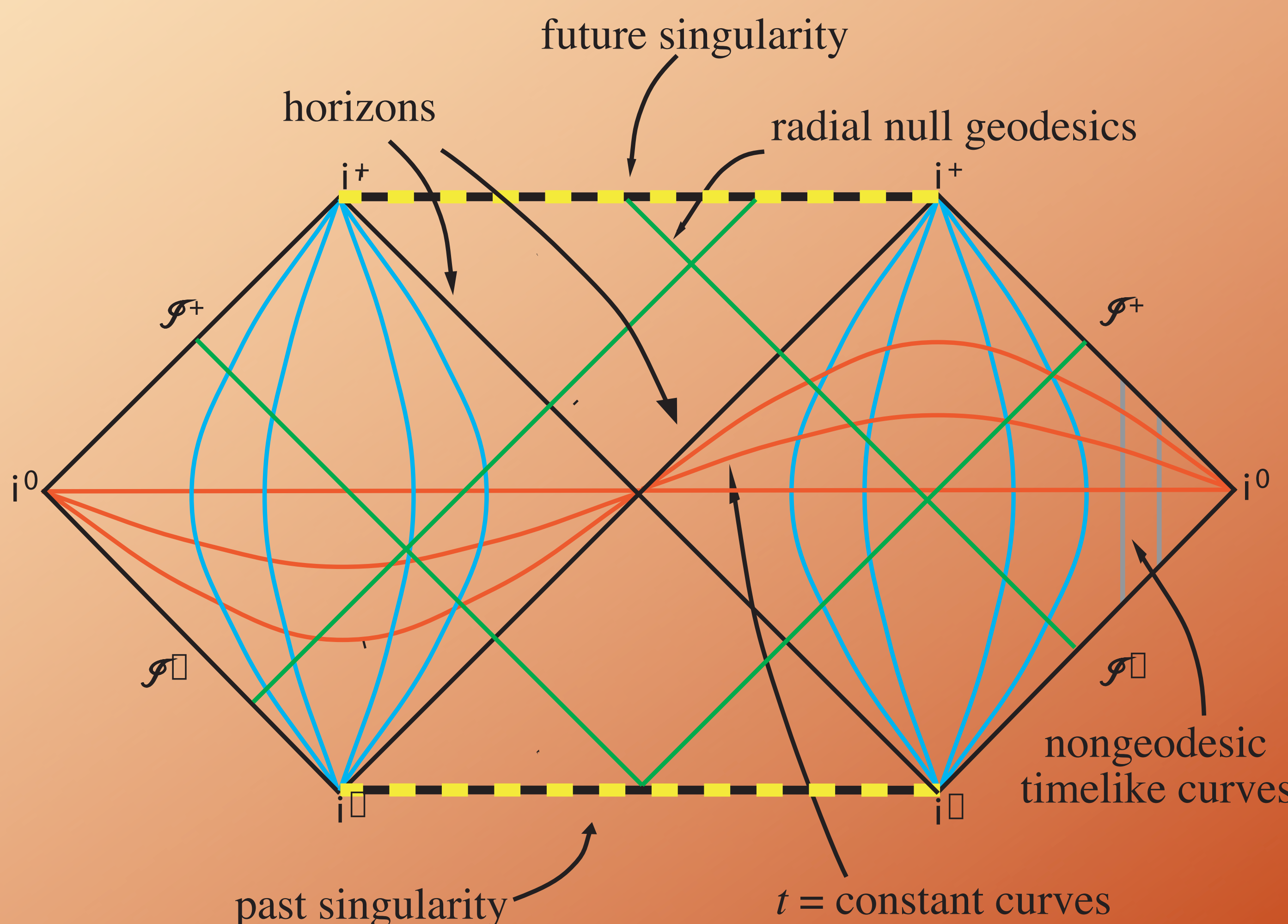
Thus,

$$ds^2 = -dU\,dV$$

Rindler spacetime



Schwarzschild spacetime



The Schwarzschild metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{(1 - \frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Eddington- Finkelstein coordinates: introduce

$$r^* = \int \frac{dr}{1 - \frac{2M}{r}} = r + 2M\ln(r - 2M)$$

Note $r \to 2M$ corresponds to $r^* \to -\infty$.

