π in the sky

mathematicae scientiae discipulus

"student of the science of mathematics"

On the Cover

The pillars of intellectual achievement are earned through dedication and commitment to the journey of learning.

Contact Information

Editorial: *David Leeming, Managing Editor, Pi in the Sky*

On behalf of the Pi in the Sky Editorial Board, I wish to thank Ivar Ekeland for being Editor-in-Chief for Issues #7 to #11. Starting with this issue there will be no Editor-in-Chief for Pi in the Sky.

In this issue, we continue our efforts to make the magazine more readable for mathematically oriented high school and college students. We have two articles (Fractal Dimension and From the Orthic Triangle) written by high school students, and another article (A Mathematical Fountain) co-authored by a high school student. In addition, we have included three 'Quickie' problems which are intended to be solvable by most students. The more challenging problems will continue to appear on our Challenge Problems page. Our thanks to Danesh Forouhari for providing the three Quickies for this issue.

Pi in the Sky is now truly an international magazine. We now have over 6000 subscribers in fifty-seven countries, including high schools and universities. You may obtain a free subscription to Pi in the Sky by going to our website and following the directions.

The Editorial Board is grateful to those who contributed to this issue of Pi in the Sky. We believe we have produced another very informative and entertaining issue of the magazine. However, we welcome and encourage new submissions to Pi in the Sky. More information on how to submit an article can be found on page 32 of this issue.

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In Issue #11, we offered a prize of $100 each for the best solution, by a high school student, to four of our Challenge Problems (#4 – #7). We did not receive any solutions from students so we are extending the offer until February 28, 2009. The four problems are not reprinted in this issue due to a lack of space, but can be found by searching at www.pims.math.ca/pi and going to Issue #11.

FRACTAL DIMENSION: MEASURING INFINITE COMPLEXITY

by Tejas Parasher

By now, fractals are practically everywhere. Their popularity has exploded tremendously in the last decade or so. Literally hundreds of fractal websites have mushroomed all over the Internet, and it is not uncommon to come across popular fractals such as the Mandelbrot Set (pictured below) in publications completely unrelated to mathematics.
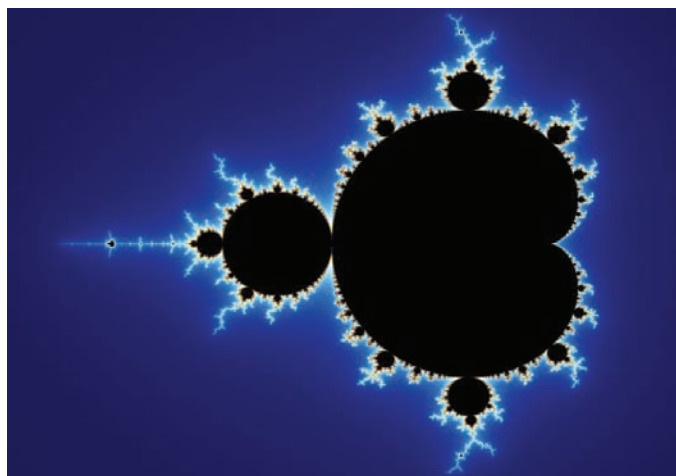
Figure 1: The Mandelbrot Set

This popularity can probably be attributed to the incredibly complex beauty of fractals. Also, unlike many other concepts, appreciating fractals does not require a thorough understanding of the math behind them. In fact, almost anyone with a grasp of basic algebra and the right computer program can easily create their own fractals.

However, fractals *are* a mathematical concept, and so there is a logical system underlying even their seemingly ridiculous complexity. This article will attempt to explore a key idea undelrying the mathematical study of fractals: that of *fractal dimension*.

