Szemerédi Regularity Lemma and its applications

Miklós Simonovits

miki@renyi.hu

Alfréd Rényi Math. Inst, Budapest

Introduction

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Regularity Lemma is one of the most useful methods in asymptotic graph theory.

Why?

Because many problems are easy for random graphs and Regularity Lemma lets us approximate all dense graphs by randomlike graphs.

Structure

- Introduction: Where does it come from?
  - Coming from number theory
  - Very useful in extremal graph theory
  - Used in geometry
  - used in some parts of Theoretical computer Science
- The lemma itself (Orsay version)
- Why can it be used?
- Some applications in extremal graph theory

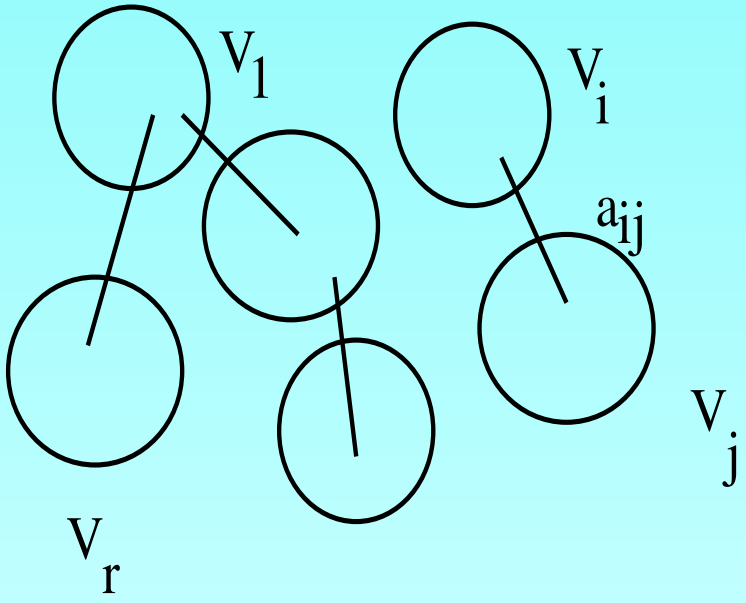
Some references

- E. Szemerédi: Regular partitions of graphs, Colloques Internationaux C.N.R.S. Nº 260 - Problèmes Combinatoires et Théorie des Graphes, Orsay (1976), 399-401.
- J. Komlós, M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, Bolyai Society Mathematical Studies 2, Combinatorics, Paul Erdős is Eighty (Volume 2) Keszthely (Hungary) (1993), Budapest (1996), 295-352.
- Y. Kohayakawa, Szemerédi's regularity lemma for sparse graphs. Foundations of computational mathematics (Rio de Janeiro, 1997), 216–230, Springer, Berlin, 1997.
- R. Diestel, Graph theory. Second edition. Graduate Texts in Mathematics, 173.
- P. Frankl, V. Rödl: The uniformity lemma for hypergraphs. Graphs Combin. 8 (1992), no. 4, 309–312.

But: What is RANDOMLIKE?

Generalised Random Graphs:

r = the number of classes is fixed,
an $r \times r$ "density" matrix $A = (a_{ij})$ is fixed, where
a_{ij}'s are probabilities: $a_{ij} \in [0, 1]$.



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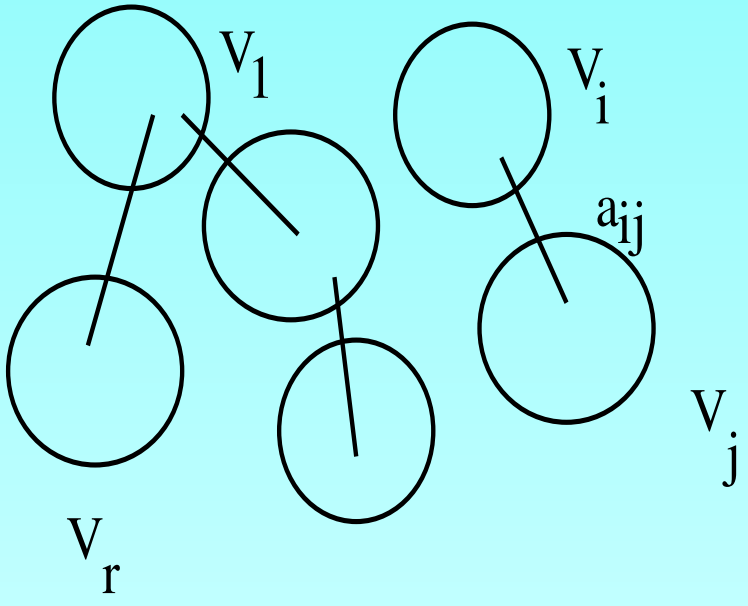
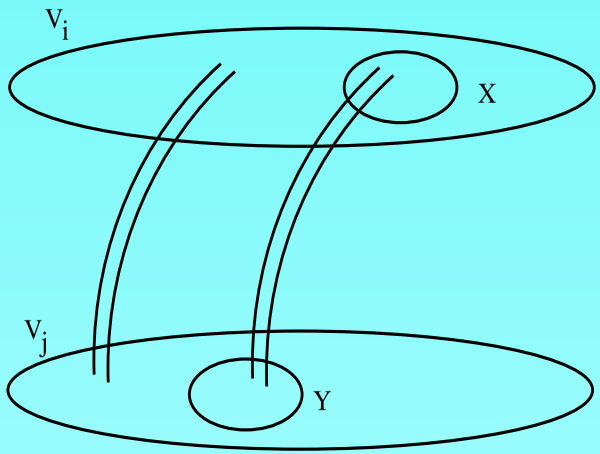


$n = n_1 + \ldots + n_r$.
$V = V_1 \cup \ldots \cup V_r$,
$|V_i| = n_i$

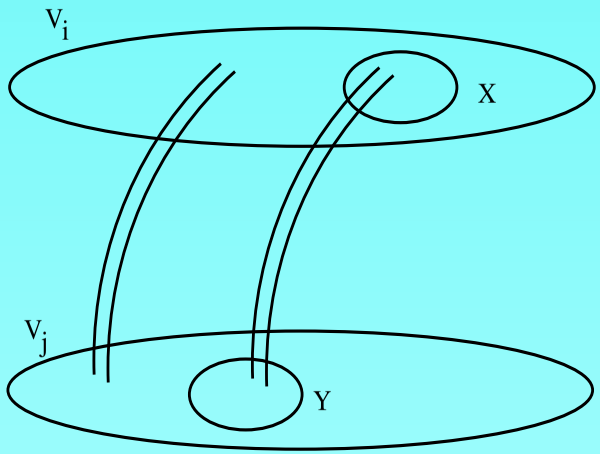
We join the vertex
$x \in V_i$ to $y \in V_j$
with probability a_{ij},
independently.

