Selecting unconditional basic sequences

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Let X be a Banach space. Let $X^{(\mathbb{N})} \subset X^{\mathbb{N}}$ be the set of linearly independent sequences. Given $\xi = (x_i) \in X^{(\mathbb{N})}$ and $\lambda = (\lambda_i) \in \ell_\infty$, let $g_\xi(\lambda) \in [0, \infty]$ denote the $\inf$ of those M such that

$$\| \sum_i \lambda_i a_i x_i \| \leq M \| \sum_i a_i x_i \| \quad \forall a \in c_{00}.$$

Note that $g_\xi(\lambda) \geq \|\lambda\|_\infty$.

The sequence $\xi = (x_n)$ is said to be an *unconditional basic sequence* if it is M-unconditional for some $M < \infty$, i.e., if $g_\xi(\lambda) \leq M \|\lambda\|_{\ell_\infty}$ for all $\lambda \in \ell_\infty$.

Let $\mathcal{M}_\xi = \{\lambda \in \ell_\infty : g_\xi(\lambda) < \infty\}$. Then ξ is unconditional iff $\mathcal{M}_\xi = \ell_\infty$.

Put $\mathcal{M}_X = \cup_{\xi \in X^{(\mathbb{N})}} \mathcal{M}_\xi$ (the set of multipliers for sequences in X).

The Gowers' dichotomy theorem says that X contains a subspace Y such that either Y contains an unconditional basic sequence or Y is hereditarily indecomposable (H.I.), i.e., no subspace of Y is decomposable.

The following result is a slight extension.

Theorem 1. *Let X be a Banach space. Then the following are equivalent*

(i) $\forall Y \subseteq X \ \exists \xi \in Y^{(\mathbb{N})} \quad \mathcal{M}_\xi = \ell_\infty$,

(ii) $\forall Y \subseteq X \quad ((-1)^n)_{n=1}^\infty \in \mathcal{M}_Y$,

(iii) $\forall Y \subseteq X \quad \mathcal{M}_Y \setminus c \neq \emptyset$.

Clearly, $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$, while $(iii) \Rightarrow (ii)$ is a minor supplement.

The implication $(ii) \Rightarrow (i)$ is the main point as it is in the Gowers' dichotomy theorem. Indeed, (ii) means exactly that no subspace of X is H.I. For, $((-1)^n) \in \mathcal{M}_Y$ iff $\exists Z \subseteq Y$ such that $Z \sim Z_1 \oplus Z_2$, i.e., Z is a decomposable subspace of Y. Hence X fails (ii) iff some $Y \subseteq X$ is H.I.

By $(ii) \Rightarrow (i)$, if X has no subspace Y that fails (ii), then X contains a ξ with $\mathcal{M}_\xi = \ell_\infty$ (ξ is an unconditional basic sequence). So does every subspace of X.

We shall present a new proof which can be adapted to obtain some further results.

For $\xi \in X^{(\mathbb{N})}$ we let $\mathrm{uc}(\xi)$ (or $\mathrm{uc}_{\mathbb{C}}(\xi)$ if X is a space over the field $\mathbb{C}$) denote its unconditionality constant, i.e., the number

$$\sup\{g_\xi(\lambda) : \|\lambda\|_{\ell_\infty} = 1\}.$$

Let $g_X : \ell_\infty \to [0, \infty]$ be defined by

$$g_X(\lambda) = \inf_{\xi \in X^{(\mathbb{N})}} g_\xi(\lambda).$$

There is a quantitative form of $(ii) \Rightarrow (i)$.

Let $\mathbf{m} \in \ell_\infty$, $\mathbf{m}_n := (-1)^n$ for each n. Also fix a complex sequence $\mathbf{u} \in \ell_\infty$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \dots\}$ is a dense subset of the unit circle and each $\mathbf{u}_i$ is repeated infinitely many times.