Define

$$v = t + r^* \quad u = t - r^*$$

Then

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

- no longer singular at $r = 2M$

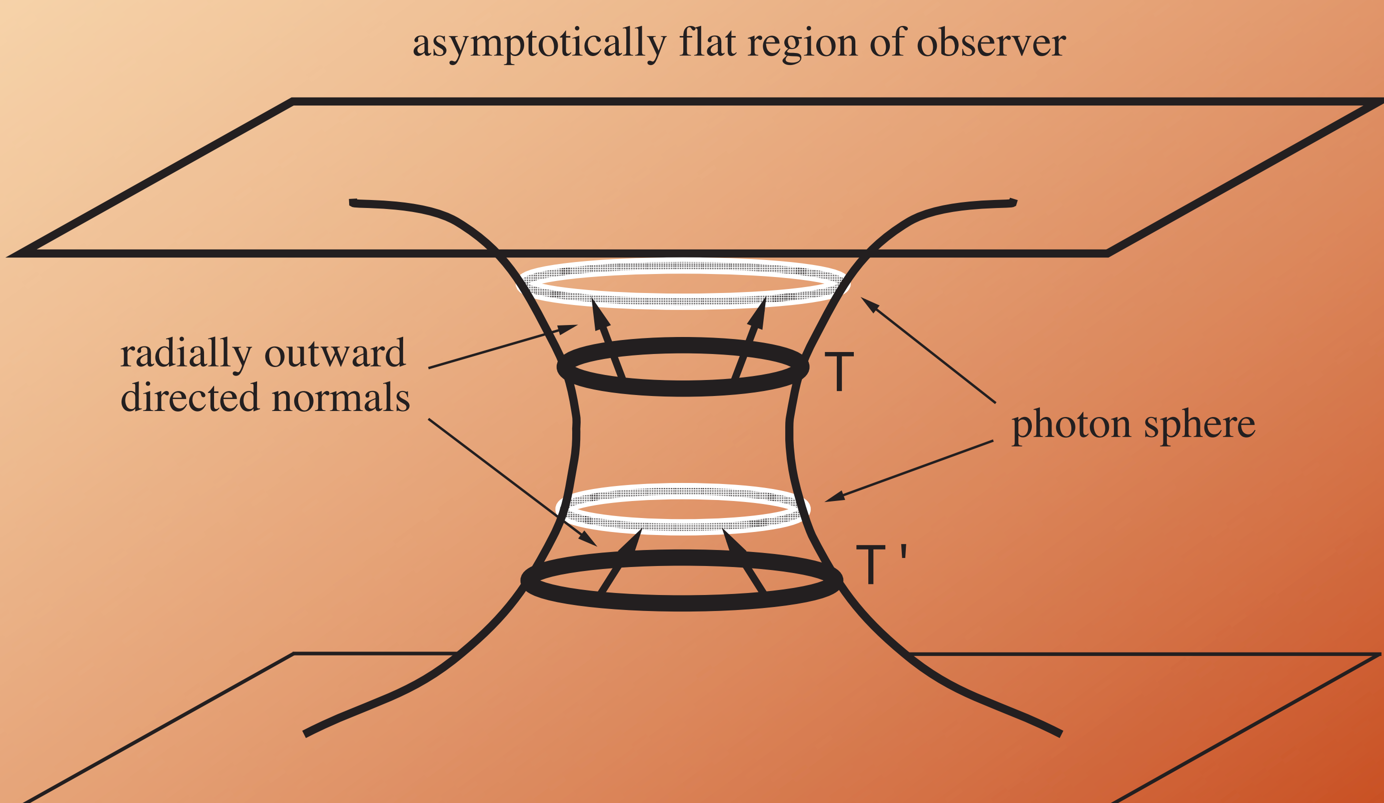
Blackhole Area Theorem

- The area of a blackhole always increase or remanins constant to the future.