The density



$$d(X,Y) = \frac{e(X,Y)}{|X| \cdot |Y|}.$$

The density



$$d(X,Y) = \frac{e(X,Y)}{|X| \cdot |Y|}.$$

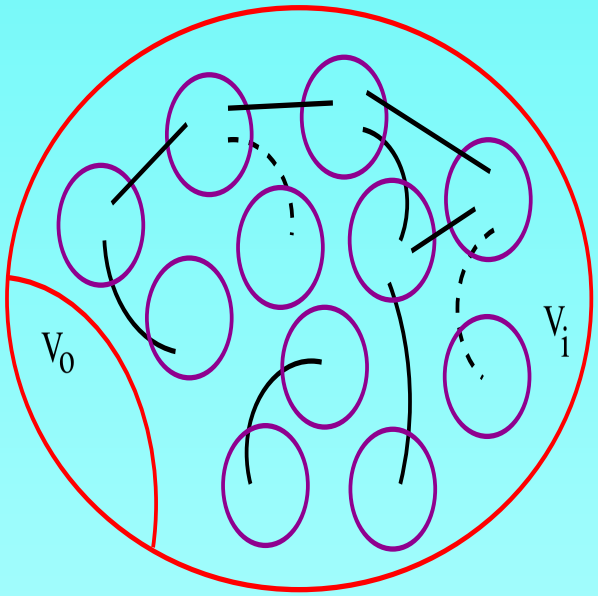
For generalised random graphs: for every $X \subset V_i$ and $Y \subset V_j$ satisfying

$$|X| > \varepsilon |V_i| \quad \text{and} \quad |Y| > \varepsilon |V_j|,$$

We have

$$d(X,Y) \approx d(V_i, V_j) = a_{ij}.$$

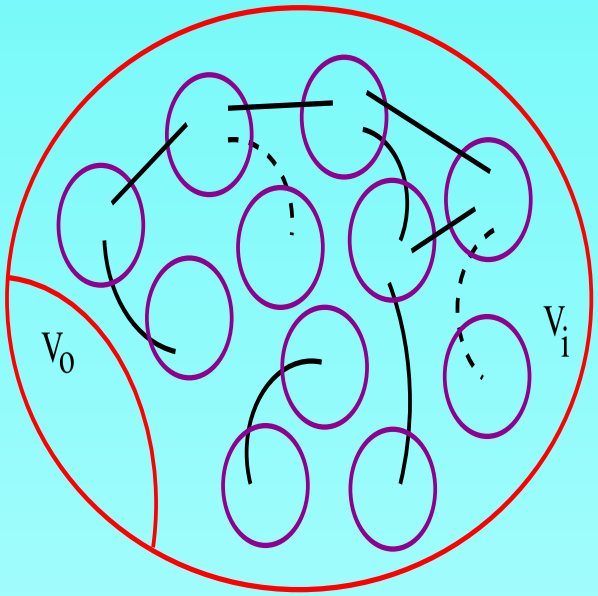
Heuristically



The Regularity Lemma says that every dense graph can be partitioned into a small number of regular pairs and a few leftover edges.

In other words:
approximate by generalised random graphs?

Heuristically



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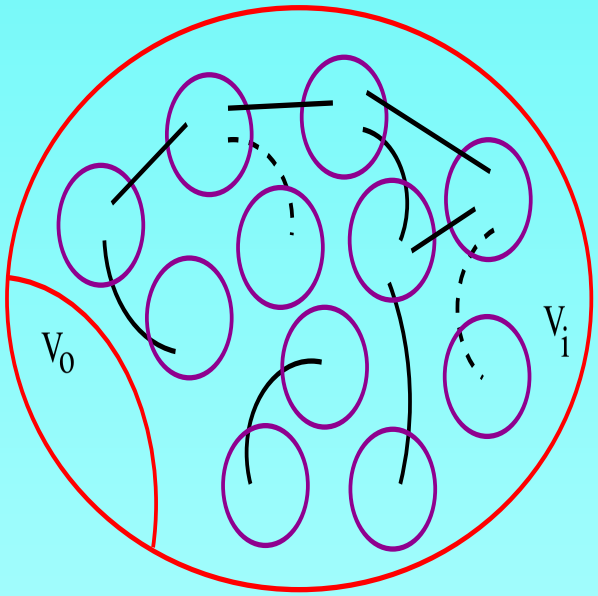

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Actually, V_0 can be redistributed: ...

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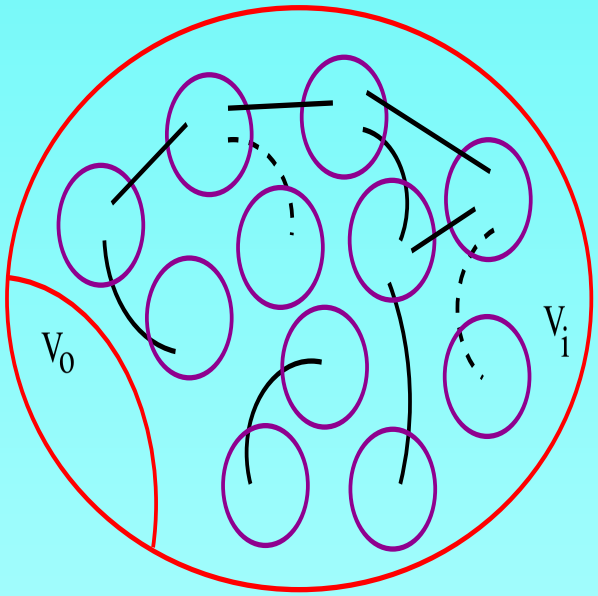

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But what is the meaning of "regular pair" ?

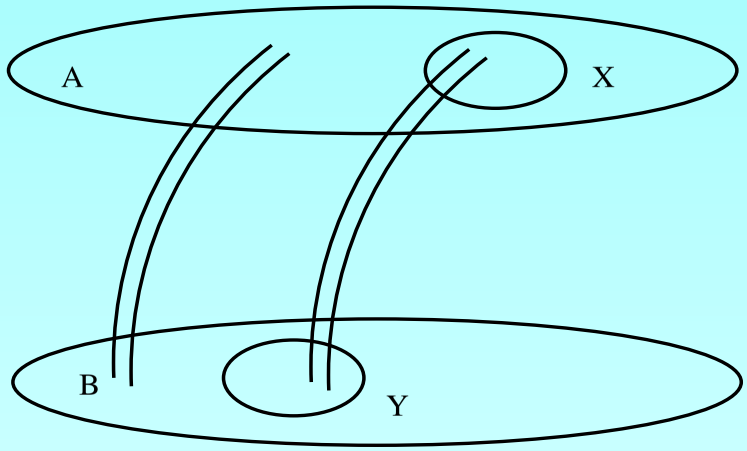
Def: ε-Regular pairs

Let $\varepsilon > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is ε-regular if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \quad \text{and} \quad |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$



Regularity Lemma (Szemerédi 1978)

For every $\varepsilon > 0$ and κ there exist two integers $M(\varepsilon, \kappa)$ and $N(\varepsilon, \kappa)$ with the following property: for every graph G with $n \geq N(\varepsilon, \kappa)$ vertices there is a partition of the vertex set into $k + 1$ classes

$$V = V_0 + V_1 + V_2 + \ldots + V_k$$

such that

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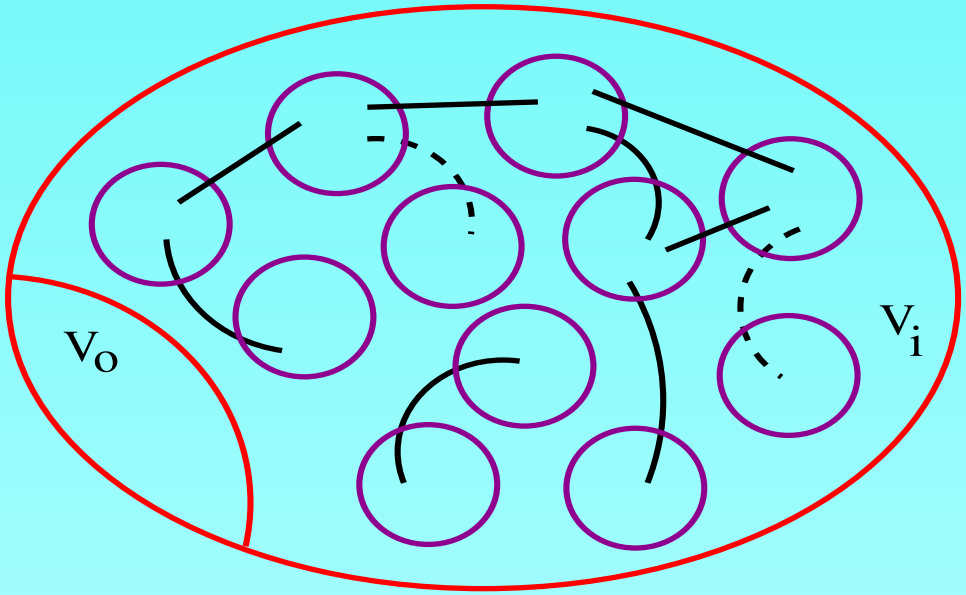
such that

- $\kappa \leq k \leq M(\varepsilon, \kappa)$,
- $|V_0| < \varepsilon n$,
- $|V_1| = |V_2| = \ldots = |V_k|$,
- all but at most εk^2, of the pairs (V_i, V_j) are ε-regular.

