Limited Data Tomography in Science and Industry

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CD E. Candès and D. Donoho, Proc. SPIE 4119 (2000).

FFRS A. Faridani, E.L. Ritman, and K.T. Smith, *SIAM J. Appl. Math.* **52**(1992), 459–484, +Finch II: **57**(1997) 1095–1127.

KLM P. Kuchment, P. Lancaster, M. Mogilevskaya, *Inverse Problems* **11**(1995), 571-589.

Ma W. Madych, *Appl. Comput. Harmon. Anal.* **7**(1999), 54-100.

Na Frank Natterer, *The Mathematics of Computerized Tomography*, Wiley, New York, 1986.

NaW Frank Natterer, Frank Wuebbling, *Mathematical Methods in Image Reconstruction*, SIAM, 2001.

NDCK F. Noo, M. Defrise, R. Clackdoyle, and H. Kudo, Phys. Med. Bio. **47**(2002), 2525-2546

NAC *Mathematics and Physics of emerging biomedical Imaging* National Academy Press, 1996

Q1 E.T. Quinto, *Inverse Problems* **4**(1988), 867-876.

Q2 E.T. Quinto, *SIAM J. Math. Anal.* **24**(1993), 1215-1225.

Q3 E.T. Quinto, *Inverse Problems* **14**(1998), 339-353.

RK A. Ramm and A. Katsevich *Pseudolocal Tomography*, CRC Press, 1996.

R-FLBW F. Rashid-Farrokhi, K. Liu, C. Berenstein and D. Walnut, IEEE Trans. Image Proc. 6(1997), 1412-1430.

X-ray Computerized Tomography (CT):

$f(x) =$ density of the object at $x \in \mathbb{R}^2$

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Total Attenuation of X-rays along the line ℓ:

$$Rf(\ell) := \int_{x \in \ell} f(x) ds \, ,$$

the **Radon Transform** [Radon 1917].

$$\text{Measured data: } \frac{I_{in}}{I_{out}} \sim e^{Rf(\ell)}.$$

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Recover an approximation of $f(x)$ from X-ray CT data over a finite number of lines.

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With uniformly distributed data (lines throughout the object on fairly evenly spaced angles), good, stable reconstruction methods exist [Na, NaW].

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When data are not uniformly distributed.

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  - All singularities are visible!

Observations on ROI CT:

- Madych: showed an equivalence of Lambda CT [FFRS], wavelet methods [R-FLBW], and pseudolocal CT [RK].

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- Candès and Donoho: curvelets and ridgelets, a very promising refinement of wavelets.
- Clackdoyle, *et al.* [NDCK]: Exact inversion for large regions of interest with enough data.

