Adaptive Finite Element Methods

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The Problem

$$-\Delta u = f$$

$$u = 0 \text{ on } \partial\Omega$$

Ω is a polygonal domain in $I\!R^2$

1. Standard Finite Element Methods
2. Adaptive Finite Element Methods

IS THERE ANY ADVANTAGE TO ADAPTIVE METHODS

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- I know of no AFEM ($d \geq 2$) which is proven to outperform Standard Finite Element Methods for problems in $\mathbb{R}^d$, $d \geq 2$

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- Give the ‘mother’of all adaptive algorithms

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- $|||u - u_P||| = \inf_{S \in \mathcal{S}_P} |||u - S|||$
- $||| \cdot |||$ energy norm (H^1-norm)

Typical Adaptive Algorithm: $P_k \longrightarrow P_{k+1}$

1. Compute u_{P_k}
2. Compute error indicators $e(\Delta)$, $\Delta \in P_k$. $e(\Delta)$ has two terms: (i) jumps in u_{P_k} accross edges of Δ, (ii) approximation of f by constants on Δ
3. Use local error indicators to mark cells $\mathcal{M}_k$ in P_k: bulk chasing.
4. Refine cells in $\mathcal{M}_k$ to obtain P'_k
5. Remove hanging nodes: further markings $\mathcal{M}'_k$
6. Refine cells in $\mathcal{M}'_k$: $P'_k \longrightarrow P_{k+1}$

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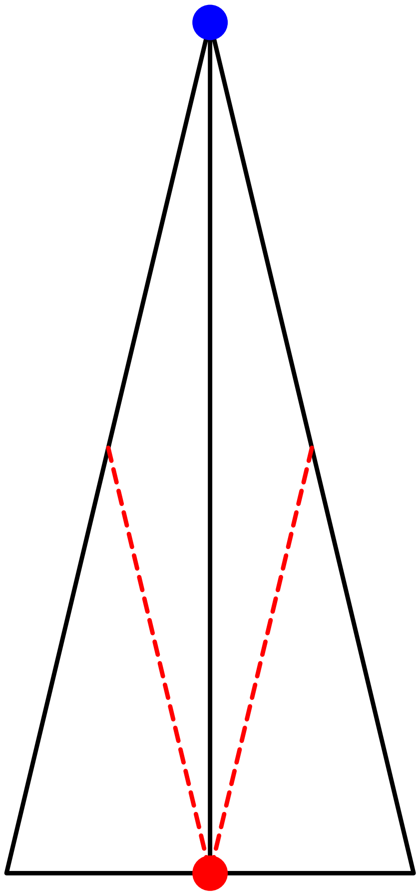
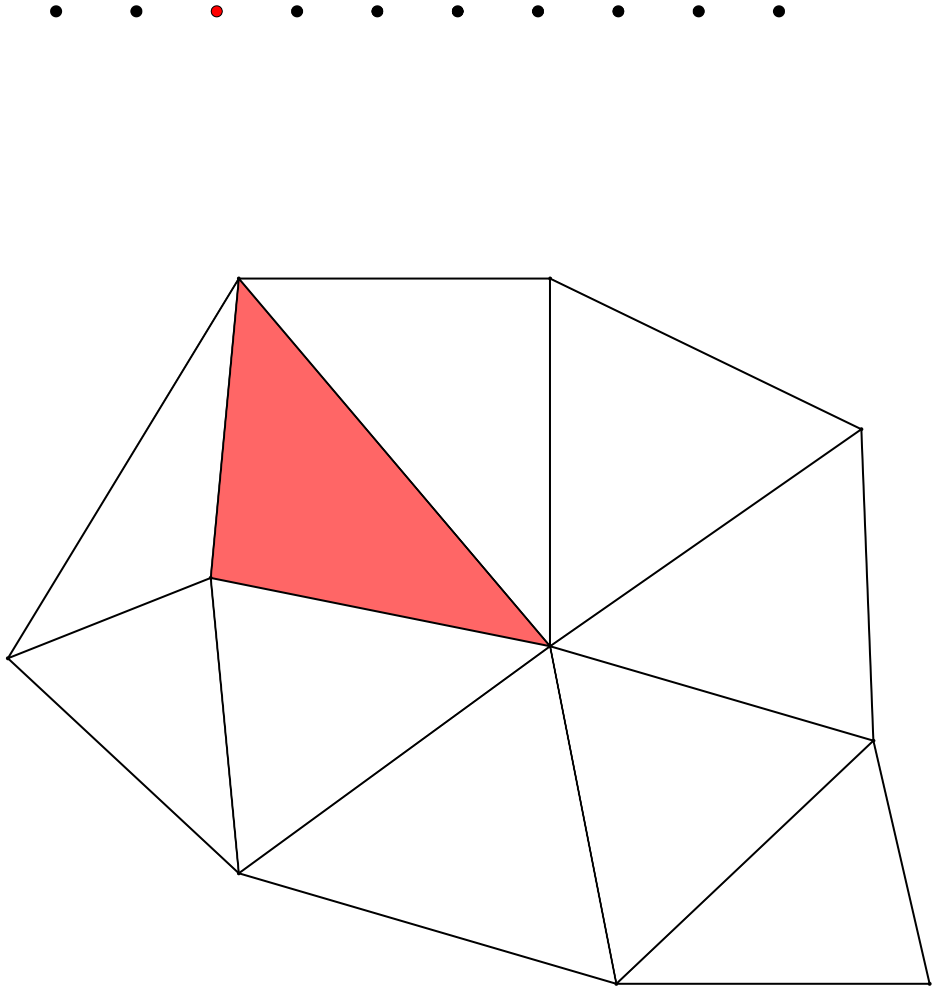