Definition of a Fractal

The term *fractal* was coined by IBM mathematician Benoit Mandelbrot in the early 1970s from the Latin *fractus* (literally, 'fragmented' or 'broken' and meaning a set with *fractional dimension*). Mandelbrot, drawing on the work of earlier mathematicians, showed that

once one has a suitable definition of dimension, many interesting sets turn out to have dimensions that are not whole numbers.   A well-known example, the Koch Curve, is given below:
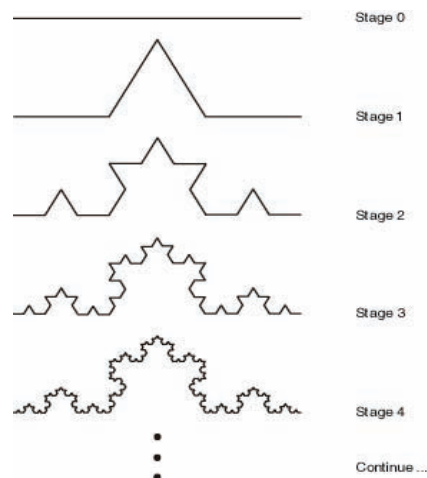
Figure 2: Generation of the Koch Curve.
The mathematically correct curve can only be reached after an infinite number of steps.

The Koch Curve is built in stages. You start with a line segment (Stage 0). In each stage after that, each line segment is replaced by four line segments (arranged as shown in Stage 1). Hence, as the stages progress, the complexity of the curve will increase. The Koch Curve is what is left when this process is carried out infinitely many times. If the curve in Stage 0 has length 1 then the curve in Stage 1 is made up of four parts of length 1/3 and so has length 4/3. At the next stage, the curve has sixteen parts of length 1/9 so has length 16/9. The final Koch curve being the limit of these stages has infinite length so that if one were to run along the perimeter of the Koch Curve at the speed of light, it would still take an infinite amount of time to finish.

Since fractals such as the Koch Curve fall outside the domain of Euclidean geometry (based on line segments, angles, circles, etc.), mathematicians have created a new set of tools *Fractal Geometry* to study the properties of fractals.

Fractal Dimension*

Let us assume that we need N sets of size r each to cover bounded set A, where r is a positive real number (for example, we need 64 square tiles of size 1 m^2 to cover a square floor with a side-length of 8m). As the size r decreases, this number N increases. (Thinking of tiling a bathroom floor: as the size of the tiles gets smaller and smaller, the number of them needed to cover the entire floor will get progressively larger). In the case of a square of side 8m, the number of tiles of size r is given by $N \approx 64/r^2$. In the case of a line segment of length 10m, the number of tiles of size r is approximately $10/r$. We notice that the denominator in these expressions for N is a power of r. Also, for the two-dimensional set (the square), the denominator is r^2 whereas for the one-dimensional set (the line segment), the denominator is r (otherwise known as r^1).

Because of this we aim to *define* the fractal dimension (more technically the *box dimension*) to be the power occurring in the denominator. Starting with the set A, we cover it in tiles of size r and acount the number N that we need. In the case where

$$N \propto \left(\frac{1}{r}\right)^D \qquad (1)$$

we say that the set has *fractal dimension* D

To actually calculate D, we use logarithms: taking logs of (1), we get

$$\log N \approx D \log(1/r). \qquad (2)$$

We use limits (letting r shrink towards 0) to obtain the actual value of D:

$$D = \lim_{r \to 0} \frac{\log N}{\log\left(\frac{1}{r}\right)} \qquad (3)$$

Now we can apply (3) to find the fractal dimension of the Koch Curve mentioned earlier. The limit curve fits inside a single square of side 1; or 4 squares of side 1/3 (arranged to be centered around the line segments in Stage 1); or 16 squares of side 1/9 arranged to be centered around the line segments in Stage 2 etc. In this way, the curve is covered by 4^n squares of size $1/3^n$. In other words, if $r = 1/3^n$, then $N = 4^n$. We have $\log N / \log (l/r) = \log 4 / \log 3$.

Therefore, we obtain from (3) the fractal dimension of the Koch Curve is given by:

$$D = \lim_{r \to 0} \frac{\log N}{\log\left(\frac{1}{r}\right)} = \frac{\log 4}{\log 3} = \log_3 4 \approx 1.261859... \quad (4)$$

Another Fractal

Another well-known fractal is the Mandelbrot set. This is obtained in a less direct way. One starts with the map $z \to z^2 + c$, where z is a complex number of the form $x + iy$.

For a given value of c, we initially set z to 0 and then *iterate* the map. That is, we plug in $z=0$ to the equation to get the next value of z (simply c itself). We then plug in the value to get the next value of z (that is $c^2 + c$). We keep going with this iteration process infinitely many times. There are two kinds of behaviour that take place depending on the value of c: either the sequence *diverges to infinity* or it remains forever in a circle of radius 2 about the origin (this takes some proof). The Mandelbrot set consists of those values of c for which the iteration stays bounded.