Corollary 1. *If X is a real Banach space, then*

$$\inf_{\xi \in X^{(\mathbb{N})}} \mathrm{uc}(\xi) \leq \sup_{Y \subseteq X} g_Y(\mathbf{m}).$$

If X is a complex Banach space, then

$$\inf_{\xi \in X^{(\mathbb{N})}} \mathrm{uc}_{\mathbb{C}}(\xi) \leq \sup_{Y \subseteq X} g_Y(\mathbf{u}) \leq 2 \sup_{Y \subseteq X} g_Y(\mathbf{m}).$$

Moreover, if $g_Y(\mathbf{m}) = g_X(\mathbf{m})$ (resp. if $g_Y(\mathbf{u}) = g_X(\mathbf{u})$) for each $Y \subseteq X$, then one has $\inf = \sup$ in the respective estimate.

All the preceding theorems are also true in the context of operator spaces, provided that the operator norm $\|\cdot\|$ is replaced by the completely bounded $\|\cdot\|_{\mathrm{cb}}$.

Notation and preliminaries

Let X be a Banach space, $\dim X = \infty$. Let $\mathcal{G}(X)$ denote the Grassmann set of all closed linear subspaces of X. Also let

$$\mathcal{G}_{\text{fin}}(X) = \{E \in \mathcal{G}(X) \,:\, \dim E < \infty\},$$

$$\mathcal{G}_{\infty}(X) = \mathcal{G}(X) \setminus \mathcal{G}_{\text{fin}}(X).$$

Definition 1. *A subset S of the Banach space X is said to be* plentiful *in X if $\forall Y \in \mathcal{G}_{\infty}(X)\, \exists Z \in \mathcal{G}_{\infty}(Y)$ such that $Z \subseteq S$.*

Clearly, the intersection of finitely many plentiful subsets of X is plentiful in X, too.

If E, $F \in \mathcal{G}_{\text{fin}}(X)$, then $E+F \in \mathcal{G}_{\text{fin}}(X)$, hence $\mathcal{G}_{\text{fin}}(X)$ is an Abelian semigroup.

If $E \in \mathcal{G}_{\text{fin}}(X)$ then $E+U \in \mathcal{G}_{\infty}(X)$ for $U \in \mathcal{G}_{\infty}(X)$, hence $\mathcal{G}_{\text{fin}}(X)$ acts on the set $\mathcal{G}_{\infty}(X)$.

For each positive $k \in \mathbb{N}$, there are analogous operations on the semigroup $\mathcal{G}_{\text{fin}}(X)^k$ and the set $\mathcal{G}_{\infty}(X)^k$.

We let $\vec{0}_k$ denote the neutral element $(\{0\})_{i=1}^k$ of the semigroup $\mathcal{G}_{\text{fin}}(X)^k$.

If $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}(X)^k$, $w \in X$ and $1 \leq i \leq k$, we let $\vec{U} +_i w$, denote the element $(V_1, \ldots, V_k) \in \mathcal{G}(X)^k$ such that $V_j = U_j$ for $j \neq i$ and $V_i = \operatorname{span}(U_i \cup \{w\})$.

The topology on $\mathcal{G}_{\mathrm{fin}}(X)$ is given by the Hausdorff metric ρ, i.e., if E, $F \in \mathcal{G}_{\mathrm{fin}}(X)$, then

$$\rho(E, F) = \inf\{t \geq 0 \,:\, B_E \subseteq B_F + tB_X \\ \text{and} \;\; B_F \subseteq B_E + tB_X\},$$

where $B_Z = \{z \in Z \,:\, \|z\| \leq 1\}$ denotes the closed unit ball of Z. It is well known that, if the Banach space X is separable, then so is the metric space $(\mathcal{G}_{\mathrm{fin}}(X), \rho)$.

We let I_Y denote the identity operator on the Banach space Y.

Selecting sequences in φ-stable spaces

Let $k \in \mathbb{N}$, $k \geq 1$, be fixed and let $\varphi : \mathcal{G}_\infty(X)^k \to [1,\infty]$ be non-decreasing, i.e.,

$$\varphi(\vec{U}) \leq \varphi(\vec{V})$$

whenever $\vec{U}$, $\vec{V} \in \mathcal{G}_\infty(X)^k$ and $U_i \subseteq V_i$ for each i. Let $\tilde{\varphi} : \mathcal{G}_{\text{fin}}(X)^k \times \mathcal{G}_\infty(X) \to [1,\infty]$ be defined by the formula

$$\tilde{\varphi}(\vec{E}, Y) = \inf\{\varphi(\vec{E} + \vec{U}) \,:\, \vec{U} \in \mathcal{G}_\infty(Y)^k\}. \tag{1}$$

