Examples of Mahler Measures as Multiple Polylogarithms

The many aspects of Mahler's measure – Banff International Research Station for Mathematical Innovation and Discovery (BIRS), Banff, Alberta, Canada

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1. Mahler Measure

Definition 1 *For* $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, *the (logarithmic)* Mahler measure *is defined by*

$$m(P) := \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \tag{1}$$

$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \tag{2}$$

Jensen's formula provides a simple expression for the Mahler measure in the one-variable case. The several-variable case is more complicated. Many examples with explicit formulae have been produced. (See [4], [10], [12], [13], [14], [15], [16])

The simplest example with two variables is due to Smyth [13]:

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} \mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1) \tag{3}$$

Where

$$\mathrm{L}(\chi_{-3}, s) := \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

is the L-series in the character of conductor 3:

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

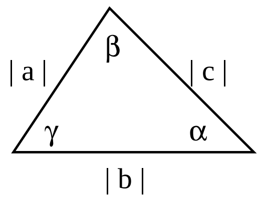
The analogous example with three variables is also due to Smyth:

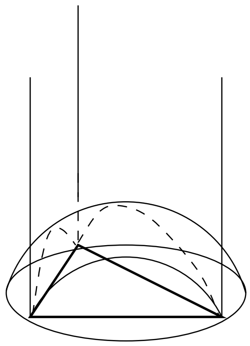
$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3) \tag{4}$$

The general linear case with two variables is due to Cassaigne and Maillot, [10] : for $a, b, c \in \mathbb{C}$,

$$\pi m(a + bx + cy) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{\mathrm{i}\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases} \tag{5}$$

Here $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively. See picture.





D stands for the Bloch–Wigner dilogarithm (see definition later). The term with the dilogarithm can be interpreted as the volume of the ideal hyperbolic tetrahedron which has the triangle as basis and the fourth vertex is infinity (see [11], [17]).

2. Examples of higher weight

We have obtained ([9]) examples of polynomials in several variables whose Mahler measures depend on polylogarithms. The first column of the table shows the polynomials. Here α is a complex number different from zero. The second column indicates the values of the first column for the case $\alpha = 1$.

$\pi m((1+y)+\alpha(1-y)x)$	$2\mathrm{L}(\chi_{-4},2)$
$\pi^2 m((1+w)(1+y)+\alpha(1-w)(1-y)x)$	$7\zeta(3)$
$\pi^3 m((1+v)(1+w)(1+y)+\alpha(1-v)(1-w)(1-y)x)$	$7\pi\zeta(3)+4\sum_{0\le j<k}\frac{(-1)^j}{(2j+1)^2k^2}$
$\pi^2 m((1+x)+\alpha(y+z))$	$\frac{7}{2}\zeta(3)$
$\pi^3 m((1+w)(1+x)+\alpha(1-w)(y+z))$	$2\pi^2\mathrm{L}(\chi_{-4},2)+8\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^3k}$
$\pi^4 m((1+v)(1+w)(1+x)+\alpha(1-v)(1-w)(y+z))$	$93\zeta(5)$
$\pi^2 m((1+w)(1+y)+(1-w)(x-y))$	$\frac{7}{2}\zeta(3)+\frac{\pi^2}{2}\log 2$

Here χ_{-4} is the real odd character of conductor 4, i.e.

$$\chi_{-4}(n)=\begin{cases} 1 & \text{if} \quad n\equiv 1 \bmod 4 \\ -1 & \text{if} \quad n\equiv -1 \bmod 4 \\ 0 & \text{otherwise}\end{cases}$$

Let us observe that all the presented formulae share a common feature. If we assign weight 1 to any Mahler measure and to π, then all the formulae are homogeneous, meaning all the monomials have the same weight, and this weight is equal to the number of variables of the corresponding polynomial.

3. Polylogarithms

We need the following definitions (see [6], [7], [8])

Definition 2 Multiple polylogarithms *are defined as the power series*

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \sum_{0<n_1<n_2<\ldots<n_m} \frac{x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}}{n_1^{k_1} n_2^{k_2} \ldots n_m^{k_m}} \tag{6}$$

which are convergent for $|x_i| < 1$. The weight of a polylogarithm function is the number $w = k_1 + \ldots + k_m$ and its length is the number m.

Definition 3 Hyperlogarithms *are defined as the iterated integrals*

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_1} \circ \underbrace{\frac{\mathrm{d}t}{t-a_2} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_2} \circ \ldots \circ \underbrace{\frac{\mathrm{d}t}{t-a_m} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_m}$$

where k_i are integers, a_i are complex numbers, and

$$\int_0^{b_{l+1}} \frac{\mathrm{d}t}{t-b_1} \circ \ldots \circ \frac{\mathrm{d}t}{t-b_l} = \int_{0\le t_1 \le \ldots \le t_l \le b_{l+1}} \frac{\mathrm{d}t_1}{t_1 - b_1} \cdots \frac{\mathrm{d}t_l}{t_l - b_l}$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$.