With a computer program, you can colour-code these two types of sequences, and thus arrive at the following familiar figure:
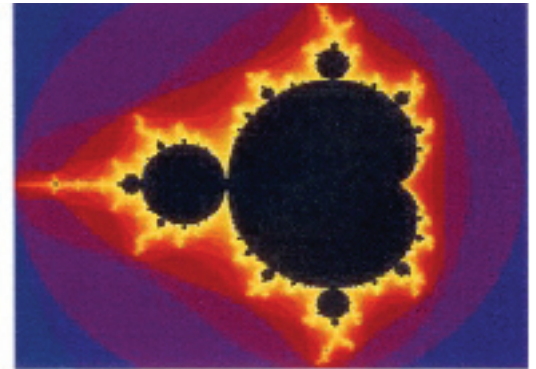

Figure 3:
The Mandelbrot Set. Black represents converging sequences in the complex plane, while the other colours represent those going to infinity

Zooming in on the Mandelbrot Set (shown below), one will find 'mini-copies' of the Mandelbrot set at a variety of scales.
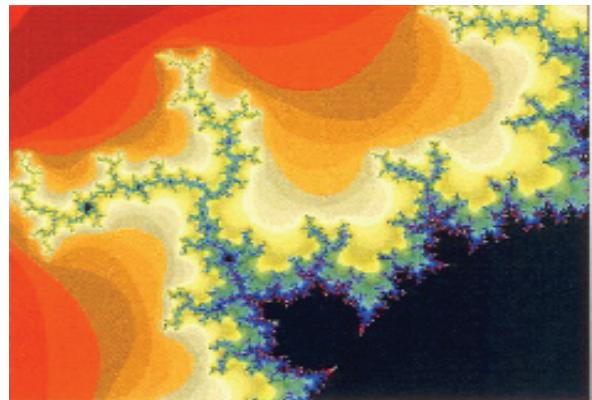

Figure 4: Zooming into the Mandelbrot Set, one can see 'mini-versions' of the original set all along the perimeter.

Conclusion

The above discussion on fractal dimension is just an inkling of the burgeoning field of *fractal geometry*. Fractal geometry studies in detail various mathematical features of these fascinating shapes. But despite the abstract–seeming nature of fractal geometry, this field is anything but confined to pure mathematics. In fact, possibly the most remarkable thing about fractals is their overwhelming number of applications. As pointed out by James Gleick in his popular book <u>Chaos: Making a New Science</u>, fractals offer "a way of seeing order and pattern where formerly only the random, the erratic, the unpredictable...had been observed". Some common and wide–ranging applications of fractals are in computer graphic design, image compression, biological distribution, and both macro and micro–level natural structures.

In conclusion, then, a more thorough understanding of fractals and their properties could not only have unforeseen impacts in "pure" mathematics but also in many other fields.

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Figures 5 & 6: Two fractal landscapes. The scene at left has been created completely mathematically using fractal software called 'Mountain 3d'. Conversely, the photograph at right, of oak branches, can be called a naturally-occurring fractal.

(From Wikipedia, the free encyclopedia)

In fractal geometry, the <u>fractal dimension</u>, D, is a statistical quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer and finer scales. There are many specific definitions of fractal dimension and none of them should be treated as the universal one. From the theoretical point of view the most important are the Hausdorff dimension, the packing dimension and, more generally, the Rényi dimensions. On the other hand the box-counting dimension and correlation dimension are widely used in practice, partly due to their ease of implementation.

Although for some classical fractals all these dimensions do coincide, in general they are not equivalent. For example, what is the dimension of the Koch snowflake? It has topological dimension one, but it is by no means a curve-- the length of the curve between any two points on it is infinite. No small piece of it is line-like, but neither is it like a piece of the plane or any other. In some sense, we could say that it is too big to be thought of as a one-dimensional object, but too thin to be a two-dimensional object, leading to the question of whether its dimension might best be described in some sense by number between one and two. This is just one simple way of motivating the idea of fractal dimension.