Definition 2. *X is said to be φ-*stable *if $\tilde{\varphi}(\vec{E}, Y) = \tilde{\varphi}(\vec{E}, X)$ for each $\vec{E} \in \mathcal{G}_{\text{fin}}(X)^k$ and $Y \in \mathcal{G}_\infty(X)$.*

Lemma 1. *If the space X is φ–stable and $\tilde{\varphi}(\vec{E}, X) < d$, then the set*

$$\{w \in X \,:\, \tilde{\varphi}(\vec{E} +_i w, X) < d\}$$

is plentiful in X for $1 \leq i \leq k$.

Proof. Let $Y \in \mathcal{G}_\infty(X)$ and $d > \tilde{\varphi}(\vec{E}, X)$ be fixed. Since X is φ-stable, we have $\tilde{\varphi}(\vec{E}, Y) < d$. Thus we can find $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}_\infty(Y)^k$ with $\varphi(\vec{E} + \vec{U}) < d$.

If $i \in \{1, \ldots, k\}$ is fixed, then for each $w \in U_i$ we have $\tilde{\varphi}(\vec{E} +_i w, X) < d$, because

$$\tilde{\varphi}(\vec{E} +_i w, X) \leq \varphi((\vec{E} +_i w) + \vec{U}) = \varphi(\vec{E} + \vec{U}).$$

Thus the set $\{w \in X \,:\, \tilde{\varphi}(\vec{E} +_i w, X) < d\}$ is plentiful, because $U_i \in \mathcal{G}_\infty(Y)$ and $Y \in \mathcal{G}_\infty(X)$ was arbitrary. Since i can be any number in $\{1,\ldots,k\}$, this completes the proof of Lemma 1.

Theorem 2. *Let X be φ-stable and $\tilde{\varphi}(\vec{0}_k, X) < d$. Then for each $\eta > 1$ there is a basic sequence $(x_i)_{i=1}^{\infty}$ in X (with basis constant $\leq \eta$) such that for each $n \geq 1$ and each $f : \{1,\ldots,n\} \to \{1,\ldots,k\}$, if*

$$(\vec{E})_f = (\mathrm{span}\,\{x_j \,:\, 1 \leq j \leq n,\ f(j) = i\})_{i=1}^{k},$$

then $\tilde{\varphi}((\vec{E})_f, X) < d$.

Proof. Note that if $Y \in \mathcal{G}(X)$ and $\dim X/Y < \infty$, then Y is plentiful in X. The sequence $(x_i)_{i=1}^{\infty}$ will be constructed by induction, using Lemma 1 and the finite intersection property of plentiful subsets.

We fix an $\eta > 1$ and let $Y_1 = X$. Suppose that $n \geq 1$ and Y_n, $x_1, \ldots, x_{n-1}$ have been defined so that $\dim X/Y_n < \infty$.

Using Lemma 1 we select a non-zero $x_n \in Y_n$ so that for each $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$ one has

$$\tilde{\varphi}((\vec{E})_f, X) < d.$$

Then we choose $Y_{n+1} \subseteq Y_n$ so that $\dim X/Y_{n+1} < \infty$ and $\|x\| \leq \eta\|x + y\|$ holds for each $x \in \operatorname{span}\{x_1, \ldots, x_n\}$ and $y \in Y_{n+1}$.

Standard arguments show that so defined sequence (x_i) has the required properties. This completes the proof of Theorem 2.

The construction used in this proof can be easily adapted to a more general situation.

Suppose now that $\mathcal{N}$ is a set, $(k_s)_{s\in\mathcal{N}} \in \mathbb{N}^{\mathcal{N}}$ and we are given a family of functions $(\varphi_s)_{s\in\mathcal{N}}$ such that $\varphi_s : \mathcal{G}_\infty(X)^{k_s} \to [1,\infty]$ for $s \in \mathcal{N}$. Let us fix $(n_s) \in \mathbb{N}^{\mathcal{N}}$ such that $\{s \in \mathcal{N} : n_s \le n\}$ is finite for each $n \in \mathbb{N}$. (If $\mathcal{N}$ is finite, then one may take $n_s = 1$ for $s \in \mathcal{N}$.)