The meaning:



- There may be a small uncovered part.
- most of the connections are regular
- very few connections may be irregular.
- But the number of parts is bounded!

Why do we like generalised (?) R.G.s

and why ... Regular partitions?

Because it is easy to build up certain subgraphs in them.
Because if the density between two classes is positive, we can handle them as if they were fully joined.

The cluster graph

Parameters: $\varepsilon, \eta > 0, \eta \geq \varepsilon$.
The vertices of the cluster graph are the classes $V_1, \ldots, V_k$ and the edges are the regular pairs, with density $\geq \eta$.

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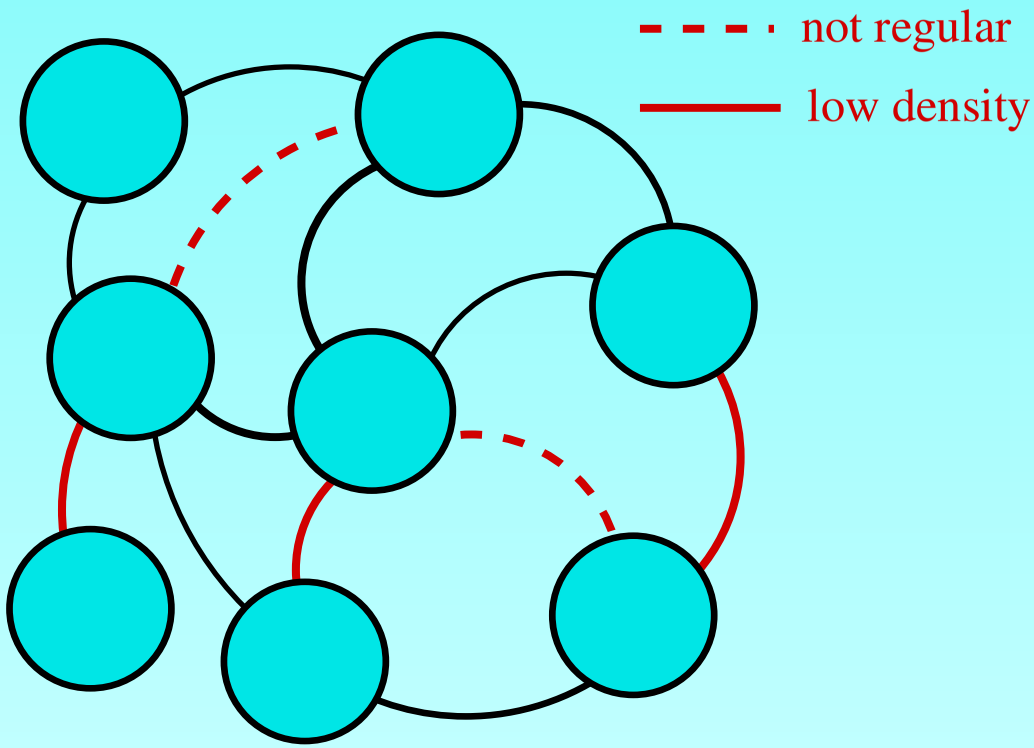

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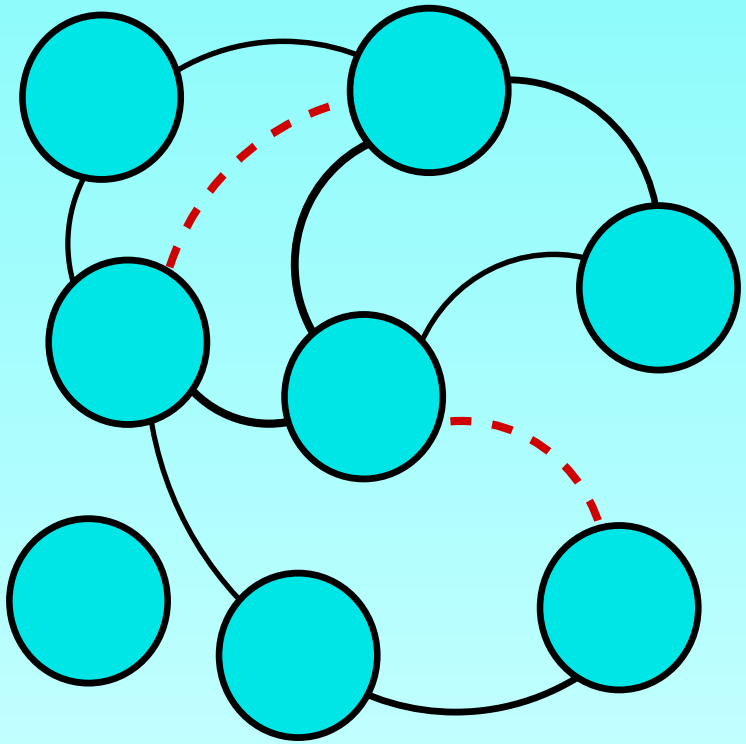

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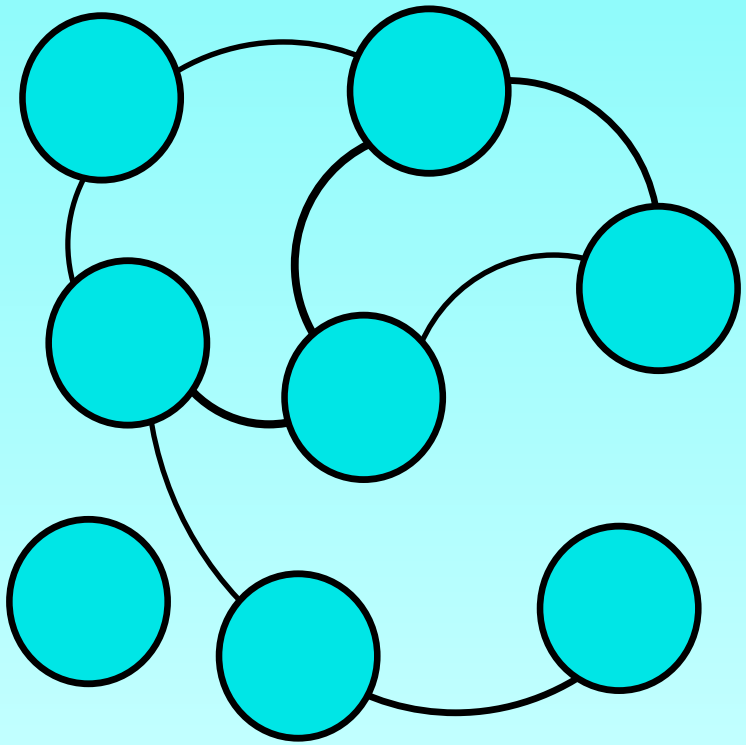

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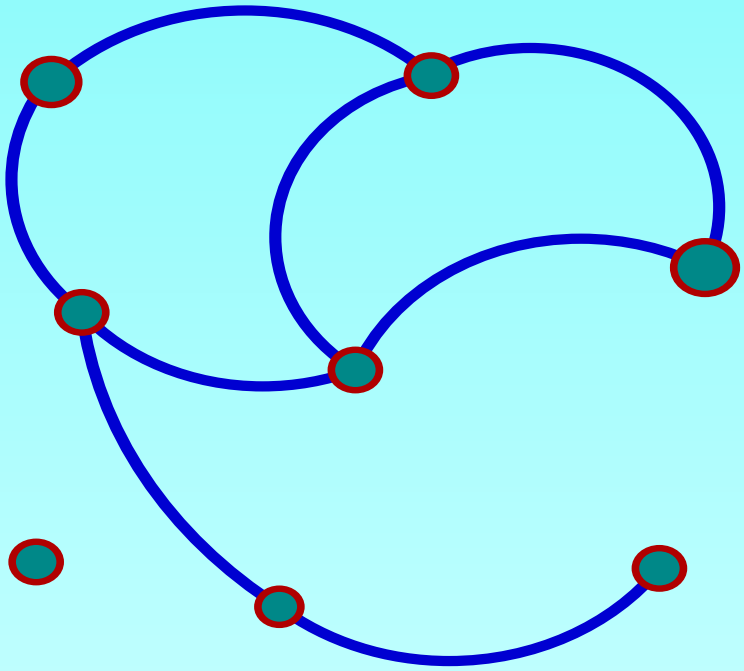

The cluster graph

How to use? Analogy to generating f.