Cardinal Sins of the Infinite

by Keith F. Taylor, Dalhousie University

PART I

Once there was a land where the laws of physics were not as stringent as they are in our unverse; however, mathematics is the same everywhere.

In this land there was a Genie who decided he needed a couple of pets for entertainment. He managed to acquire an ape and a monkey from a shop in the magic market. The shop owner pointed out that the ape was powerful, cranky, and greedy while the monkey, although small, was quite clever.

The Genie brought his new pets home and placed them in a large valley with steep walls. The valley seemed ideal because it had lots of fruit available for food and his pets could not escape. He built himself a viewing platform jutting out over an edge of the valley and would come each day to enjoy observing his new possessions.

The ape's life was completely focussed on food. He ate voraciously and resented the monkey eating anything. In the ape's mind, each grape the monkey ate was one less for him. The ape located a large cave and started to hoard food in it. He settled down to a routine of spending much of his day stationed at the mouth of his cave eating and guarding his stash of food. He would make occasional high speed trips around the valley to gather any fruit that was ripe enough and bring it back to his cave.

Meanwhile the monkey was struggling to survive. Anytime she tried to climb a tree for a banana, the ape would see her and charge. With good luck, the monkey found a cave that had a small entrance and large interior. Most of the time the monkey hid in her cave and the ape could not follow through the small entrance. Her cave provided some security, but food was a problem. Foraging trips were very dangerous and the ape was taxing the productivity of the valley all by himself.

In order to survive, the monkey would wait until the ape left his cave to gather up any new fruit and then she would slip into his cave and steal something. By moving quickly and just grabbing a single piece of fruit, the monkey found that she could get back to her own cave without the ape realizing he had been robbed.

The Genie was not happy with how things were going. It became clear that the ape's greed was destroying the little ecosystem in the valley. So the Genie went back to the magic market with the intention of buying so much food that the ape could never eat it all up. At a coconut shop, he found exactly what he wanted.

Soon he delivered a huge load of coconuts to the valley. For inventory purposes the coconuts were individually numbered by positive integers and all the positive integers were used – it was a large load indeed! The Genie settled back onto his platform to enjoy watching his pets with more food than they could ever possibly use up. Unfortunately, he again underestimated the ape's greed.

When the ape saw the new supply, he went to the pile and gathered up coconuts numbered 1 to 10 and carried them back to his cave. This took him 30 minutes. He went back for another load and realized he would need to become more efficient if he wanted to move such a large number of coconuts. This trip he gathered up the next hundred coconuts, numbered from 11 to 110, and carried them back to his cave in just 15 minutes. He was quite pleased with his improvements and resolved to gain similar efficiencies each trip. He did not notice that while he was getting his second load, the monkey had slipped into his cave and stole coconut number 1 and brought it back to her cave.

This process continued. On each trip, the ape carried back 10 times as many coconuts, always numbered by the next integers in order, and made the trip in half of the time of the previous trip. Meanwhile, each time the ape went over to the pile for another load, the monkey slipped into his cave and took the coconut

with the smallest integer on it back to her cave.

The Genie was absolutely transfixed by the spec-tacle. He stared without blinking as the ape carried larger and larger loads at higher and higher speeds. He noticed that the monkey was able to better her times as well, but this was nothing compared to the ape's improvements in strength as well as speed. The Genie stared, motionless, for 59 minutes and 59.99 seconds. Then he blinked. An hour had passed since the ape picked up the first load. The Genie was astonished with what had happened. Why?

(to be continued).
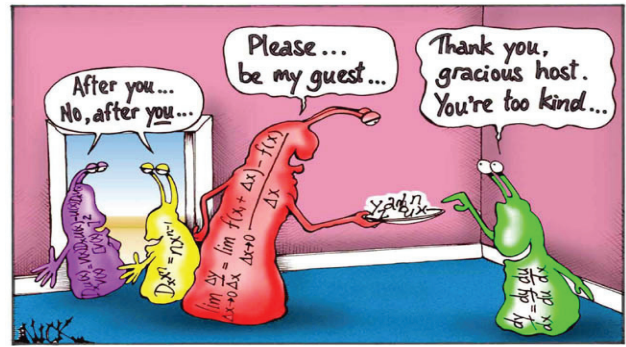

Deferential Equations

There were three medieval kingdoms on the shores of a lake. There was an island in the mid-dle of the lake, over which the kingdoms had been fighting for years. Finally, the three kings decided that they would send their knights out to do battle, and the winner would take the island.