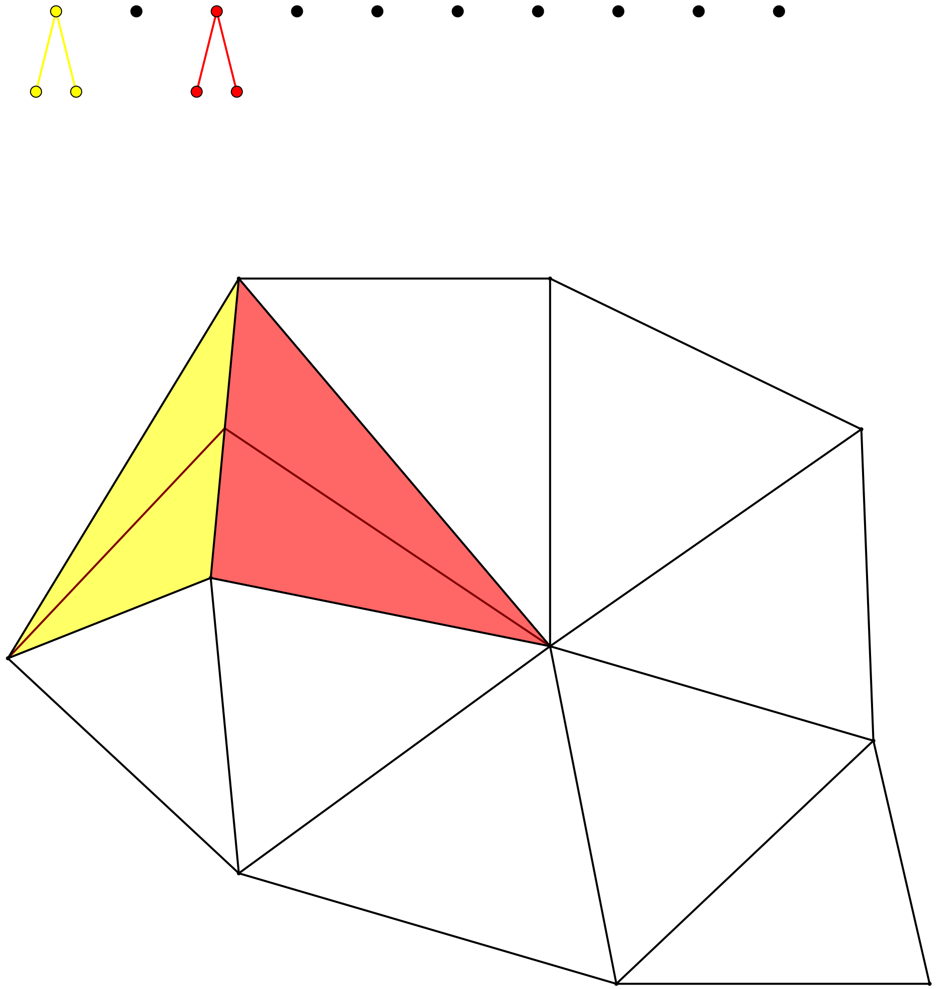
newest vertex of big triangle

newest vertex of new triangles

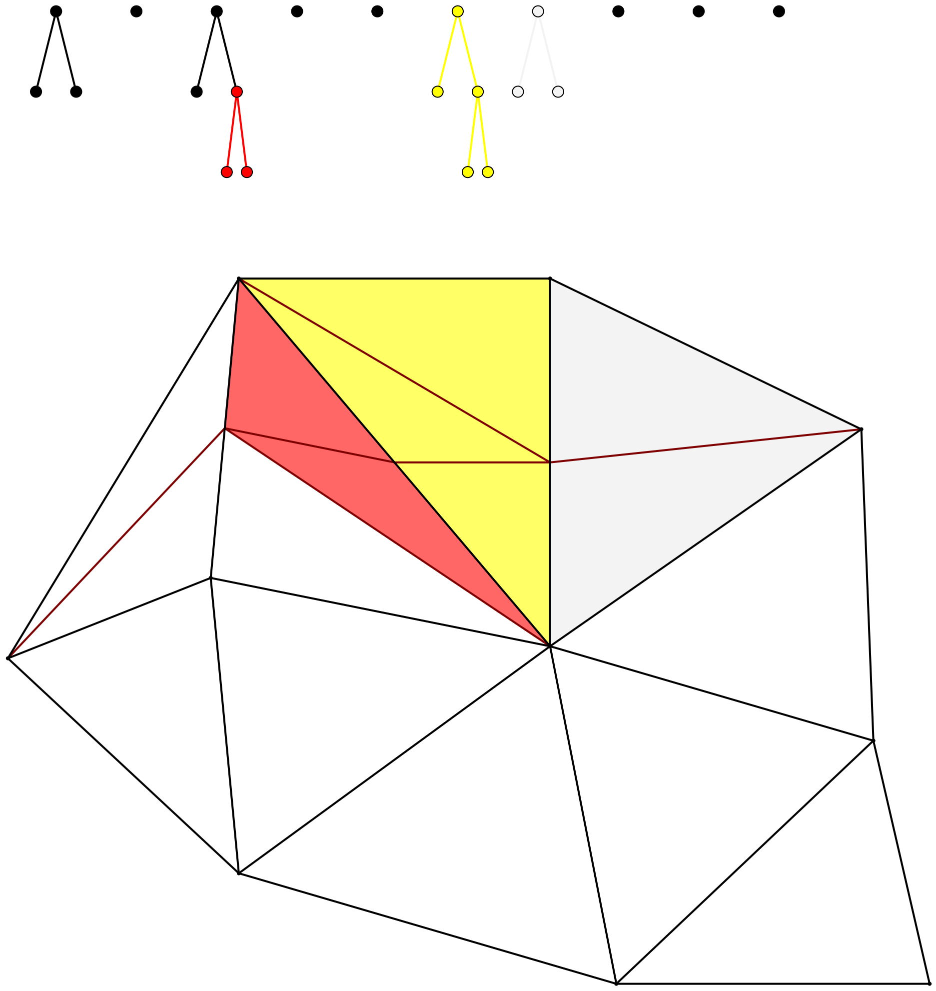
Tree structure



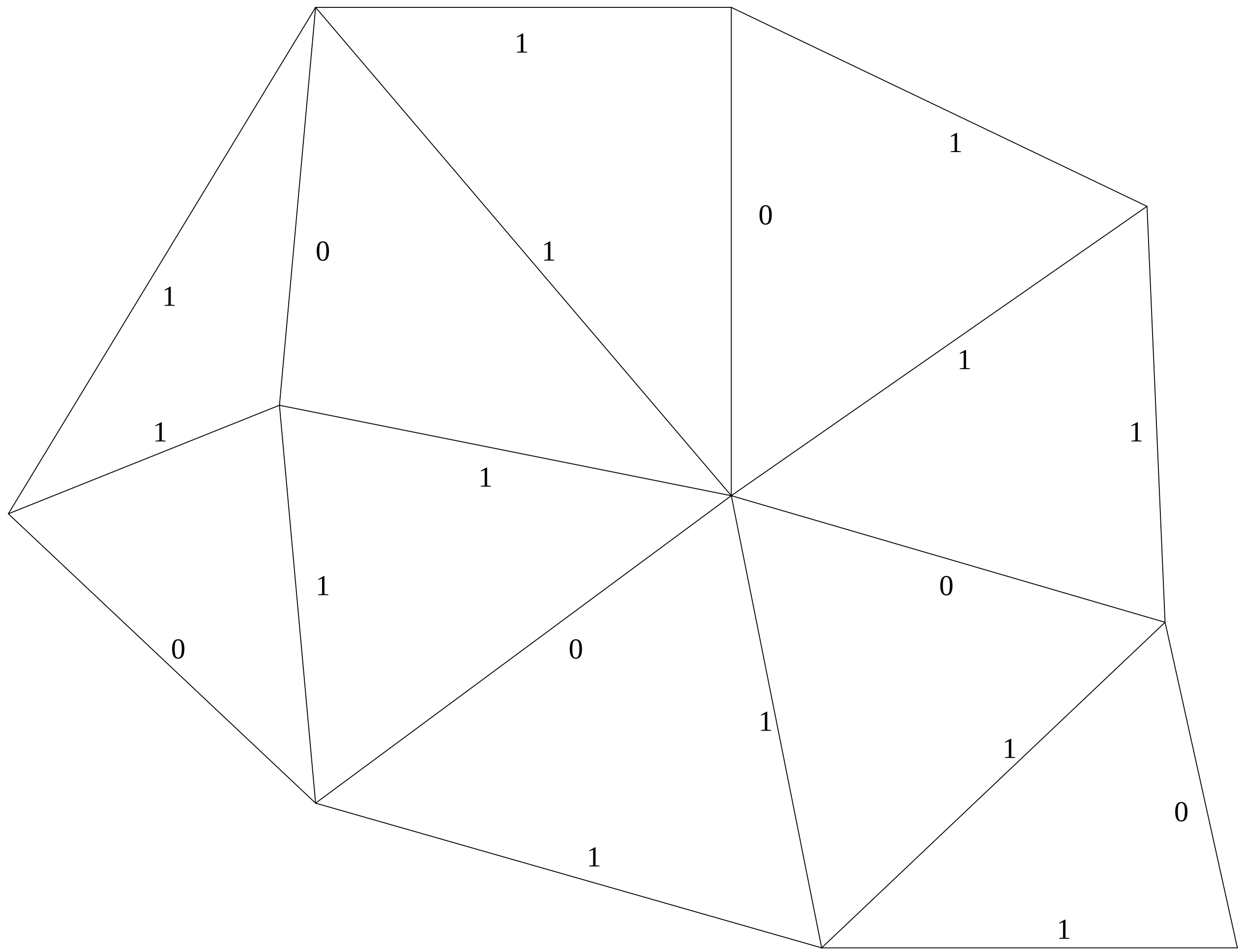
Refinement grows the tree



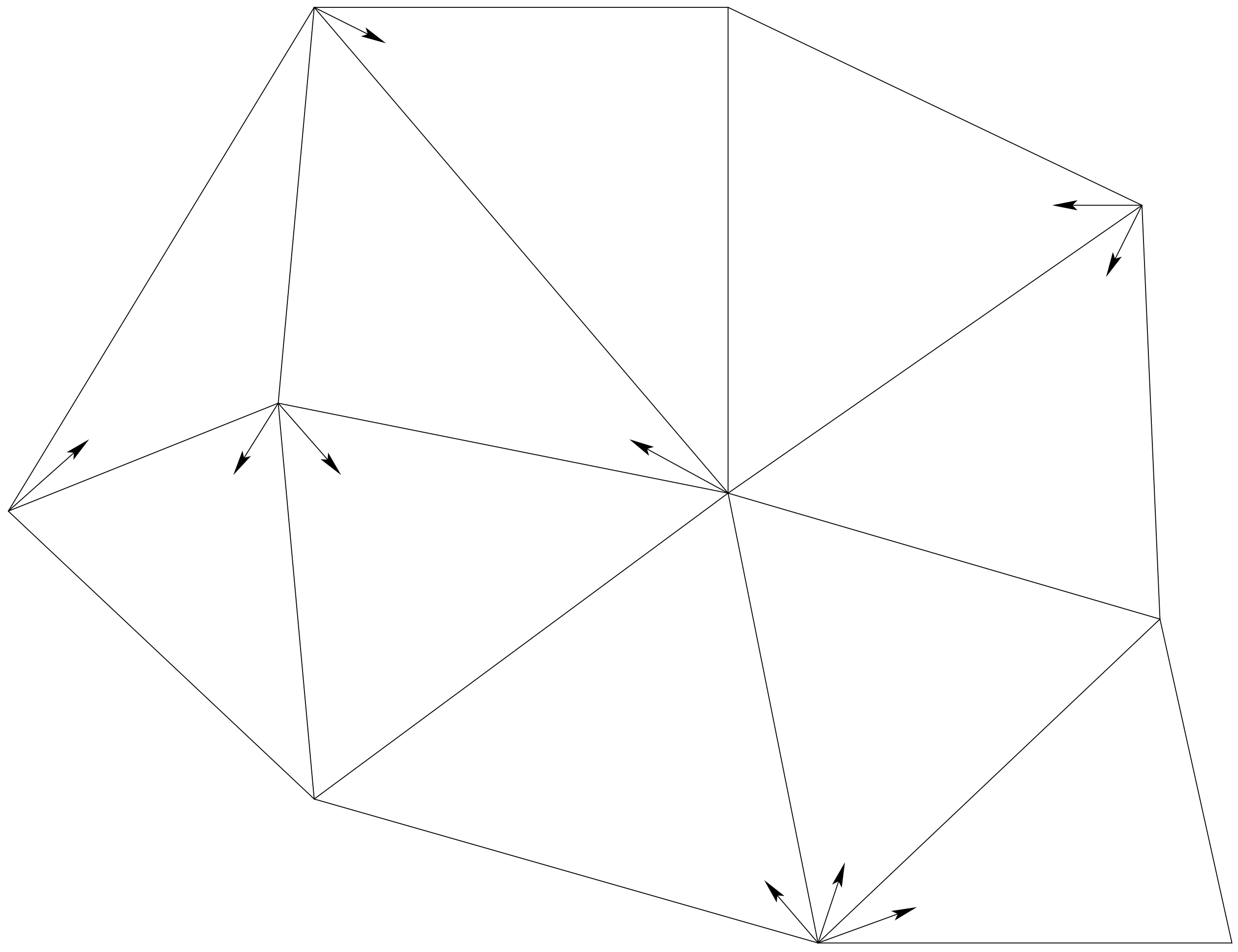
Grow more



Initial Labeling of edges



Initial assignment of newest vertices



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- Equivalent to $\sigma_n(u) \leq \epsilon$ with $n = C\epsilon^{-1/2}$

Binev-Dahmen- DeVore Algorithm

For each $\epsilon > 0$, algorithm produces P_ϵ such that

1. $|||u - u_{P_\epsilon}||| \leq \epsilon$

2. If $u \in \mathcal{A}^s$ then $\#(P_\epsilon) \leq C_0 |u|_{\mathcal{A}^s} \epsilon^{-1/s}$