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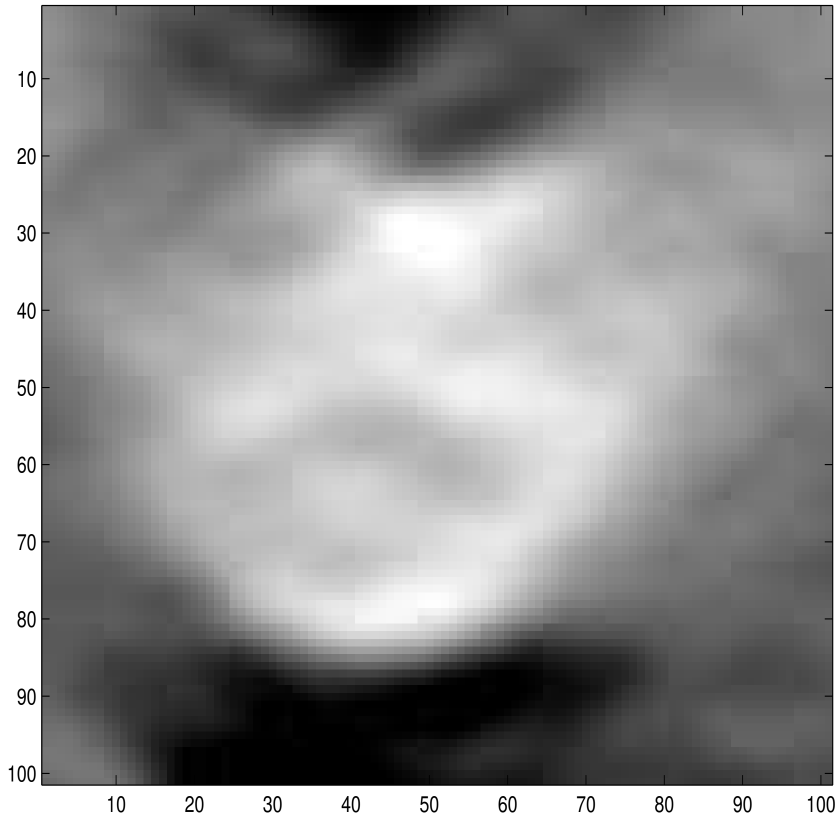
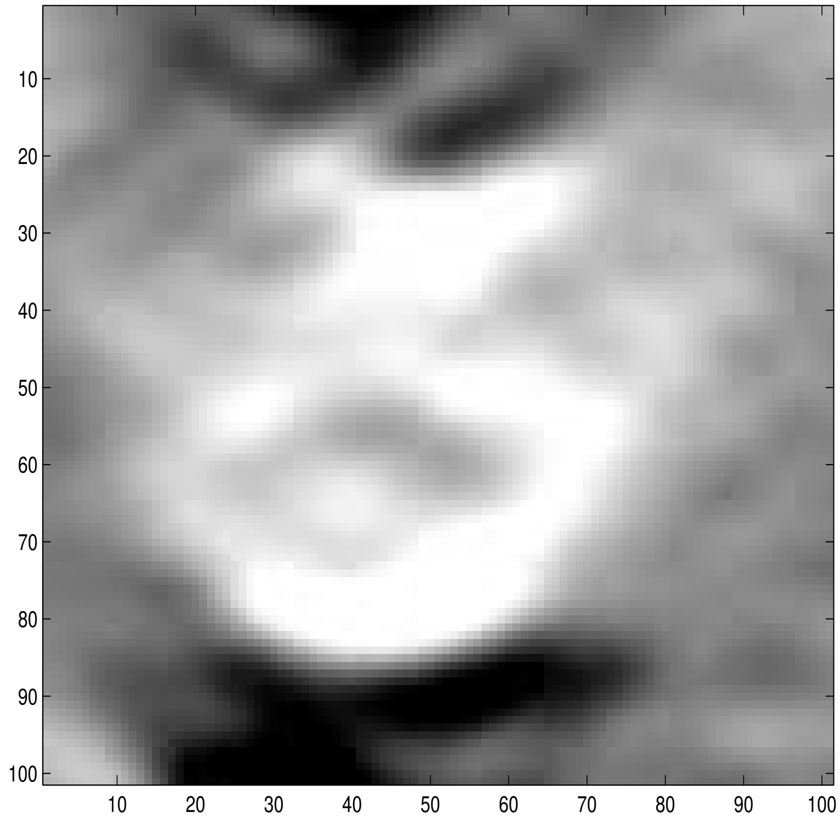
Limited angle ROI Sidec Reconstruction I: Parallel beam CT data of a virus, 25 more-or-less evenly spaced angles in a 120° range, 256 data lines/slice in 256 parallel slices.

I: Virus, 25 angles, $30°$ to $150°$:

Reconstruction with one window:



Different Window:



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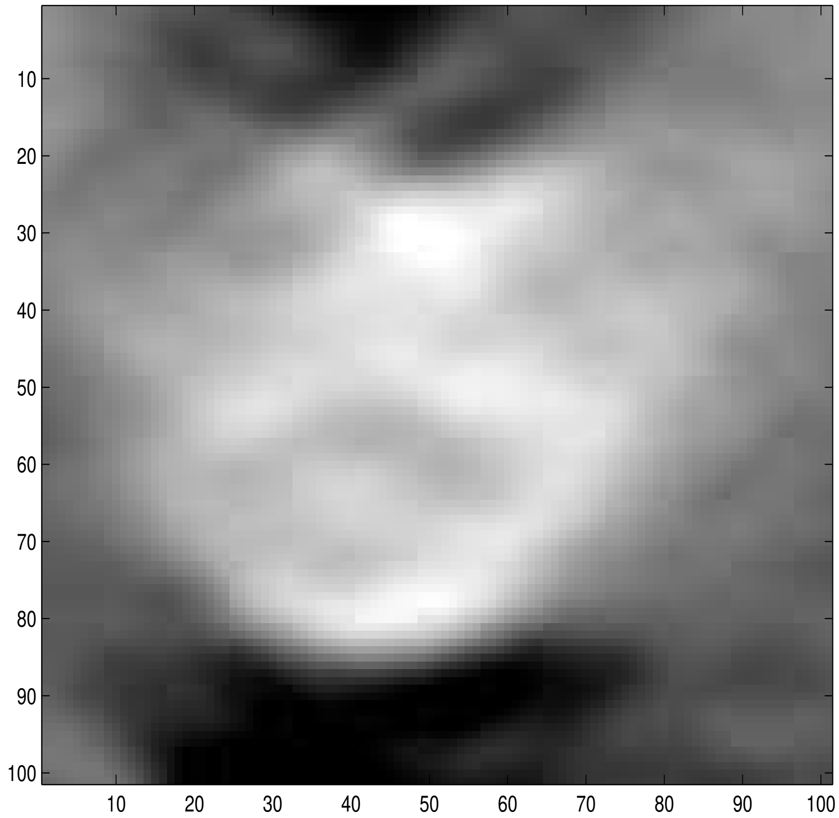

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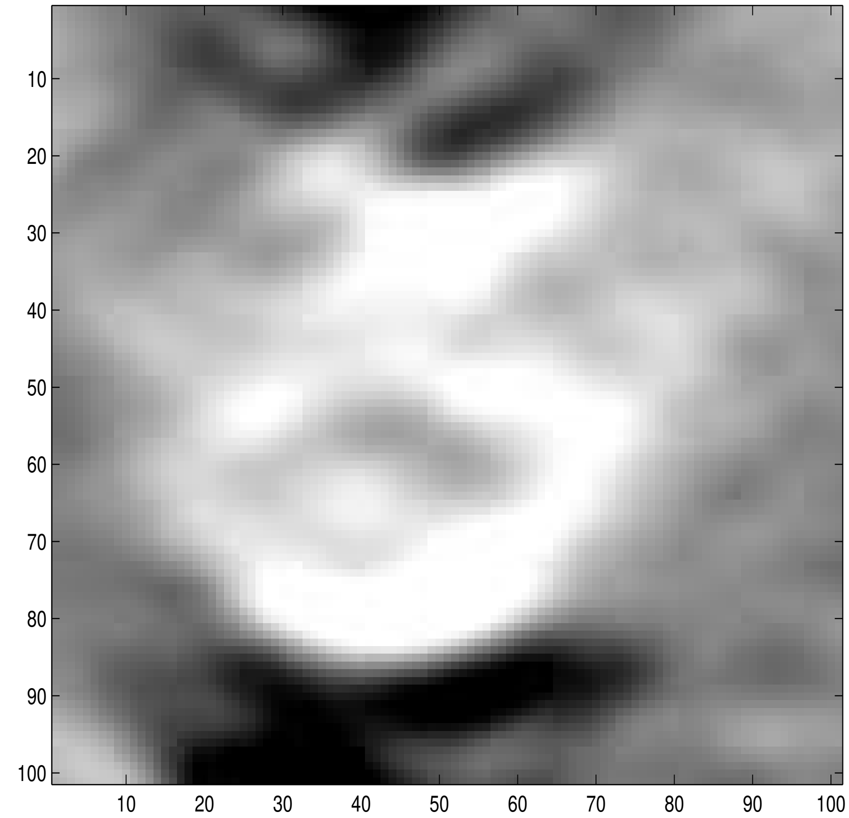

Note the singularities..... Recons. 2?

Big Question:

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- Learn what the Radon transform does to singularities.