The night before the battle, the knights and their squires pitched camp and readied themselves for the fight. The first kingdom had 12 knights, and each knight had five squires, all of whom were bus-ily polishing armor, brushing horses, and cooking food. The second kingdom had twenty knights, and each knight had 10 squires. Everyone at that camp was also busy preparing for battle. At the camp of the third kingdom, there was only one knight, with his squire. This squire took a large pot and hung it from a looped rope in a tall tree. He busied himself preparing the meal, while the knight polished his own armor.

When the hour of the battle came, the three king-doms sent their squires out to fight (this was too trivial a matter for the knights to join in).

The battle raged, and when the dust had cleared, the only person left was the lone squire from the third kingdom, having defeated the squires from the other two kingdoms, thus proving that the squire of the high pot and noose is equal to the sum of the squires of the other two sides.

A differential equation is a mathematical equa-tion for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. Dif-ferential equations play a prominent role in en-gineering, physics, economics and other disciplines. Visualization of airflow into a duct modelled us-ing the Navier-Stokes equations, a set of partial differential equations.

Differential equations arise in many areas of science and technology; whenever a deterministic relationship involving some continuously chang-ing quantities (modeled by functions) and their rates of change (expressed as derivatives) is known or postulated. This is well illustrated by classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's Laws allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function of time. In many cases, this differential equation may be solved explicitly, yielding the law of motion.

Differential equations are mathematically stud-ied from several different perspectives, mostly con-cerned with their solutions, functions that make the equation hold true. Only the simplest differ-ential equations admit solutions given by explicit formulas. Many properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained for-mula for the solution is not available, the solution may be numerically approximated using comput-ers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by dif-ferential equations, while many numerical meth-ods have been developed to determine solutions with a given degree of accuracy.

(From Wikipedia, the free encyclopedia)

FINDING A PARENT FOR AN ORTHIC ORPHAN

By Klaus Hoechsmann

1. Introduction.

The purpose of this article to take a run at the next one, which begins with the words: "Since each triangle has a unique orthic triangle ... and *vice versa* ...". We are going to look into the "*vice versa*", wondering in particular (a) what it means, (b) why we should believe it, and (c) what it is good for. One would guess that "orthic" has to do with right angles, since "orthogonal" is one of the ways of saying "at right angles", the other one being "perpendicular". Since *vice versa* means the *the other way round*, translation software might nonsensically proclaim that every orthic triangle has a unique triangle. Uh? A human translator might try: "Every triangle is orthic for a unique [other] triangle ", which is essentially right but awkward. To distinguish that other triangle verbally, we shall temporarily -- just for this article -- refer to it as an "orthogenic parent".

It is usually said that ΔDEF is *orthic* for ΔABC, if D, E, and F are the feet of the altitudes of the latter. Recall that an altitude is a line segment orthogonally connecting a vertex to a point on the opposite side called its "foot". Unfortunately, this would make ΔDEF -- as shown in Figure 1 -- orthic for both the acute ΔABC in the top diagram of that figure and the obtuse ΔBAC in the bottom one -- and there goes the uniqueness. One could safeguard it by careful rules of labelling, but it is easier to write acuteness into the definition.
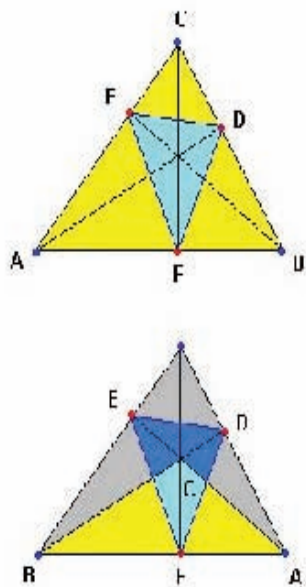
Figure 1

Thus, we shall say that ΔABC is an *orthogenic parent* of ΔDEF if its altitudes AD, BE, CF intersect its sides inside the segments BC, AC, AB, respectively. Forcing the feet of the altitudes of ΔABC to lie on its boundary bars obtuse triangles from becoming orthogenic parents.