Theorem 3. *Suppose that X is φ_s-stable and $\tilde{\varphi}_s(\vec{0}_{k_s}, X) < d_s$ for $s \in \mathcal{N}$. Let $\eta > 1$. Then there is a basic sequence $(x_i)_{i=1}^\infty$ in X with basis constant η such that for each $n \ge 1$, each $s \in \mathcal{N}$ and each function $f : \mathbb{N} \to \mathbb{N}$, if $(\vec{E})_{f,s,n}$ denotes*

$$(\operatorname{span}\{x_j \,:\, n_s \le j \le n,\ f(j) = i\})_{i=1}^{k_s},$$

then one has $\tilde{\varphi}_s((\vec{E})_{f,s,n}, X) < d_s$.

Indeed, a proof of this theorem can be easily obtained by adapting that of Theorem 2, because for each n the inductive choice of Y_n and $x_n \in Y_n$ is subject only to a finite number of conditions (those conditions that pertain to those $s \geq 1$ such that $n_s \leq n$).

Namely, for each n there is a finite set of functions $\mathcal{F}_n$ which has the property that, if (3) is satisfied for $s \in \mathcal{N}$ such that $n_s \leq n$ and for each $f \in \mathcal{F}_n$, then for this n condition (3) is automatically satisfied for each $s \in \mathcal{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$.

The existence of φ-stable subspaces

We present here some general results which in particular can be applied to the functions Φ of the form $\tilde{\varphi}$.

A partially ordered set $(S, \leq)$ is said to satisfy (CCB) (i.e., countable chains in S are bounded in S) provided that, for each sequence (s_i) in S such that $s_i \leq s_{i+1}$ for each i, there is an $s \in S$ such that $s_i \leq s$ for each i.

A subset $S' \subseteq S$ is said to be cofinal in S if for each $s \in S$ there is an $s' \in S'$ such that $s \leq s'$.

If Φ is a function defined on the product $T \times S$ where T is a set, then by $\mathrm{St}(\Phi)$ we denote the set of Φ-stable elements, i.e.,

$$\mathrm{St}(\Phi) = \{s \in S \,:\, \forall t \in T\, \forall s' \in S \\ (s \leq s') \Rightarrow\, (\Phi(t,s) = \Phi(t,s'))\}.$$

Lemma 2. *Assume that* $(S, \leq)$ *satisfies* (CCB) *and* T *is a separable topological space. Let* $\Phi : T \times S \to [0, \infty]$ *be a function such that* $\Phi(t, \cdot)$ *is nondecreasing for each* $t \in T$*, while* $\Phi(\cdot, s)$ *is continuous for each* $s \in S$*. Then* $\mathrm{St}(\Phi)$ *is cofinal in* S*.*

Proof. First we consider Case 1 which occurs if $\mathrm{Card}(T) = 1$. In this case the assertion follows easily from (CCB). Case 2 occurs when $T = \{t_0, t_1, \ldots\}$ is countable.

In this case we again make use of the (CCB) condition after applying consecutively Case 1 to each $\Phi(t_i, \cdot)$. In the general case, where T is a topological space with a dense countable subset T', we first apply Case 2 to the function $\Phi|_{T' \times S}$ and then we make use of the continuity assumption.

The following lemma is an easy consequence.

Lemma 3. *Assume that $(S, \leq)$ satisfies (CCB) and suppose that (Φ_i) is a countable family of functions such that $\Phi_i : T_i \times S \to [0, \infty]$ and the set $\mathrm{St}(\Phi_i)$ of Φ_i-stable elements is cofinal in S for each i. Then the set $\cap_i \mathrm{St}(\Phi_i)$ of those s that are (Φ_i)-stable for each i is cofinal in S, too.*

Lemma 4. *Let X be a Banach space, $\dim X = \infty$. Define a partial order $\preceq$ on $\mathcal{G}_\infty(X)$ by the formula: $Y \preceq Z$ if and only if $Z \subseteq Y + E$ for some $E \in \mathcal{G}_{\mathrm{fin}}(X)$. Then $(\mathcal{G}_\infty(X), \preceq)$ satisfies (CCB).*

Proof. Let $(X_n)_{n=1}^\infty$ be a sequence in $\mathcal{G}_\infty(X)$ such that $X_n \preceq X_{n+1}$ for each n. Put $Y_n = \cap_{1\le i\le n} X_i$. Then $Y_n \in \mathcal{G}_\infty(X)$, because $\dim(X_n/Y_n) < \infty$ for each n.