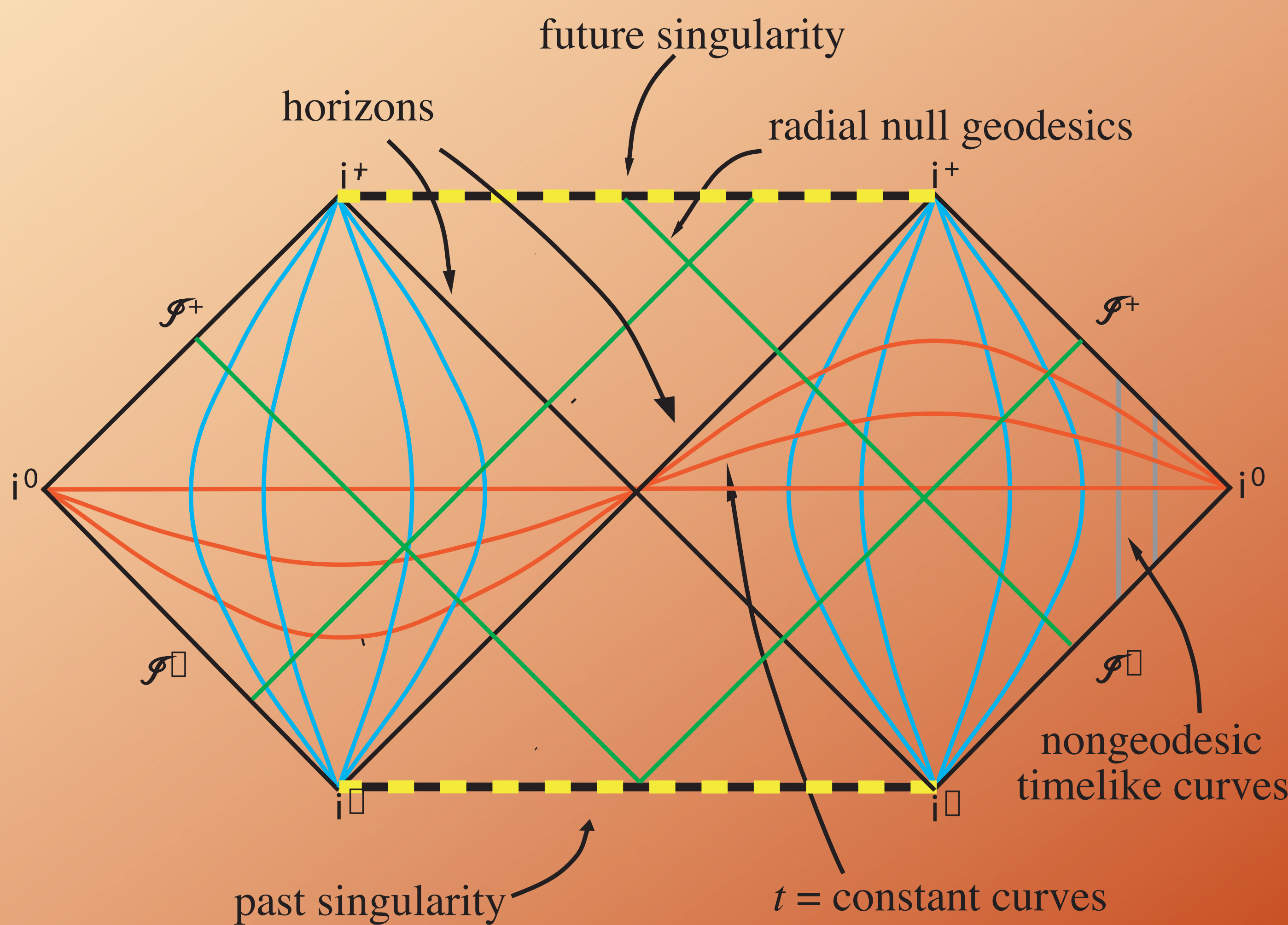
- What does this mean in terms of causal structure?
- How is this related to field theory?