Regarding (a), one could claim that *any triangle has a unique orthogenic parent*. More explicitly this statement asserts the solvability of the following problem:

Given three points D, E, F, find an acute triangle ABC which has them as feet of its altitudes.

In fact, we shall *construct* a solution, and in the process, its uniqueness will become apparent. This will answer Question (b) asked at the beginning, and will occupy the next two paragraphs. As for Question (c), we give the same example which you can find at the end of the next article, but we go more slowly.

2. Cyclic Quadrilaterals

This paragraph will recall some elementary facts about angles and circles. Imagine four points M, P, N, Q (in that order) lying on a circle with center O. The four line segments MP, PN, NQ, QM going around the circle are the *sides* of what is called a *cyclic quadrilateral* (gray), because "cyclic" means "circular". The segments PQ and MN crossing each other are known as the *diagonals* of this quadrilateral. Each of the six segments mentioned so far is also referred to as a *chord* of the circle. As Figure 2 shows, there are two cases depending on whether O lies in the gray zone or not.

Figure 2

The sides of the quadrilateral form the bases of four isosceles triangles, each with apex at O. At each of its vertices, the quadrilateral has an angle formed by the base angles of two neighbouring triangles. In the upper diagram, four different colours are used for those base angles, and it is clear that any two vertice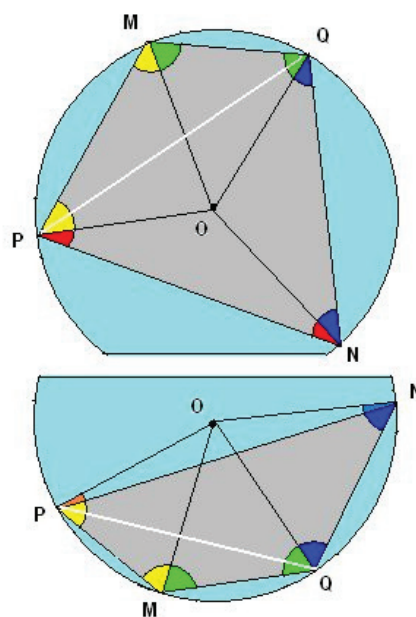s on same diagonal have exactly one angle of each colour. Therefore, < M + < N = < P + < Q, and since all four angles must add up to 360 degrees, each pair of *opposite* angles (such as < M + < N) must yield 180 degrees.

The equation < M + < N = < P + < Q (with its conclusion) also holds in the lower diagram, provided the angles are counted correctly. The problem is that ΔNOP (with red base angles in the upper diagram) now lies outside the quadrilateral, and must have its base angles *subtracted* instead of added at P and N. Thus < P consists of a full yellow angle (like the one at M) *minus* the base angle OPN, and < N consists of a full blue angle (like the one at Q) *minus* the base angle PNO. (If we allow negative angles, the two cases are not really different, since < MPO and < OPN "turn" in the same direction for one and opposite directions for the other.) In other words, *opposite angles in a cyclic quadrilateral are supplementary* in every case. What we really want, however, is a criterion for deciding whether some point R lies on a given circle. Here is what we get:

If points R and M are on opposite sides of a chord PQ, the circumcircle of ΔPMQ will contain R if and only if ‹ PMQ and ‹ QRP are supplementary.........(2.1)

Indeed, if R does lie on that circle, we can let it play the role of N in our previous considerations, and supplementarity is assured. Suppose, then, that we know only that the angles PMQ and QRP are supplementary, but nothing about the position of R. Let N be the point of intersection of the line QR with the circle in question. If N were not equal to R, the angles QRP (given) and QNP (from previous result) could not both be supplementary to angle PMQ.

Note that *all* points N on the circular arc on one side of the chord PQ must yield the same angle QNP, namely the one which is supplementary to PMQ (imagine M fixed). Hence the criterion just derived can be reformulated by substituting "equal" for "supplementary" as the last word, and placing R and M on the same side of PQ instead of opposite sides. Therefore

If points R and M are on the same side of a chord PQ, the circumcircle of ΔPMQ will contain R if and only if ‹ PMQ and ‹ QRP are equal ....(2.2)

3. Orthogenic Construction.

A constellation resembling the one in Figure 1 will now be constructed from scratch. Given any ΔDEF, draw a line through each vertex at right angles to the respective angle bisector. Since the latter are pairwise non-parallel, so are these new lines. They will therefore intersect in pairs, giving us three new points, which we call A, B, and C in such a way that D, E, and F lie on BC, AC, and AB, respectively (check that this can be done!).