3. Number of computations used is $\leq C_0 |u|_{\mathcal{A}^s} \epsilon^{-1/s}$

COROLLARY: The BDD AFEM beats Standard Finite Element Methods for a wide class of problems

How to do Marking

Bulk Chasing: Doerfler, Morin-Nochetto-Siebert

1. Uses error indicators $e(\Delta)$
2. Choose smallest set $\mathcal{M}_k$ such that

$$\sum_{\Delta \in \mathcal{M}_k} e(\Delta) \geq 1/2 \sum_{\Delta \in P_k} e(\Delta)$$

3. There exists $0 < \lambda < 1$: Given target accuarcy $\epsilon > 0$ and P_k which resolves f to accuracy $\gamma\epsilon$,. then either

$$|||u - u_{P_{k+1}}||| < \epsilon$$

or

$$|||u - u_{P_{k+1}}||| \leq \lambda |||u - u_{P_k}|||$$

Binev-Dahmen-DeVore Algorithm

$P_k \longrightarrow P_{k+1}$:

1. $P_{k,0} := P_k$

2. $P_{k,j-1} \longrightarrow P_{k,j}$, $j = 1, \ldots, K$:

$$|||u - u_{P_{k,j}}||| \leq \lambda |||u - u_{P_{k,j-1}}|||$$

3 $P_{k,K} \longrightarrow P_{k+1}$ by Coarsening

Two main results

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- How to do Coarsening

Control removal of hanging nodes

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- $\mathcal{P}_k \to P_{k+1}$ uses $\mathcal{M}_k$ and $\mathcal{M}'_k$
- Theorem:

$$\#(P_n) \leq \#(P_0) + C(\#(\mathcal{M}_0) + \cdots + \#(\mathcal{M}_{n-1}))$$

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- Not simple induction: we do not have

