SOLUTIONS TO PROBLEMS IN PI IN THE SKY 16

Problem 1
Find the number of primes that are less then 100 and can be written as a difference of perfect cubes.

Solution
The solutions are 2,7,19,37 and 61.

Problem 2
The sum of twenty positive integers is 462. Find the largest possible common divisor of these numbers.

Solution
If d is a common factor, then the sum of the numbers is a multiple of d and is at least $20d$. Since $462 = 21 \times 22$, the largest possibility for d is $d = 22$, obtained by taking nineteen 22's and one 44.

Problem 3
Let a, b be real numbers and $P(x) = ax^2 + 1007x + b$ such that the inequatity $1 + x(x+1) + P(x)(P(x)+1) \leq xP(x)$ has at least one solution. Find $P(1)$.

Solution
Let x be the solution. The inequality is equivalent to

$$(x+1)^2 + (P(x)+1)^2 + (P(x)-x)^2 \leq 0.$$

Hence the only possibility is $x = -1$, $P(-1) = -1$. Now $P(-1) = a + b - 1007$ while $P(1) = a + b + 1007$, so that $P(1) = P(-1) + 2014 = 2013$.

Problem 4
Find the values of a for which the equation

$$|x| + |x-1| + \ldots + |x-2012| = a$$

has exactly one positive integer solution.

Solution
First we remark that for any value of a, the equation cannot have a solution in the range $[0, 2012]$ and a second solution in the range $[2013, \infty)$. To see this, notice that if $k \in [0, 2012]$ and $\ell \geq 2013$, then

$$|k| + |k-1| + \ldots + |k-2012| \leq 0 + 1 + \ldots + 2012 \text{ while}$$
$$|l| + |l-1| + \ldots + |l-2012| > 0 + 1 + \ldots + 2012.$$

This means that we can ask: for which values of a is there a unique solution in the range $[0, 2012]$? and for which values of a is there a unique solution in the range $[2013, \infty)$?

Secondly, we see that if x is a solution in the range $[0, 2012]$, then so is $2012 - x$. Hence the only possibility with a solution in the first range is $x = 1006$, so $a = 1006 + 1005 + \ldots + 1 + 0 + 1 + \ldots + 1006 = 1006 \times 1007$.

If $x > 2012$, then $|x| + |x - 1| + \ldots + |x - 2012| = 2013x - 1006 \times 2013 = 2013(x - 1006)$

The possible values of a are therefore 1006×1007 and $2013n$ for $n \geq 1007$.

Problem 5
In $\triangle ABC$, $m(\hat{A}) = 90^\circ$ and $BC = 1$. The incircle of $\triangle ABC$ is tangent to AB and AC at E and F respectively. Find the distance from the midpoint of BC to EF(produced).

Solution
Let M be the midpoint of BC, G the point where the circle is tangent to BC and B', C' and M' the projections of B ,C and M respectively on EF. Since $\triangle AEF$ is isosceles, $m(A\hat{E}F) = m(A\hat{F}E) = 90^\circ - m(\hat{A})/2 = 45^\circ$. In $\triangle BB'E$, $BB' = BE \sin 45^\circ$. Similarly $CC' = CF \sin 45^\circ$. Now in the trapezoid $BB'C'C$, we have

$$\begin{aligned} MM' &= \tfrac{1}{2}(BB' + CC') = \tfrac{1}{2}\sin(45^\circ)(BE + CF) \\ &= \tfrac{1}{2}\sin(45^\circ)(BG + CG) = (BC/2)\sin 45^\circ. \end{aligned}$$

Hence $MM' = \sqrt{2}/4$.

Problem 6
In $\triangle ABC$, $\hat{C}$ is obtuse and $\hat{A} = 2\hat{B}$ and the lengths of its sides are positive integers. Find the triangle $\triangle ABC$ of the above type with the minimal perimeter.

Solution
Let D lie on the line BC be such that AD bisects the angle A. Let $a = BC$, $b = AC$ and $c = AB$. Since $\triangle ACD$ and $\triangle BCA$ are similar triangles and $\triangle ABD$ is isosceles, we have $CD/BD = CD/AD = b/c$. Since $CD + BD = a$, we obtain $AD = DB = ac/(b+c)$. On the other hand, by similarity of $\triangle ABC$ and $\triangle DAC$, we obtain $AD/b = c/a$. Hence we arrive at $a^2 = b(b + c)$.

Assume that $\triangle ABC$ has minimal perimeter. We see from the above that if b and c have a common divisor d, then a is also divisible by d. Hence we see that $\gcd(b, c) = 1$. From the equality above, we see $a = kh$, $b = k^2$ and $c = h^2$, where $\gcd(h, k) = 1$.

Since C is obtuse, we get $c^2 > a^2 + b^2$, which yields $3k^2 < h^2$. The triangle inequality gives $c < a + b$, which implies $h^2 < hk + 2k^2$. Hence we conclude that $3k^2 < h^2 < hk + 2k^2$. If $k = 1, 2$ or 3, there is no integer h satisfying the constraints. If $k = 4$, then $h = 7$ satisfies the constraints (yielding $a = 28$, $b = 16$ and $c = 33$). This gives a triangle with perimeter 77. If $k \geq 5$, then $h \geq 9$ and the resulting triangle has perimeter greater than 77. Hence the triangle with sides 28, 16 and 33 has minimal perimeter.