Since points on an angle-bisector are equidistant from both legs of the angle, the intersection of two such bisectors in a triangle is equidistant from two pairs of legs, i.e., from all three sides. Hence all three bisectors meet at the same point P, and we get a picture as in Figure 3 -- *but it is not clear that the bisectors* DP, EP, and FP *will align with the segments* PA, PB, and PC.

To investigate that missing link in our construction, we need to take a detour. Stepping back from bisectors, we drop perpendiculars PD, PE, and PF from any point P inside an arbitrary ΔABC (possibly obtuse) onto its three sides BC, AC, and AB. The question is under what conditions these perpendiculars line up with PA, PB, and PC.
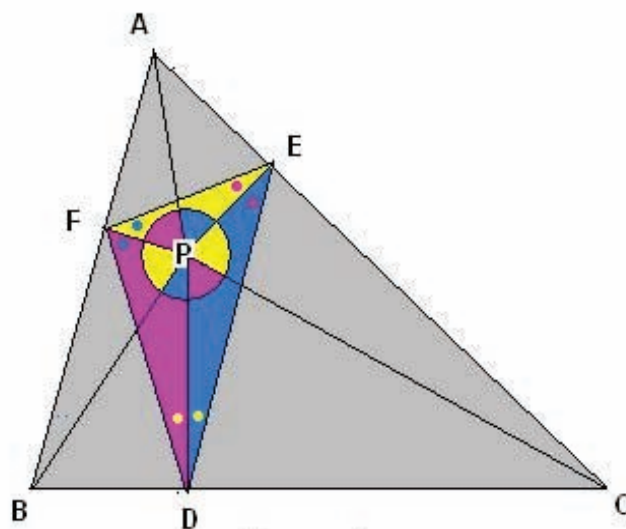
Figure 3

In terms of Figure 3, such alignments clearly require that equal colours imply equal angles among the coloured sectors sorrounding P. Note that the equality of, say, the yellow sectors on either side of AP+PD does not suffice to force the straightening of this broken line: we need all three colours. To clarify this, let us take AP+PD as a reference and denote by x, y, z the blue, yellow and red angles to the right of it, while their same-colour analogues on the other side are labelled x', y', z'. Then AP+PD will be straightened by

the equality $x+y+z = x'+y'+z'$, while,

BP+PE will go with $x'+y+z = x+y'+z'$, and

CP+PF with $x'+y'+z = x+y+z'$.

Putting $u = x-x'$, $v = y-y'$, $w = z-z'$, these three equations amount to $u+v+w = 0$, $u-v-w = 0$, $u+v-w = 0$, whence $u=v=w$. In other words, the three alignments occur if and only if $x=x'$, $y=y'$, $z=z'$. A fine criterion but somewhat useless -- so, let us continue the detour.

To transfer these sector equalities to the angles of ΔDEF, consider the decomposition shown in Figure 4. Having two opposite right angles, each of the three pieces is a cyclic quadrilateral, as a consequence of (2.1). Consider the top one AFPE. The angle of the red sector is complementary to the red-dotted angle at A, which in turn equals the red-dotted angle at E because they sit on a common "chord" FP, as required by (2.2). Likewise, the blue-dotted angle at F is complementary to the blue sector by the same reasoning around the "chord" EP.