- Have a combinatorial problem
- choose the generating function (more options)
- transform comb. info into analytic info
- solve the analytic problem
- transform it back to the combinatorial problem

The scheme:

- Have a problem, transform it into a graph question.
- choose the parameters carefully
- create a reduced graph.
- transform the original information to the cluster graph
- derive some extra info on the cluster graph
- transform this back to the original situation

The number theoretic origins: How it started?

Theorem. (van der Warden 1927) Let k and t be arbitrary positive integers. If we color the integers in t colors, at least one color-class will contain an arithmetic progression of k terms.

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Erdős-Turán: Is this a density problem?

For fixed k, is it true that for any $\varepsilon > 0$, if in $[0, n]$ we choose a set A of $\geq \varepsilon n$ integers, and $n > n_0(k, \varepsilon)$ then A contains a k-term arithmetic progression?

K. F. Roth solved this for $k = 3$.

$r_k(n)$, Szemerédi 1975

For every integer $k > 2$ and $\varepsilon > 0$ there exists a threshold $n_0 = n_0(k, \varepsilon)$ such that if, for some $n \geq n_0$, $A \subset \{1, \ldots, n\}$ and $|A| > \varepsilon n$, then A must contain an arithmetic progression of k terms.

$$\boxed{r_k(n) = \max \ldots}$$

Szemerédi Thm: $r_k(n) = o(n)$.

Main tool: Szemerédi Regularity lemma, + a very complicated "induction":

. . . Hilbert cubes = generalised arithmetic progressions

Furstenberg: ergodic theoretical solution.

T. Gowers

Erdős's Conjecture

Is it true that if $A = \{a_1, a_2, \ldots\}$ is a sequence of integers and

$$\sum \frac{1}{a_i} = \infty,$$

then A contains arbitrary long arithmetic progressions?

In particular:

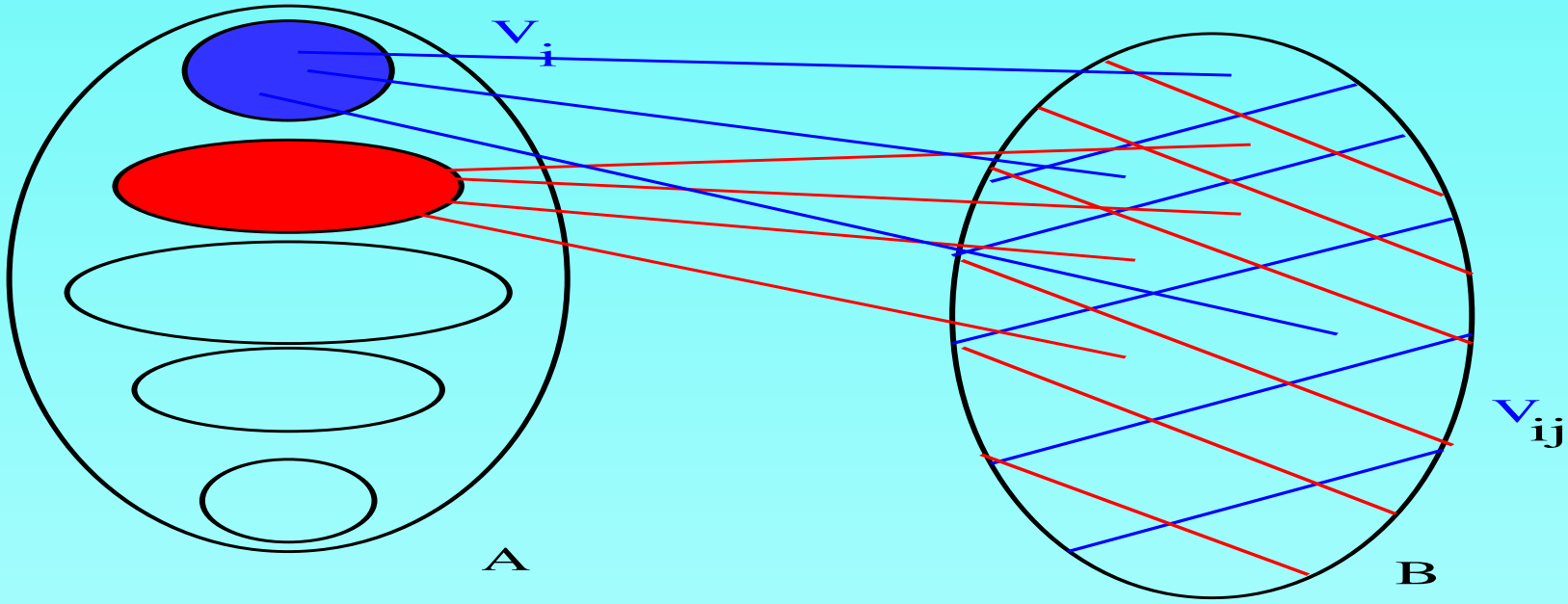
Is it true that the sequence of primes contains arbitrary long arithmetic progressions?

A historical detour: The Original Version

For every $\varepsilon_1, \varepsilon_2, \delta, \varrho, \sigma > 0$ there exist n_0, m_0, N, M, such that for every bipartite graph (A, B, E) with $|A| = n \geq N$, $|B| = m \geq M$ there exist sets $V_i \subset A$ $(i < n_0)$ and $V_{ij} \subset B$ $(i < n_0, j < m_0)$ for which

- $|A - \cup_{i<n_0} V_i| < \varrho n$, and $|B - \bigcup_{j<m_0} V_{ij}| < \sigma m$, for every $i < n_0$, and
- for every $i < n_0,\ j < m_0$ and for every $T \subset V_i,\ S \subset V_{ij}$, if $|T| > \varepsilon_1 |V_i|$ and $|S| > \varepsilon_2 |V_{ij}|$, then $d(T, S) > d(V_i, V_{ij}) - \delta$ and
- $|N(u) \cap V_i| < (d(V_i, V_{ij}) + \delta)|V_i|$ for each $u \in V_{ij}$.

Bipartite, many partitions:



- Applies to bipartite graphs $G(A, B, E)$.
- We have a subdivision V_i of A
- For each V_i we have a separate subdivision $B = \cup U_{ij}$
- . . . are ε-regular connections.

Sketch of the proof of R.L.

Given a partition $\mathcal{P} = \{V_0, \ldots, V_k\}$,

$$\mathbf{ind}(\mathcal{P}) := \frac{1}{\binom{k}{2}} \sum_{i<j} d(U_i, U_j)^2 < \frac{1}{2}.$$

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Using a "defect" form of the Cauchy-Schwarz inequality, if the partition does not satisfy the assertion of Regularity Lemma, then we find a "better" partition with

$$\mathbf{ind}(\mathcal{P}*) > \mathbf{ind}(\mathcal{P}) + \frac{\varepsilon^5}{120}.$$

What is the defect form?