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The Fourier Transform:

$$\begin{aligned}(\mathcal{F}f)(\xi) = \tilde{f}(\xi) &= \int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx, \\ f(x) &= \frac{1}{(2\pi)^2}\int_{\xi\in\mathbb{R}^2} e^{ix\cdot\xi}(\mathcal{F}f)(\xi)d\xi\end{aligned} \tag{1}$$

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Key idea: *Rapid decrease (or $L^2(\mathbb{R}^2,(1+|\xi)^2)^s)$ integrability) of $\mathcal{F}f \quad \sim$ smoothness (or L^2 derivatives) of f.....*

Singularities done rigorously:

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  - **Sobolev Singular Support:** $f \notin H^s$ at x_0 iff for every cut-off φ near x_0, $\mathcal{F}(\varphi\, f) \notin L^2(\mathbb{R}^2, (1+|\xi)^2)^s)$.

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  - **Sobolev Wavefront Set....**

Sobolev Wavefront Set:

Definition 1. Let $x_0 \in \mathbb{R}^2$ and $\xi_0 \in \mathbb{R}^2 \setminus 0$. The function f is *in* H^s at x_0 in direction ξ_0 **iff** $\exists$ a cut-off function φ near x_0 such that

$$\mathcal{F}(\varphi f)(\xi) = \int_{x \in \mathbb{R}^2} e^{-ix\cdot\xi} \varphi(x)\, f(x)\, dx \qquad (2)$$

is in $L^2(\mathbb{R}^2, (1 + |\xi)^2)^s)$ in some open cone from the origin containing ξ_0.
On the other hand, $(x_0, \xi_0) \in \mathrm{WF}^s(f)$ **iff** f is **not** in H^s at x_0 in direction ξ_0.

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MORAL: If f has a jump singularity on a smooth curve C, then vectors (co)normal to C are in $\mathrm{WF}^4(f)$.

Definitions for R:

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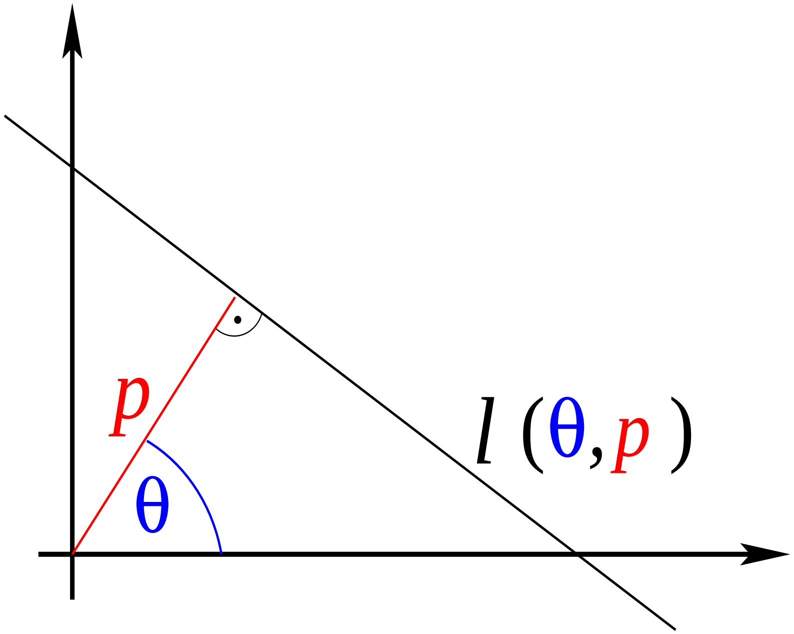
$$\ell(\theta, p) = \{x \in \mathbb{R}^2 \big| x \cdot \overline{\theta} = p\} :$$



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Theorem 2. [Q1993] *There is a one-to-one correspondence* (T) *between singularities (wavefront) of* f *and those of* Rf*.*
Let $(\theta_0, p_0) \in [0, 2\pi] \times \mathbb{R}$*. If for some* $x_0 \in \ell(\theta_0, p_0)$*,* $(x_0, \overline{\theta}_0) \in \mathrm{WF}^s(f)$*, then* Rf *is not in* $H^{s+1/2}$ *near* (θ_0, p_0)*. If* f *is* H^s *in directions* $\pm\overline{\theta}_0$ *at every point on* ℓ_0*, then* Rf *is in* $H^{s+1/2}$ *near* (θ_0, p_0)*.*
Undetected singularities (those not perpendicular to the line being integrated on) are smoothed by R*.*

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- Picture! $\longrightarrow$