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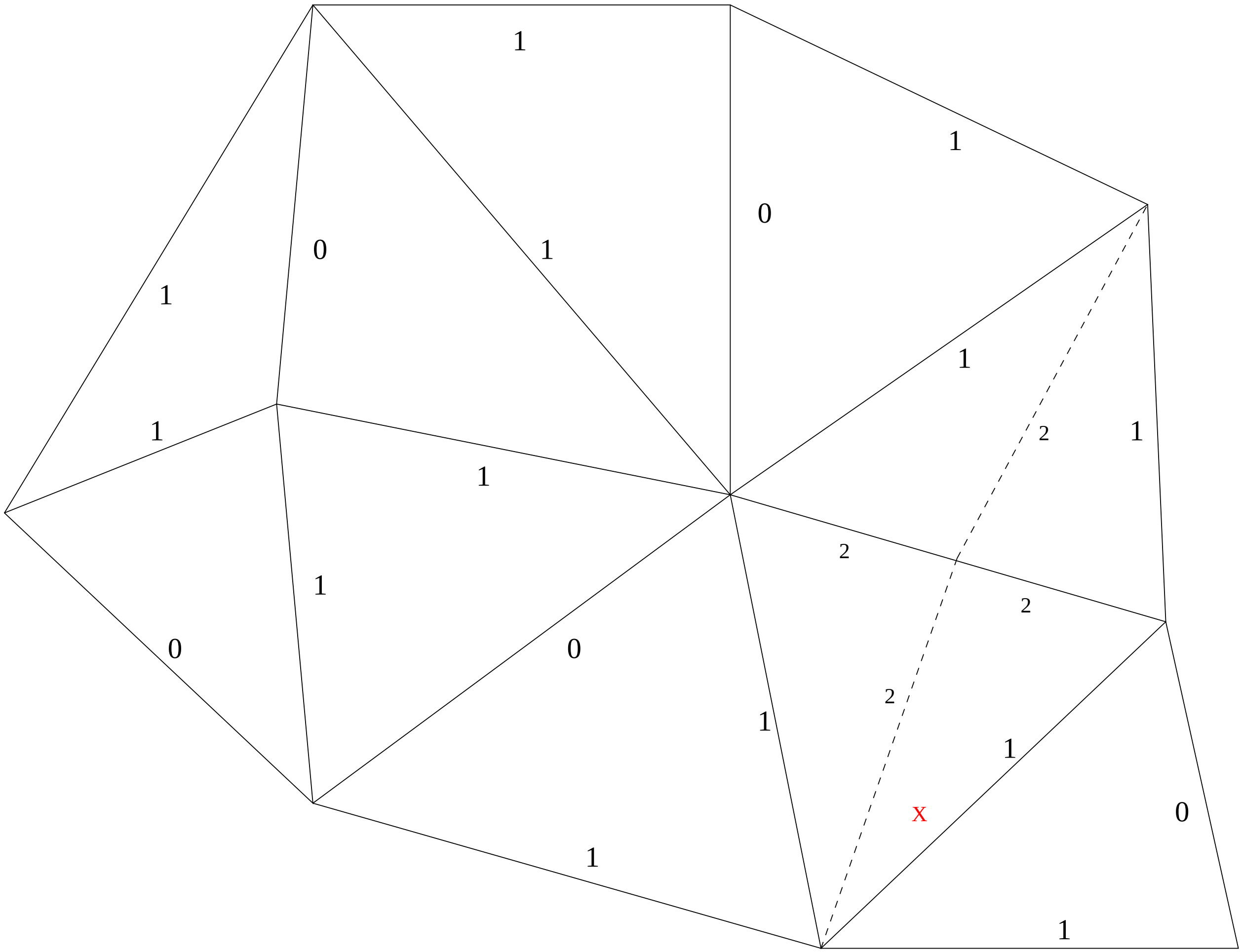
- Need to look at entire history of how $\Delta \in P_n$ was created