Lemma. (Improved Cauchy-Schwarz inequality)

If for the integers $0 < m < n$,

$$\sum_{k=1}^{m} X_k = \frac{m}{n} \sum_{k=1}^{n} X_k + \delta,$$

then

$$\sum_{k=1}^{n} X_k^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} X_k \right)^2 + \frac{\delta^2 n}{m(n-m)}.$$

Iterating, but how many times?

We have: If $\mathcal{P}_0$ is not good, $\rightarrow \mathcal{P}_1 \ldots$:

$$\mathbf{ind}(\mathcal{P}_{i+1}) > \mathbf{ind}(\mathcal{P}_i) + \frac{\varepsilon^5}{120}.$$

Iterating ℓ times we get a contradiction in c/ε^5 steps:

$$\mathbf{ind}(\mathcal{P}_\ell) > \mathbf{ind}(\mathcal{P}_0) + \ell\frac{\varepsilon^5}{120}.$$

since $\mathbf{ind}(\mathcal{P}) \leq 1/2$.

The obtained upper bound on
the number of classes is HUGE!!!

Turán type extremal problems

$\mathcal{L}$ = the family of forbidden graphs

G is $\mathcal{L}$ -free, if $L \not\subseteq G$ for any $L \in \mathcal{L}$.
$\forall L \in \mathcal{L}$, $e(L) > 0$.

$\mathcal{P}(n, \mathcal{L})$= class of $\mathcal{L}$-free graphs with vertex set $[n]$.

$$\mathbf{ext}(n, \mathcal{L}) = \max\{e(G) : G \in \mathcal{P}(n, \mathcal{L})\}$$

and

$$\mathbf{EXT}(n, \mathcal{L}) = \{G : G \in \mathcal{P}(n, \mathcal{L}), \quad e(G) = \mathbf{ext}(n, \mathcal{L})\}.$$

Extremal Graph Theory:

For a given family $\mathcal{L}$, determine or estimate $\mathbf{ext}(n, \mathcal{L})$, and describe $\mathbf{EXT}(n, \mathcal{L})$ as $n \to \infty$.

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Erdős's theorem on C_4 (1938):

$$\mathbf{ext}(n, C_4) \leq Cn\sqrt{n}.$$

Turán's theorem, (1941)
Erdős-Stone theorem (1946):

$$\mathbf{ext}(n, K_{p+1}(m, \ldots, m)) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

Extremal Graph Theory:

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Theorem. (Erdős-Stone-Simonovits Theorem) If $\mathcal{L}$ is a family of graphs,

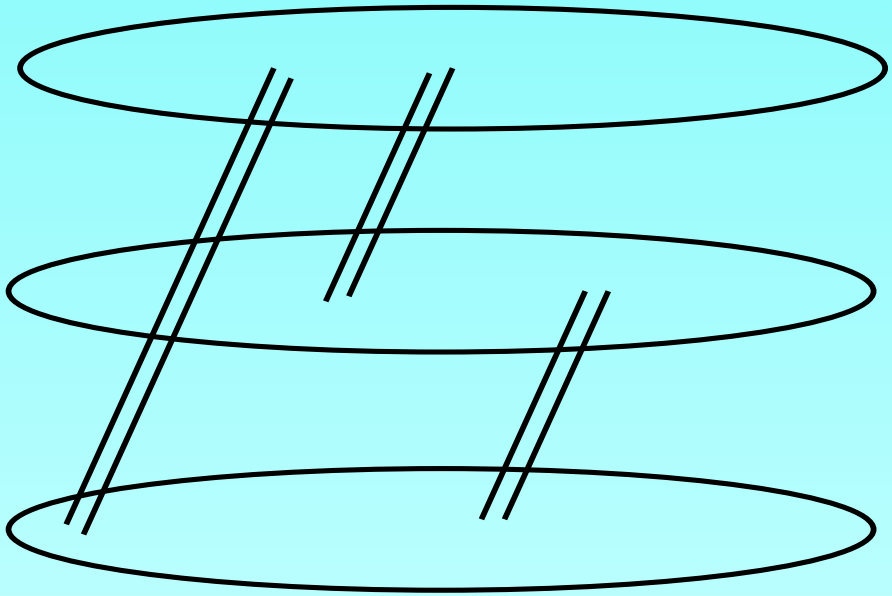
$$p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1.$$

then

$$\mathbf{ext}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right) \binom{n}{2} + o(n^2).$$

Motivation: Turán's graph

Turán's graph $T_{n,p}$:
n vertices are partitioned into p classes as equally as possible and x, y are joined iff they belong to different classes.



Induced Matching

Problem. (Brown-Erdős-T. Sós, 1973) Consider r-uniform hypergraphs. Let

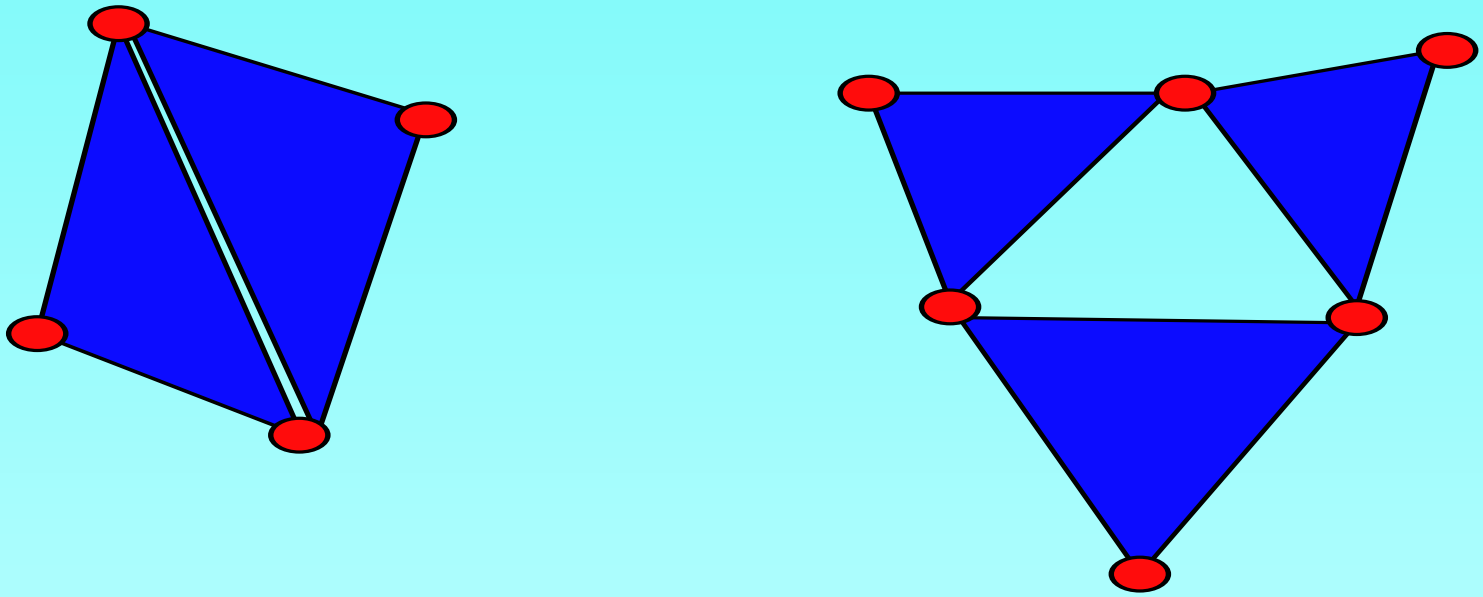
$$f(n,k,\ell) := \mathbf{ext}(n, \mathcal{L}^{(r)}_{k,\ell}).$$

Determine or estimate $f(n,k,\ell)$.