Theorem: Let (M^n, g_{ab}) be a spacetime which is ”globally hyperbolic” and has a well defined notion of Blackhole. Let $\square_1$ and $\square_2$ be two Cauchy slices with $\square_2 \subset I^+(\square_1)$. If H is the horizon, then the area of $B_2 = H \cap \square_2$ is greater than or equal to the area of $B_1 = H \cap \square_1$.

An outer trapped surface



Schwarzschild spacetime

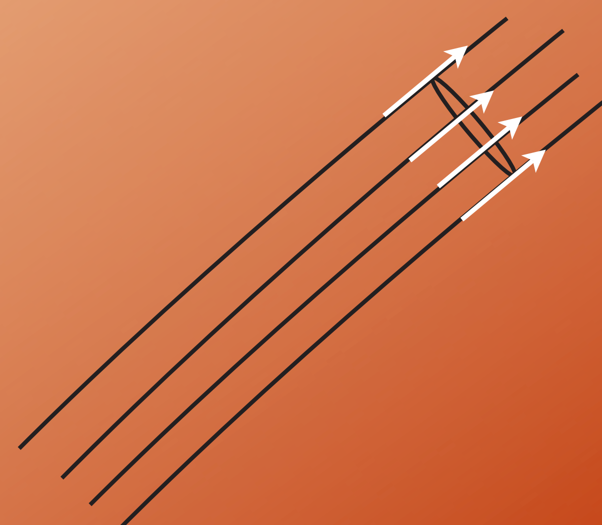


Expansion: θ

Let $B_{ab} = \nabla_a k_b$ where k_a is tangent to a congruence of null geodesics. Decompose B in the following form

$$\hat{B}_{ab} = \frac{1}{n-2}\theta \hat{h}_{ab} + \hat{\sigma}_{ab} + \hat{\omega}_{ab}$$

where the ˆ denote the restriction onto the $(n-2)$ subspace given by tensors orthogonal to k_a.



$$\theta = \hat{h}^{ab}\hat{B}_{ab}$$

$$\hat{\sigma}_{ab} = \hat{B}_{(ab)} - \frac{1}{n-2}\theta \hat{h}_{ab}$$

$$\hat{\omega}_{ab} = \hat{B}_{[ab]}$$

Raychaudhuri's Equation

$$\frac{d\theta}{d\lambda} = \frac{-1}{n-2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}k^a k^b$$

Now, for our surfaces one can take ω can be taken to be zero. If a standard energy condition is obeyed, then the rate of change of θ is negative.

So to prove area always increases or remains the same we need to show that θ is non-negative for geodesic run to the future from our first surface to our second. Assume this is not true then θ and its derivative are negative so the geodesics have a conjugate point. This a contradiction.

Causal Patch

- The DOC or domain of outer communication which is defined to be

$$DOC = I^{+}(I) \cap I^{-}(I)$$

- Theorem: The topology of the DOC is determined by $\boldsymbol{I}$.

IV: Euclidean field theory and examples

Examples are worked out for Euclidean field theory.

Note in my lectures $c = G = 1$ and the Lorentz metric signature is $- + + \cdots +$

Euclidean Field Theory

The Feynman propagator, $\square(x, y)$ is defined by

$$i\square(x, y) = \frac{\langle 0_{out}|T(\hat{\square}(x)\hat{\square}(y))|0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle}$$

where
$T(\hat{\square}(x)\hat{\square}(y)) = \hat{\square}(x)\hat{\square}(y)$ if $y \notin J^{+}(x)$ and $\hat{\square}(y)\hat{\square}(x)$ if $x \notin J^{+}(y)$.

This is the Green function for the wave operator.

Euclidean Green Function

In the case of a scalar field one finds that

$$\square(x,y) = -iG_E(x,y)$$

where $G_E(x,y)$ is the Euclidean Green function.

One advantage of the Euclidean approach is the wave operator is ellpitic which means unique inverse. Everything done so far is for pure states, namely, the vacuum.