Thus it is easy to construct by induction a linearly independent sequence (y_n) such that $y_n \in Y_n$ for each n.

Let Y denote the closed linear span of (y_n). Then $X_n \preceq Y$ for each n, because $Y \subseteq \mathrm{span}\{y_1, \ldots, y_{n-1}\} + Y_n$ and $Y_n \subseteq X_n$. This completes the proof.

Some examples of functions φ

Given a Banach space X and $k \geq 2$, for any function $h : X^k \to [0, \infty]$ we can define two functions $\mathcal{G}_\infty(X)^k \to [0, \infty]$. Namely, we let for $\vec{U} = (U_1, \ldots, U_k) \in \mathcal{G}_\infty(X)^k$

$$h^{\uparrow}(\vec{U}) = \sup\{h(x) : x \in \mathcal{U}\},$$
$$h^{\downarrow}(\vec{U}) = (\inf\{h(x) : x \in \mathcal{U}\})^{-1},$$

where $\mathcal{U} = \{(u_1, \ldots, u_k) \in U_1 \times \cdots \times U_k : \|\sum_{i=1}^k u_i\| = 1\}$.

The simplest interesting example is obtained for $k = 2$ and the function $h = h_\pm$, where $h_\pm(x_1, x_2) = \|x_1 - x_2\|$ for $x_1, x_2 \in X$.

Given any $\lambda \in \ell_\infty^k$ and a 1-unconditional norm $\mathbf{q}$ on $\mathbb{R}^k$, we put $h_\lambda(x_1,\ldots,x_k) = \|\sum_{i=1}^k \lambda_i x_i\|$ and

$$h_{\mathbf{q}}(x_1,\ldots,x_k) := \mathbf{q}(\|x_1\|,\ldots,\|x_k\|). \quad (2)$$

Definition 3. *We put* $\varphi_\pm = (h_\pm)^\uparrow$, $\varphi_\lambda = (h_\lambda)^\uparrow$, $\varphi_{>\mathbf{q},k} = (h_{\mathbf{q}})^\uparrow$ *and* $\varphi_{<\mathbf{q},k} = (h_{\mathbf{q}})^\downarrow$.

Observe that for each $(u_1,\ldots,u_k) \in U_1 \times \cdots \times U_k$ one has

$$\|u_1 - u_2\| \le \varphi_\pm(U_1,U_2)\|u_1+u_2\|, \quad (3)$$

$$\|\sum_{i=1}^k \lambda_i u_i\| \le \varphi_\lambda(\vec{U})\|\sum_{i=1}^k u_i\|, \quad (4)$$

$$\mathbf{q}(\|u_1\|,\ldots,\|u_k\|) \le \varphi_{>q,k}(\vec{U})\|\sum_{i=1}^k u_i\|, \quad (5)$$

$$\|\sum_{i=1}^{k} u_i\| \le \varphi_{<q,k}(\vec{U})\mathbf{q}(\|u_1\|, \ldots, \|u_k\|),$$

and the coefficients $\varphi(\vec{U})$ are optimal.

Proposition 1. *Let* $(S, \le) = (\mathcal{G}_\infty(X), \preceq)$ *and let* $T = \mathcal{G}_{\mathrm{fin}}(X)^k$ *with the topology induced by the metric* ρ*. If* $\varphi : \mathcal{G}_\infty(X)^k \to [0, \infty]$ *is one of the functions introduced in Definition 3, then the function* $\tilde{\varphi} : T \times S \to [0, \infty]$ *defined by (1) satisfies the assumptions of Lemma 2.*

It is easy to see that, given any function $h : X^k \to [0, \infty]$, if either $\varphi = h^\uparrow$ or $\varphi = h^\downarrow$, then $\tilde{\varphi}(\vec{E}, \cdot)$ is nondecreasing for each fixed $\vec{E} \in \mathcal{G}_{\mathrm{fin}}(X)^k$. Thus it remains to prove the continuity of $\tilde{\varphi}(\cdot, Y)$ for each fixed Y.