The first difficult case is Ruzsa-Szemerédi theorem.

Excluded graphs for (6,3)

$f(n, 6, 3)$ where we prohibit 6 vertices with 3 triplets.
No pair (x, y) is in 2 triplets, otherwise ...



$$\Longrightarrow \quad f(n, 6, 3) \leq \tfrac{1}{3}\binom{n}{2}.$$

Question: $f(n, 6, 3) = o(n^2)$?

Ruzsa-Szemerédi: $f(n, 6, 3) = o(n^2)$.

This is one of the first applications of the (old) Regularity Lemma.

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Ruzsa-Szemerédi Theorem.

$(*)$ $$n \cdot r_3(n) \leq f(n,6,3) = o(n^2).$$

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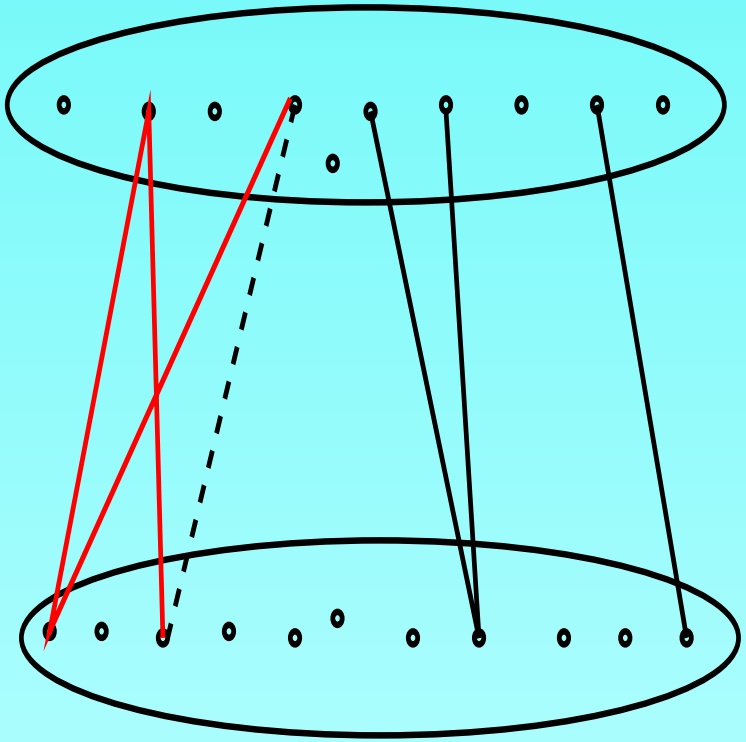
Related results, e.g., Frankl-Rödl.

Simon-Murty Conjecture



$f(n, 6, 3) = o(n^2)$
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Füredi's solution of the Simon-Murty conjecture:

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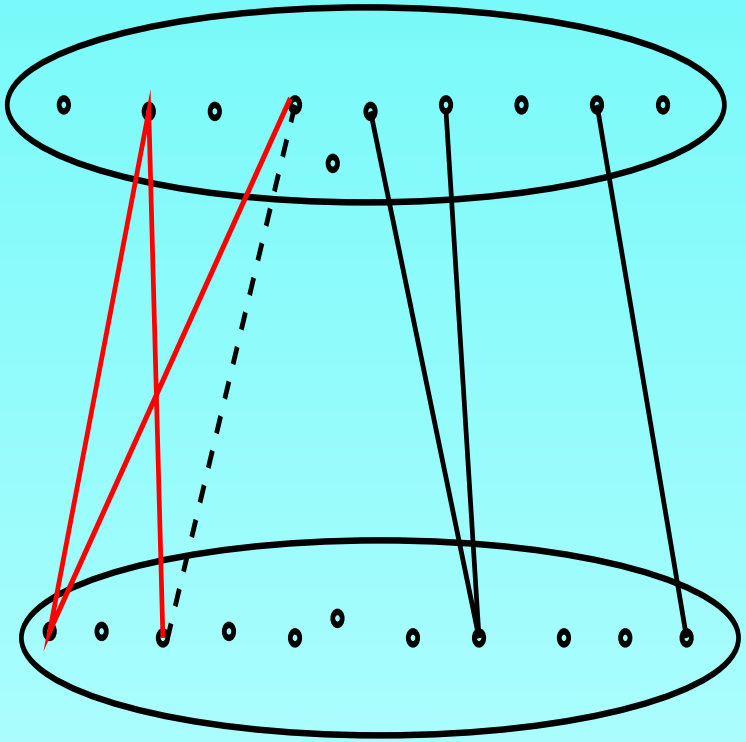

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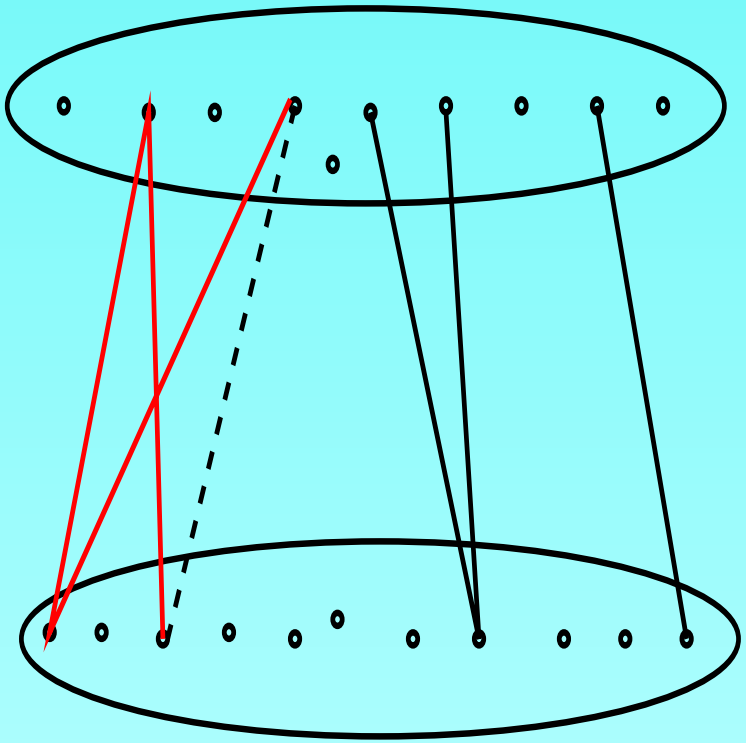
Füredi's solution of the Simon-Murty conjecture:

Is it true that if $\mathrm{diam}(G_n) = 2$ but deleting any edge from G_n the diameter increases, then $e(G_n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Simon-Murty Conjecture



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is often applicable in other cases, e.g.,

Füredi's solution of the Simon-Murty conjecture:

Is it true that if $\mathrm{diam}(G_n) = 2$ but deleting any edge from G_n the diameter increases, then $e(G_n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Sharpness is shown by $T_{n,2}$

Regularity Lemma in Turán-Ramsey theory

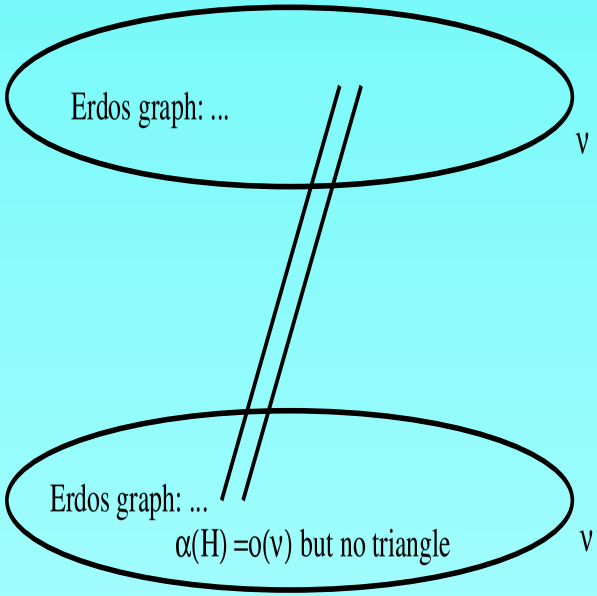
$\alpha(G) =$ independence number

Problem. (Ramsey-Turán)

$$\mathbf{RT}(n, L, m) := \max_{\substack{G_n \not\supseteq L \\ \alpha(G_n) \leq m}} e(G_n).$$

Motivation comes from applying Turán type extremal graph theorems to distance distributions.