Density Matrix

For any ordinary quantum system with a time independent Hamiltonian, H. The state of a thermal equilibrium system at temperature T where $\beta = (kT)^{-1}$ is defined via the density matrix

$$\rho = \frac{e^{-\beta H}}{Z}$$

where

$$Z = tr\left(e^{-\beta H}\right)$$

Thermal Feynman Propagator

The Thermal Feynman Propagator at temperature kT is given by

$$i\Delta_T(x,y) = tr\left[\rho T\left(\hat{\phi}(x)\hat{\phi}(y)\right)\right] = Z^{-1} tr\left[e^{-\beta H} T\left(\hat{\phi}(x)\hat{\phi}(y)\right)\right]$$

The thermal Feynman Propagator is periodic is imaginary time.

Now, continue the time coordinate $t = -i\square$ so that the above equations still hold. Take x and x' to have the same spatial coordinates but the time differs by $\square' + \square\hbar$

$$i\square_T(x',y) = Z^{-1} tr\left[e^{-\square H} T(\hat{\square}(x')\hat{\square}(y))\right] = Z^{-1} tr\left[e^{-\square H}(\hat{\square}(x')\hat{\square}(y))\right] = Z^{-1} tr\left[e^{-\square H}(\hat{\square}(x')\hat{\square}(y))\right]$$

$$i\square_T(x',y) = Z^{-1} tr\left[e^{-\square H}(\hat{\square}(x')\hat{\square}(y))\right] = Z^{-1} tr\left[e^{-\square H}(e^{\square H}\hat{\square}(x)e^{-\square H}\hat{\square}(y))\right]$$

$$i\square_T(x',y) = Z^{-1} tr\left[\hat{\square}(x)e^{-\square H}\hat{\square}(y))\right] = Z^{-1} tr\left[e^{-\square H}(\hat{\square}(y)\hat{\square}(x))\right] Z^{-1} tr\left[e^{-\square H} T(\hat{\square}(y)\hat{\square}(x))\right]$$

Finally,

$$i\square_T(x',y) = i\square_T(x,y)$$

Example of Schwarzschild

The Schwarzschild metric is

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{(1-\frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Now, take t to $i\tau$, this yields the metric

$$ds^2{}_E = (1-\frac{2M}{r})d\tau^2 + \frac{1}{(1-\frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Change coordinates using

$$R = 4M(1-\frac{2M}{r})^{\frac{1}{2}}$$

to obtain

$$ds^2{}_E = R^2 d\left(\frac{\tau}{4M}\right)^2 + \left(\frac{r}{2M}\right)^4 dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The Riemannian completion of Schwarzschild

$$ds^2{}_E = (1 - \frac{2M}{r})d\tau^2 + \frac{1}{(1 - \frac{2M}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Change coordinates using

$$R = 4M(1 - \frac{2M}{r})^{\frac{1}{2}}$$

to obtain

$$ds^2{}_E = R^2 d\left(\frac{\tau}{4M}\right)^2 + \left(\frac{r}{2M}\right)^4 dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The Riemannian metric is singular at $R = 0$. One can extend the manifold by making the τ coordinate periodic with period $8\pi M$. So a single point is added on the manifold to obtain $S^2 \times R^2$.

Temperature of Schwarzschild Blackhole

Using the thermal Feynman propagator $i\square_T(x,y)$ to obtain the temperature

$$kT = \square^{-1} = \frac{\hbar}{Period} = \frac{\hbar}{8\square M}$$

This is the Hartle-Hawking Vacuum. This also works for non-linear fields!

De Sitter Spacetime

$$ds^2 = -dt^2 + \alpha^2 \cosh^2 \frac{t}{\alpha} \left(dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)\right)$$

invariant under $O(4,1)$
Note - 4 disconnected components of the group

G =O(4,1)$_{id}$
T - time reversal, $t \to -t$
S - spatial reflection, $r \to r + \pi/2$

$G_T = \{gT,\ g \in O(4,1)_{id}\}$, G_S, G_{ST} yield the other three components

The symmetric two point function in a vacuum state $|\square>$ is

$$G^{(1)}(x,y) = <\square|\square(x)\square(y) + \square(y)\square(x)|\square>$$

If it is to be invariant under the full (disconnected) de Sitter group,

$$G^{(1)}(x,y) = G^{(1)}(d(x,y))$$

where $d(x,y)$ is the geodesic distance between points x and y.