It is easy to see that

$$\begin{aligned} \mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) &= (-1)^m \mathrm{Li}_{k_1,\ldots,k_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right) \\ \mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) &= (-1)^m \mathrm{I}_{k_1,\ldots,k_m}\left(\frac{1}{x_1 \ldots x_m} : \ldots : \frac{1}{x_m} : 1\right) \end{aligned}$$

which gives an analytic continuation to multiple polylogarithms. For instance, with the convention about integrating over a real segment, simple polylogarithms have an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

There are modified versions of these functions which are analytic in larger sets, like the Bloch-Wigner dilogarithm,

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z) \qquad z \in \mathbb{C} \setminus [1, \infty) \tag{7}$$

which can be extended as a real analytic function in $\mathbb{C} \setminus \{0, 1\}$ and continuous in $\mathbb{C}$.

4. General method for building examples

The idea behind the computations that led to the examples in the table, is the following.

1. Let $P_\alpha \in \mathbb{C}[x_1, \ldots, x_n]$ whose coefficients depend polynomially on a parameter $\alpha \in \mathbb{C}$. For instance, start with $P_\alpha(x) = 1 + \alpha x$, whose Mahler measure is $\log^+ |\alpha|$.

2. We replace α by $\alpha \frac{1-y}{1+y}$ and obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1, \ldots, x_n, y]$. In the example, $\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1-y)x$.

3. The Mahler measure of the second polynomial is a certain integral of the Mahler measure of the first polynomial.

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y} \qquad \left(\text{in the example, } = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ \left|\alpha \frac{1-y}{1+y}\right| \frac{\mathrm{d}y}{y}\right)$$

4. If the Mahler measure depends just on the absolute value of α, we can make a change of variables $u = \left|\alpha\frac{1-y}{1+y}\right|$ (to be precise, first write $y = \mathrm{e}^{\mathrm{i}\theta}$ and then set $u = |\alpha| \tan\left(\frac{\theta}{2}\right)$). We obtain,

$$m(\tilde{P}_\alpha) = \frac{2}{\pi} \int_0^\infty m\left(P_u\right) \frac{|\alpha|\, \mathrm{d}u}{u^2 + |\alpha|^2} = \frac{\mathrm{i}}{\pi} \int_0^\infty m\left(P_u\right) \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|}\right) \mathrm{d}u$$

In the example,

$$\begin{aligned} m(1+y+\alpha(1-y)x) &= \frac{\mathrm{i}}{\pi} \int_0^\infty \log^+ u \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|}\right) \mathrm{d}u \\ &= \frac{\mathrm{i}}{\pi} \int_0^1 \int_s^1 \frac{\mathrm{d}t}{t} \left(\frac{1}{s + \frac{\mathrm{i}}{|\alpha|}} - \frac{1}{s - \frac{\mathrm{i}}{|\alpha|}}\right) \mathrm{d}s \end{aligned}$$

$$= \frac{\mathrm{i}}{\pi} \left(\mathrm{I}_2\left(-\frac{\mathrm{i}}{|\alpha|} : 1\right) - \mathrm{I}_2\left(\frac{\mathrm{i}}{|\alpha|} : 1\right)\right) = -\frac{\mathrm{i}}{\pi}(\mathrm{Li}_2(\mathrm{i}|\alpha|) - \mathrm{Li}_2(-\mathrm{i}|\alpha|))$$

If we look back at the table, we have splitted the examples into three families. The first family was developed starting from $1+\alpha x$, the second family starts from $(1+x)+\alpha(y+z)$ (see [2], [16]). The last polynomial was obtained by integrating one particular case of Maillot's formula: $1 + \alpha x + (1-\alpha)y$.

5. The example with five variables

By applying the method described above, we can prove that

$$\begin{aligned} \pi^4 m((1+v)(1+w)(1+x)+(1-v)(1-w)(y+z)) &= 7\pi^2\zeta(3) + 8(\mathrm{Li}_{3,2}(1,1) - \mathrm{Li}_{3,2}(-1,1)) \\ &\quad +8(\mathrm{Li}_{3,2}(1,-1) - \mathrm{Li}_{3,2}(-1,-1)) \qquad (8) \end{aligned}$$

The values $\mathrm{Li}_{3,2}(\pm 1, \pm 1)$ are alternating Euler sums. We use formula (75) of [1], which in this particular case, states that

$$\mathrm{Li}_{3,2}(x,y) = -\frac{1}{2}\mathrm{Li}_5(x\,y) + \mathrm{Li}_3(x)\,\mathrm{Li}_2(y) + 3\mathrm{Li}_5(x) + 2\mathrm{Li}_5(y) - \mathrm{Li}_2(x\,y)(\mathrm{Li}_3(x) + 2\mathrm{Li}_3(y))$$

for $x, y = \pm 1$.

Taking into account that

$$\mathrm{Li}_k(1) = \zeta(k) \quad \text{and} \quad \mathrm{Li}_k(-1) = \left(\frac{1}{2^{k-1}} - 1\right)\zeta(k) \qquad (9)$$

we get

$$\mathrm{Li}_{3,2}(1,1) - \mathrm{Li}_{3,2}(-1,1) + \mathrm{Li}_{3,2}(1,-1) - \mathrm{Li}_{3,2}(-1,-1) = -\frac{21}{4}\zeta(2)\zeta(3) + \frac{93}{8}\zeta(5)$$

We obtain the result by using that $\zeta(2) = \frac{\pi^2}{6}$

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