Simplest interesting case is $\mathbf{RT}(n, K_\ell, o(n)) = ?$



Trivially:

$$\mathbf{RT}(n, K_3, m) \leq \tfrac{1}{2}nm,$$

$$\Longrightarrow$$

$$\mathbf{RT}(n, K_3, o(n)) = o(n^2)$$

On the other hand, Erdős-Sós:

$$\mathbf{RT}(n, K_{2p+1}, o(n)) = e(T_{n,p}) + o(n^2), \qquad \Longrightarrow$$

$$\mathbf{RT}(n, K_5, o(n)) = \left\lfloor \frac{n^2}{4} \right\rfloor .$$

So K_4 was the first open case!

Szemerédi thm:

$$\mathbf{RT}(n, K_4, o(n)) \leq \frac{n^2}{8} + o(n^2)$$

A Bollobás-Erdős construction shows that

$$\mathbf{RT}(n, K_4, o(n)) \geq \frac{n^2}{8} - o(n^2).$$

Generalizations:
Erdős-Hajnal-Sós-Szemerédi,
Erdős-Hajnal-Sim.- Sós-Szemerédi,

Using the Reduced graph

Definition. (Reduced Graph) $\eta, \varepsilon > 0$ are given, we take the ε-regular partition and form a graph where the vertices are $V_1, \ldots, V_k$ and the edges are those regular pairs where $d(V_i, V_j) > \eta$.

Why is this good for us? One reason why the Regularity Lemma can be used here is that if we know that the reduced graph contains some graph L, (e.g., a K_3), then using the $o(n)$-condition we can guarantee a larger subgraph (e.g., a K_4) in the original graph.

The upper estimate: sketch

- Fix any $\varepsilon > 0$ and $\eta = \sqrt{\varepsilon}$. Cluster-size: m.
- Create the reduced graph $H_k \leftarrow (\varepsilon, \eta)$
- Show that $K_3 \not\subseteq H_k$.
- Show: $d(V_i, V_j) \leq \frac{1}{2} + \eta$.

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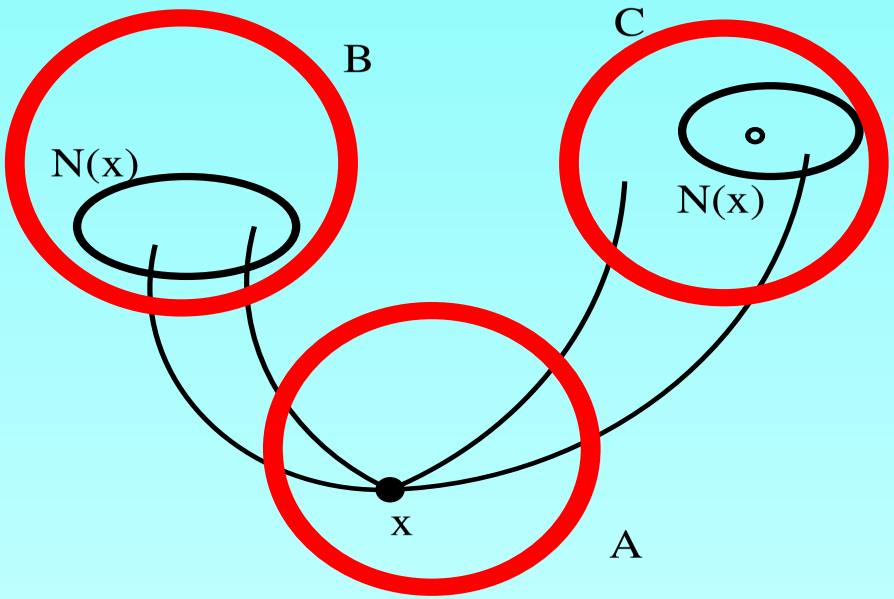

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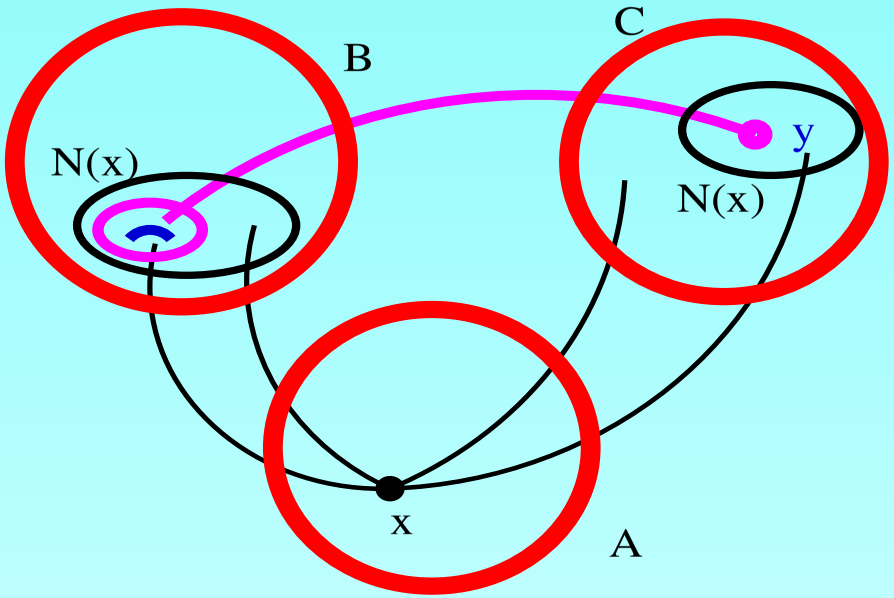

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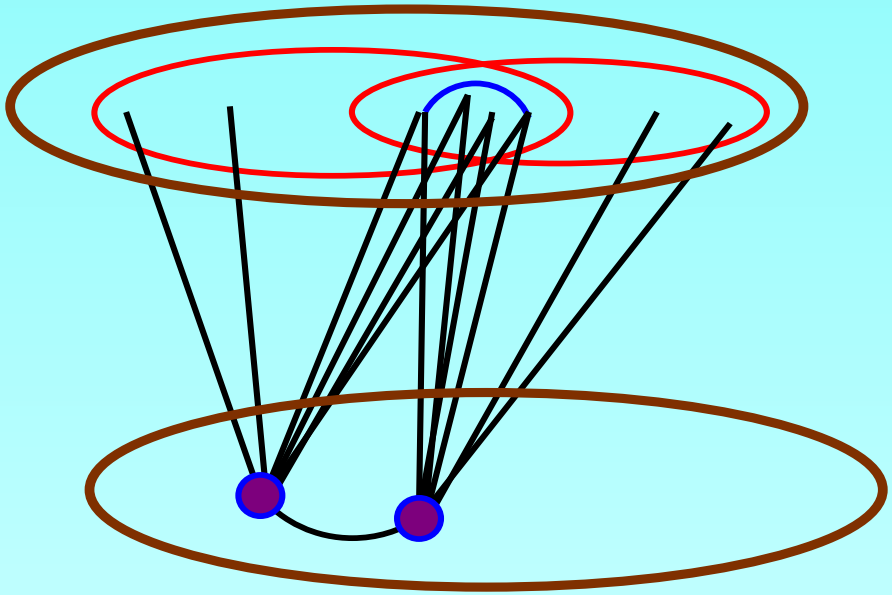