Now

$$(-\Box^2 - 4\Box\Box - m^2)G^{(1)}(d(x,y)) = 0$$

General solution: linear combination of hypergeometric functions

$$G^{(1)} = a\, {}_2F_1(c, 3-c, 2, \frac{1}{2}(1+\cos(\Box d)) + b\, {}_2F_1(c, 3-c, 2, \frac{1}{2}(1-\cos(\Box d))$$

c root of

$$c(c-3) + \Box^2 m^2 + 12\Box = 0$$

a and b real constants

Note: if x and y are null separated, $d(x,y) = 0$ and $\cos(\Box d) = \pm 1$.
Then either first or second factor has a simple pole.

Example de Sitter

The global de Sitter metric is

$$ds^2 = -dt^2 + \square^2 \cosh^2 \frac{t}{\square} \left(dr^2 + \sin^2 r(d\square^2 + \sin^2 \square d\square^2)\right)$$

One can start by taking t to $i\square$, this yields

$$ds^2{}_E = d\square^2 + \square^2 \cosh^2 \frac{i\square}{\square} \left(dr^2 + \sin^2 r(d\square^2 + \sin^2 \square d\square^2)\right) ,$$

however, $\cosh i\square = \cos \square$. So the Riemannian metric becomes

$$ds^2{}_E = d\square^2 + \square^2 \cos^2 \frac{i\square}{\square} \left(dr^2 + \sin^2 r(d\square^2 + \sin^2 \square d\square^2)\right) .$$

Now, this metric is not complete unless we add two points and $\square$ is periodic with

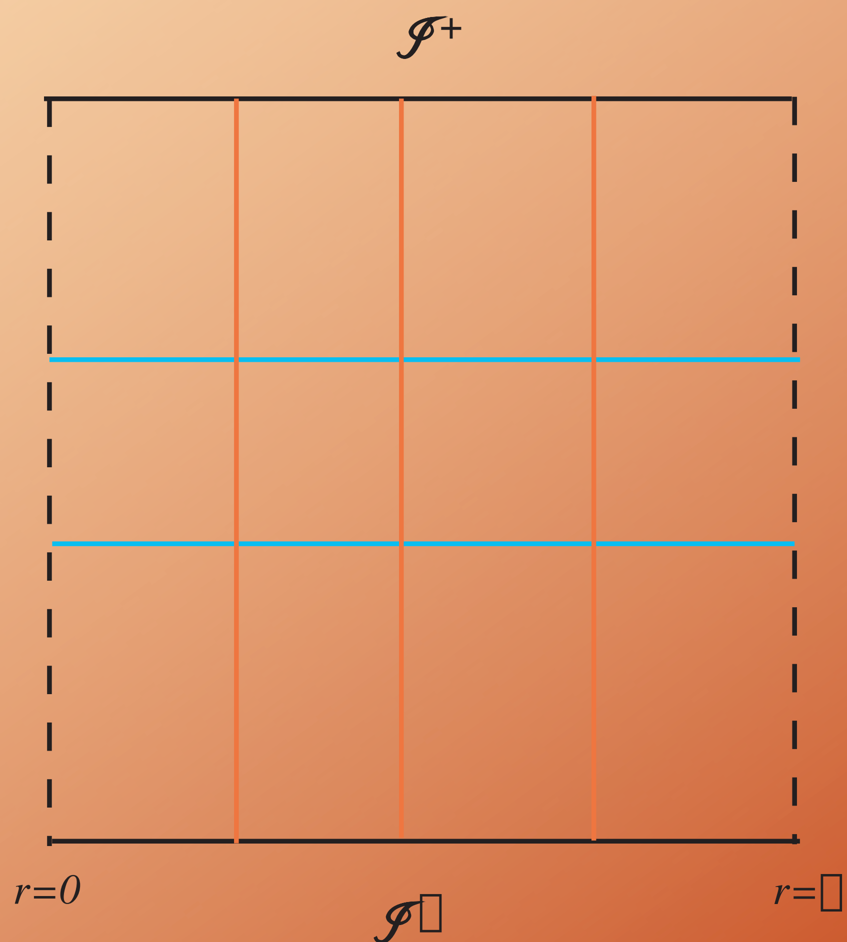
$$2\square\square$$

where $\square = \frac{1}{H}$.