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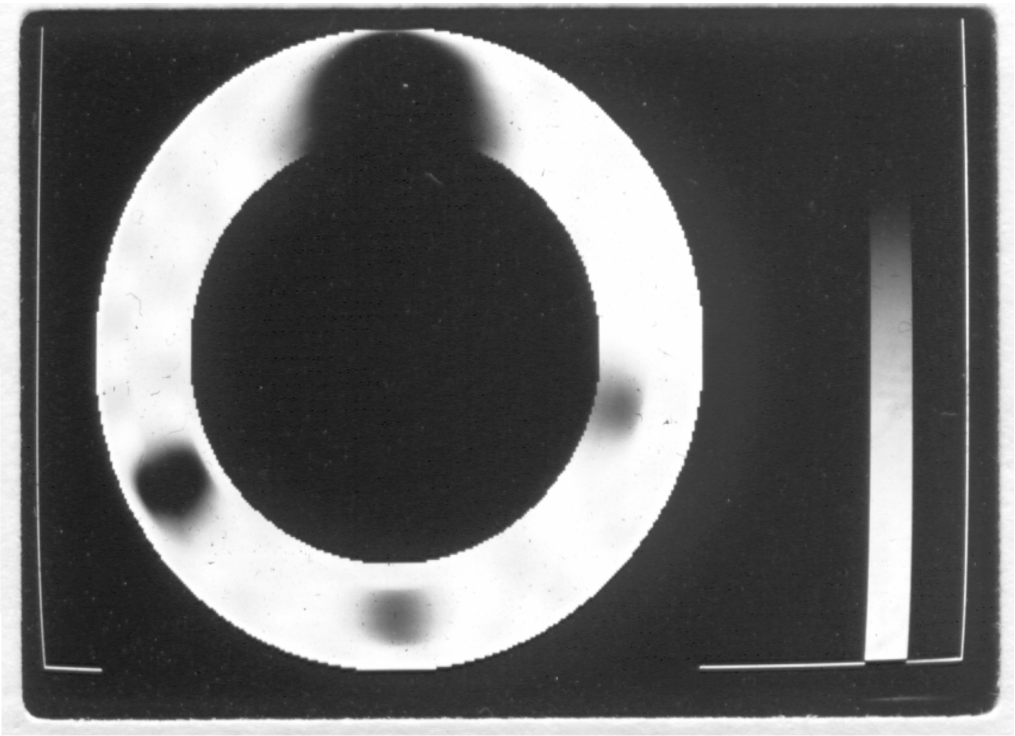
Example: ROI CT. $\longrightarrow$

Reconstructions again:

Exterior reconstruction



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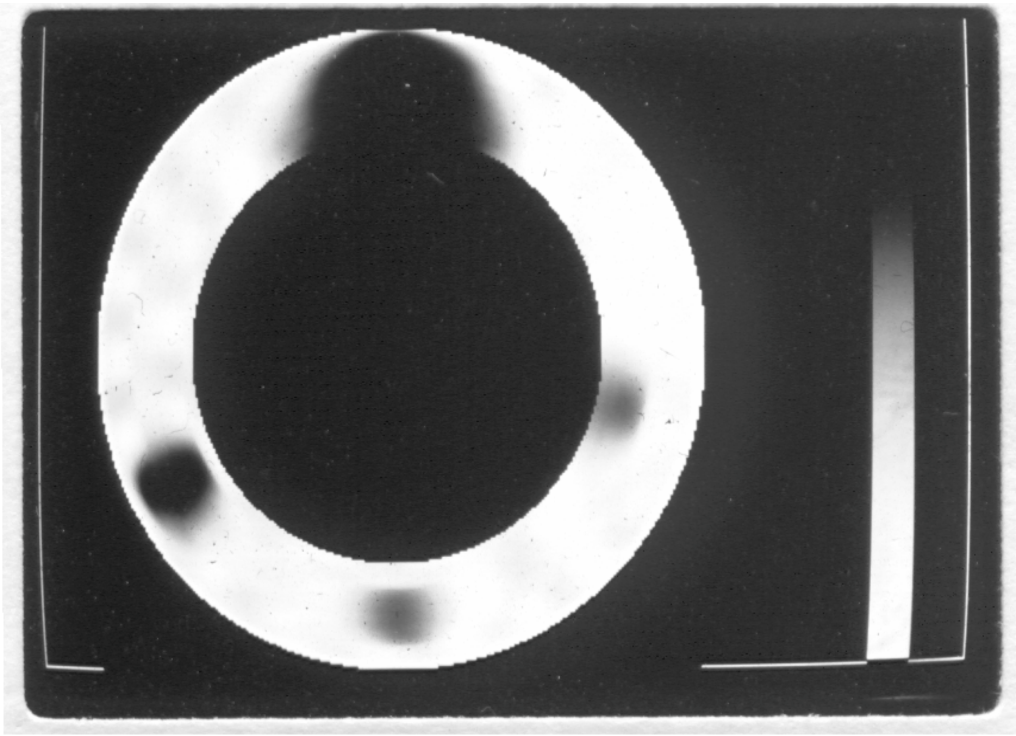
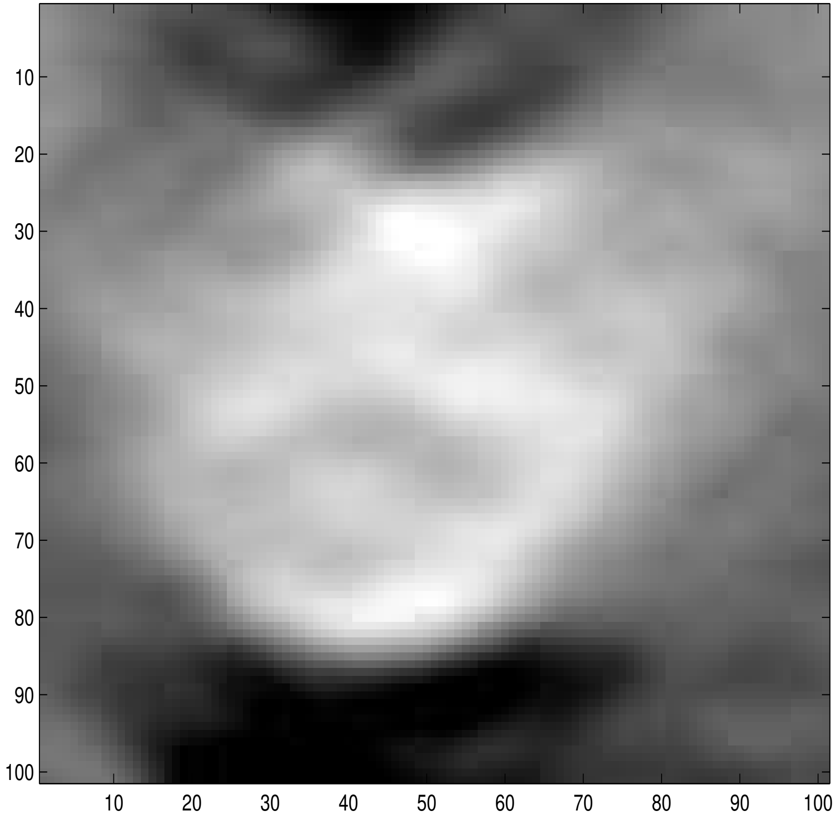


$[30^\circ, 150^\circ]$ Limited Angle ROI Reconstruction



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Lambda Operators:

$$\Lambda_p g(\theta,p) = \sqrt{-\frac{d^2}{dp^2}}\, g = \frac{1}{2\pi}\int_{\tau=-\infty}^{\infty} e^{ip\tau}\, |\tau| \left(\mathcal{F}_{p\to\tau} g\right)(\theta,\tau)d\tau,$$

$$\Lambda_x f = \sqrt{-\Delta} f = \frac{1}{(2\pi)^2}\int_{\xi\in\mathbb{R}^2} e^{ix\cdot\xi}|\xi|\tilde{f}(\xi)d\xi.$$

Filtered Back Projection:

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- Reconstructions look great!
- **Key Idea:** Λ_x is an elliptic ΨDO,
  $\mathrm{WF}_s(\Lambda_x f) = \mathrm{WF}_{s+1}(f)$.

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- R^* is replaced by R^*_α (or soft edge version with enough angles).

Λ_α as a ΨDO:

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The Caveat: However, other details of the object (“invisible” singularities) can be blurred or distorted.