Many refinements



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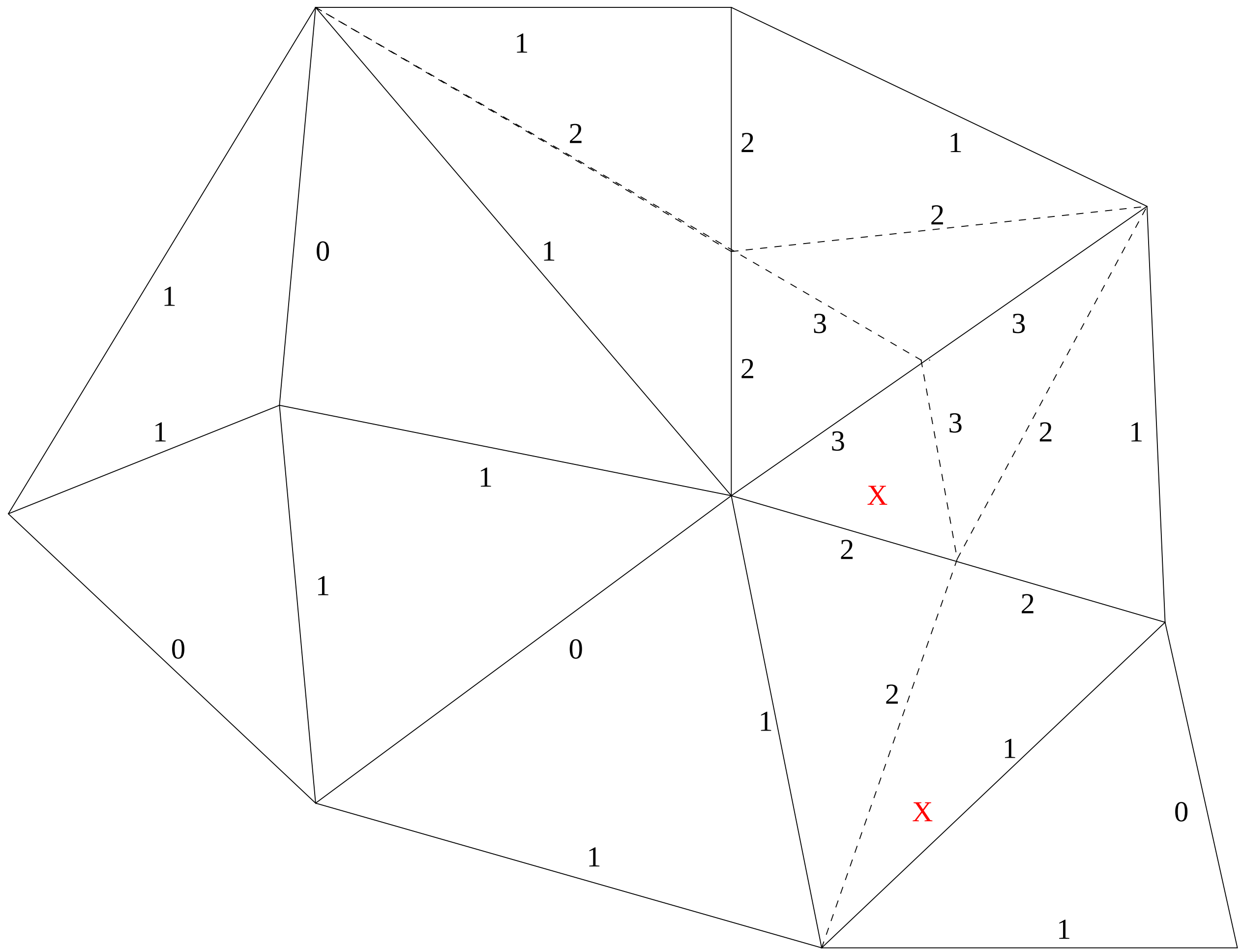
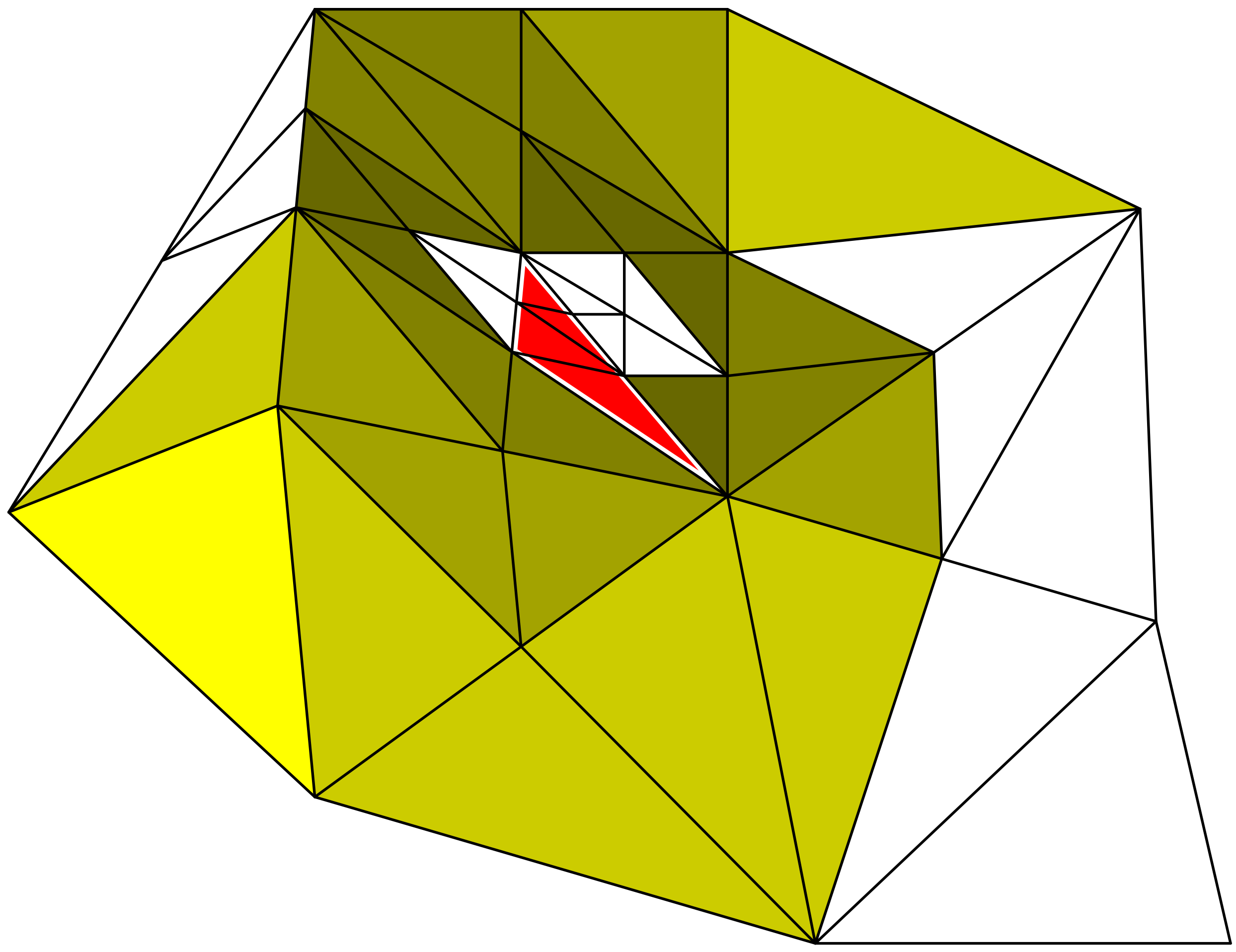