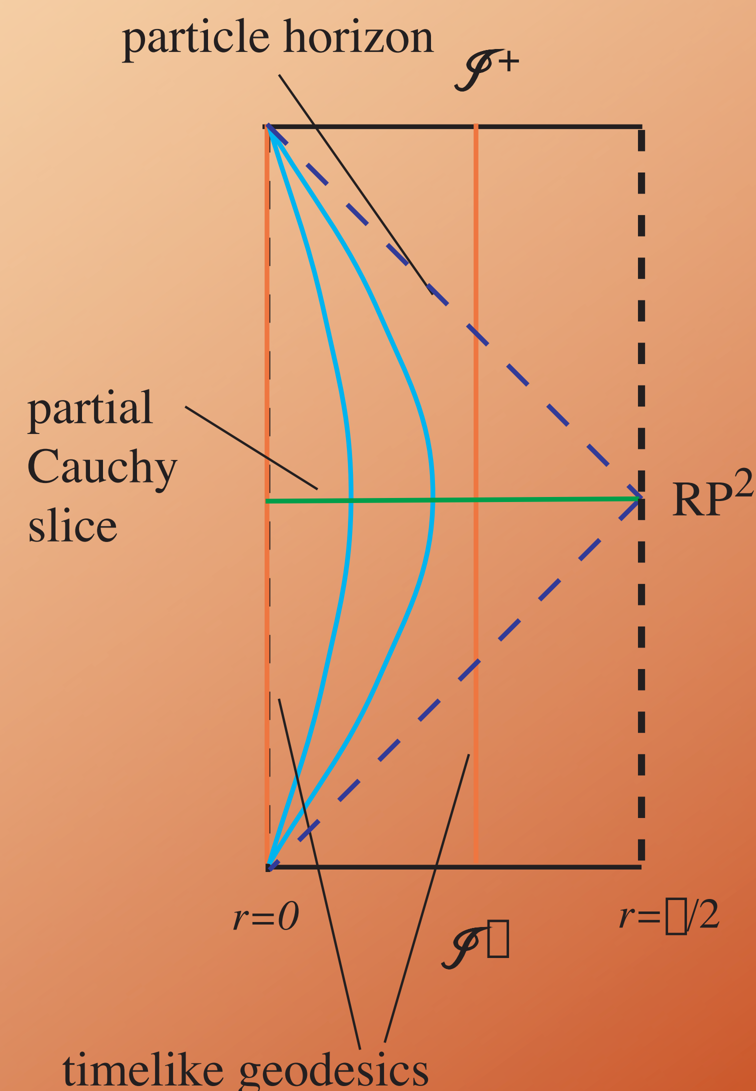
De Sitter Topology

The global topology of the Lorentzian de Sitter spacetime is $S^3 \times R$. The Riemannian metric is defined on the same manifold. However, two points are added to complete the metric. The final manifold is S^4.

de Sitter Spacetime



RP^3 de Sitter Spacetime



The global topology of the Lorentzian de Sitter spacetime is $S^3 \times R$ where the spheres are increasing in radius. The Riemannian metric is defined on the same manifold. however, two points are added to complete the metric. The final manifold is S^4.

RP^3 de Sitter

$$ds^2 = -dt^2 + \square^2 \cosh^2 \frac{t}{\square} \left(dr^2 + \sin^2 r(d\square^2 + \sin^2 \square d\square^2)\right)$$

The metric admits an isometry given by $x \to -x$. This global spacetime is $RP^3 \times R$. However, the Riemannian version is not a manifold. The reason is the spatial topology is $RP^3 \times R$ with two points added.