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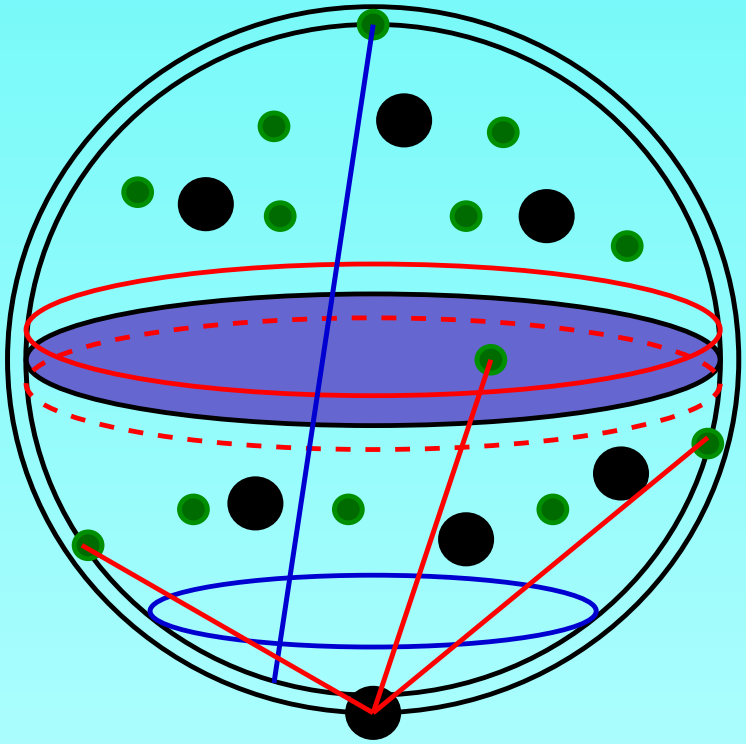
These yield:

$$e(G_n) \leq (\tfrac{1}{2} + \eta)m^2 \cdot \left\lfloor \tfrac{k^2}{4} \right\rfloor + 8(\varepsilon + \eta)(km)^2 \approx \frac{n^2}{8}.$$

Surprise: High dimensional constructions?



We have a high dimensional sphere, put on it $\frac{n}{2}$ black and $\frac{n}{2}$ green points uniformly.

Join two vertices of the same color if they are at distance $\geq 2 - \frac{\varepsilon}{\sqrt{d}}$ and join points of different colors if their distance is $\geq \sqrt{2} - \frac{\varepsilon}{\sqrt{d}}$.

$\alpha(G_n) = o(n)$ follows from an isoperimetric inequality (dim $\to \infty$)

Many colors

Theorem. (Many-Color Regularity Lemma) For any $\varepsilon > 0$ and integers r, κ there exists an M such that if the edges of a graph G_n are r-colored then the vertex set $V(G_n)$ can be partitioned into sets $V_0, V_1, \ldots, V_k$ – for some $\kappa \leq k \leq M$ – so that $|V_0| < \varepsilon n$, $|V_i| = m$ (the same) for every $i \geq 1$, and all but at most εk^2 pairs (V_i, V_j) satisfy the following regularity condition: for every $X \subset V_i$ and $Y \subset V_j$ of size $|X|, |Y| > \varepsilon m$ we have

$$|d_\nu(X, Y) - d_\nu(V_i, V_j)| < \varepsilon \qquad (\nu = 1, \ldots, r).$$

This theorem is proved exactly as the original one and can be applied to many Ramsey-Turán problems.

Quasirandomness and Regularity Lemma

Quasi-randomness

Thomason
Wilson, Frankl-Rödl
$\Longrightarrow$ Chung-Graham-Wilson

. . . and the regularity

$\Longrightarrow$ Sim. T. Sós
The Regularity Lemma yields a quasi-regular property, which makes the phenomena transparent.

Chung-Graham-Wilson thm.

For any graph sequence (G_n) the following properties are equivalent:

$\mathcal{P}_1(\nu)$: for fixed ν, for all graphs H_ν

$$N_G^*(H_\nu) = (1 + o(1))n^\nu 2^{-\binom{\nu}{2}}.$$

$\mathcal{P}_5$: For each subset

$$X \subset V, \quad |X| = \lfloor \frac{n}{2} \rfloor$$

$$e(X) = \left(\frac{1}{16} n^2 + o(n^2) \right).$$

$\mathcal{P}_6$: $\sum_{x,y \in V} \left| |S(x,y)| - \frac{n}{2} \right| = o(n^3)$.

Details: the property $\mathcal{P}_s$

Definition. ($\mathcal{P}_S$) For every $\varepsilon > 0$ and κ there exist two integers, $k(\varepsilon, \kappa)$ and $n_0(\varepsilon, \kappa)$ such that for $n \geq n_0$, G_n has a regular partition with parameters ε and κ and k classes $U_1, \ldots, U_k$, with $\kappa \leq k \leq k(\varepsilon, \kappa)$, so that

$$(U_i, U_j) \quad \text{is} \quad \varepsilon- \quad \text{regular, and} \quad \left| d(U_i, U_j) - \frac{1}{2} \right| < \varepsilon$$

holds for all but at most εk^2 pairs (i, j), $1 \leq i, j \leq k$.

Theorem. (Sim. Sós) $\mathcal{P}_S$ is quasi-random.

Algorithmic Aspects

Alon-Duke-Rödl-Yuster

It is NP-hard to decide if a partition satisfies the regularity lemma, for an $\varepsilon > 0$.

Algorithmic Aspects

Alon-Duke-Rödl-Yuster

It is NP-hard to decide if a partition satisfies the regularity lemma, for an $\varepsilon > 0$.

On the other hand,
one can construct another partition, ε-regular, in polynomial time.
Surprising?

An application:
Duke, Lefmann, Rödl: A fast approximation algorithm for computing the frequencies of subgraphs in a given graph. SIAM J. Comput. 24 (1995), no. 3, 598–620.

Sparse Regularity Lemma

Why is it needed?

Kohayakawa-Rödl

Hypergraph Regularity Lemma

Frankl-Rödl
Fan Chung

What is the “techical problem”?

... What we have in the reduced graph we do not necessarily have in the graph

When is it needed?

Hypergraph extensions of ordinary problems

Random references

- Kohayakawa, Yoshiharu; Rödl, Vojtěch; Skokan, Jozef Hypergraphs, quasi-randomness, and conditions for regularity. J. Combin. Theory Ser. A 97 (2002), no. 2, 307–352.
- Czygrinow, Andrzej; Rödl, Vojtech An algorithmic regularity lemma for hypergraphs. SIAM J. Comput. 30 (2000), no. 4, 1041–1066 (electronic).
- Czygrinow, A.; Poljak, S.; Rödl, V. Constructive quasi-Ramsey numbers and tournament ranking. SIAM J. Discrete Math. 12 (1999), no. 1, 48–63 (electronic).
- Rödl, Vojtech; Ruciński, Andrzej Ramsey properties of random hypergraphs. J. Combin. Theory Ser. A 81 (1998), no. 1, 1–33.

Many applications in number theory

Blow-Up Lemma: Komlós-Sárközy-Szemerédi

What is the extra feature?

The meaning of blow-up lemma: regular pairs behave like complete bipartite graphs from the point of view of embedding bounded degree subgraphs.

It gives a spanning subgraph.

- Komlós: The blow-up lemma. Recent trends in combinatorics (Mátraháza, 1995), 161–176, Cambridge Univ. Press, Cambridge, 2001.
- Komlós, G. Sárközy, Szemerédi: Proof of the Seymour conjecture for large graphs. Ann. Comb. 2 (1998), no. 1, 43–60.
- Komlós, G. Sarkozy, Szemerédi: An algorithmic version of the blow-up lemma. Random Struct. Algor. 12 (1998), no. 3, 297–312.
- Rödl, Rucinski . . .