Ideas:

1. Each cell in $\mathcal{M} := \cup_{k=0}^{n-1} \mathcal{M}_k$ is given A dollars to spend
2. $\Delta' \in P_n$ receives $\lambda(\Delta, \Delta')$ dollars from a given $\Delta \in \mathcal{M}$.
Here λ depends on the generations $g(\Delta) = k$, $g(\Delta') = j$ and the distance $\mathrm{dist}(\Delta, \Delta')$

$$\lambda(\Delta, \Delta') := \begin{cases} (j-k+2)^{-2}, & \mathrm{dist} \leq A2^{-k/2}; k \leq j+1, \\ 0, & \text{otherwise}. \end{cases}$$

Who's got the money?



Ideas

3. Prove each $\Delta' \in P_n$ receives at least B dollars from all of the $\Delta \in \mathcal{M}$ combined.

4. The latter requires to look at how Δ' is created, i.e. the set of markings responsible for the creation of Δ'.

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- master tree T_*: all cells Δ that can be obtained by newest vertex bisection

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- Global error for tree T: $\mathcal{L}(T)$ leaves of T

$$E(T) := \sum_{\Delta \in \mathcal{L}(T)} e(\Delta)$$

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- Theorem (Binev-DeVore): Given e, can find a near best tree with Cn computations

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- Greedy on $\tilde{e}$