Figure 4

Exactly the same thing happens in the other two fragments: The foo-dotted angle at D, E, or F is complementary to the foo-coloured sector in the same quadrilateral (check!). We can therefore say:

If PD, PE, PF are perpendiculars dropped from a point P inside a ΔABC onto its sides, the triples APD, BPE, CPF are collinear if and only if AP, BP, CP are angle-bisectors.
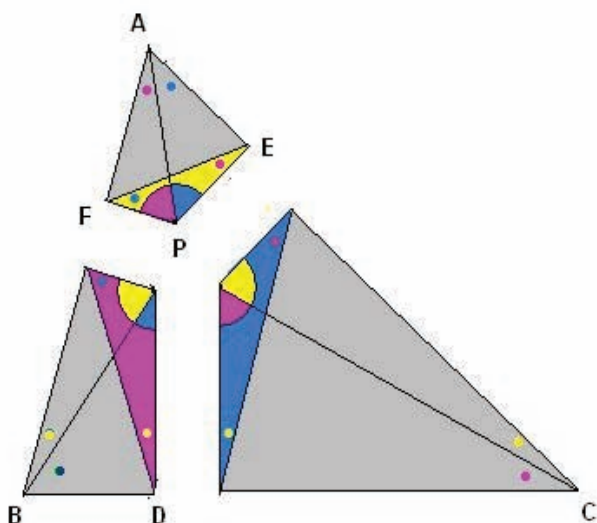
Here ends the detour, and we can get back to our construction. Since we started it with bisectors, the criterion just derived shows that the three segments AD, BE, and CF all go through P, and therefore meet the

respective sides at right angles. In other words, DEF *is the orthic triangle* of ABC and P is the *orthocentre*, which is traditionally labelled H . Moreover, we have shown that these segments (ye olde *altitudes*) bisect the angles of DEF. Finally, ABC is necessarily acute, because all the altitudes have their "feet" inside the segments BC, AC, and AB. In other words, we have constructed an orthogenic parent ABC of the given DEF. Its uniqueness is fairly obvious, but we shall not need it and therefore omit its demonstration.

4. Euler's Formula

The main purpose of all this hue and cry is an apparently modest fact involving not only the angle-bisectors of the given ΔDEF but also its *circumcircle* (the one which contains all three vertices). In case anyone has forgotten: its centre Q is the intersection of any two perpendicular bisectors, i.e., lines which cut the sides at right angles through their midpoints (all points on such a line are equidistant from the ends of the side in question). This is where ΔDEF needs an orthogenic parent ABC, for which the circumcircle of DEF becomes the so-called Nine-Point or Feuerbach Circle. In the last issue of this magazine we had called it Euler's Circle, though neither Euler nor Feuerbach (but the French geometerTerquem) proved the property which is essential here, namely that it cuts AH exactly through the middle. This makes N the mid-point of the hypotenuse of the right triangle HEA. Without reference to orthics, we can formulate the main consequence as follows.
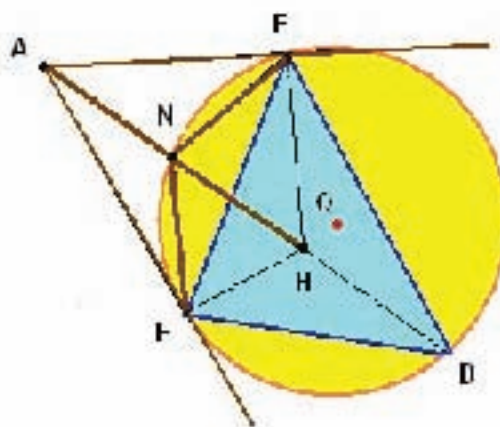

Figure 5

Given ΔDEF, let ND be the chord of its circumcircle which bisects the angle at D. Then NE = NH.

Of course the equality sign here means that the two segments are congruent, not literally identical (lest E=H). Furthermore, it is clear that NF and NA are included in the same congruence; in fact, N is the centre of the circle which makes AFHE a cyclic quadrilateral.

A curious result indeed, but without the pizzazz we have come to expect from Euler. His problem was this: if you know the radii of the circumcircle and the incircle of a ΔDEF, can you figure out the distance between their centres? Since the incircle must touch all three sides, its centre is equidistant from them, hence equal to H. Thus, Euler wanted the distance d = |HQ| expressed in terms s = |QN| and r, the distance from H to any of the sides (such as ED).

Once more (2.2) steps up to the plate and delivers the similarity of the light blue and green right triangles, bringing about the identity 2sr = |NE| |HD|, which the curious result from above turns into 2sr = |NH| |HD|. This apparently innocuous equation is crucial: putting p = |NH| and q = |HD| in the isosceles ΔNDQ as shown in the gray box under Figure 6, we immediately obtain Euler's Formula as expounded at the end of Bill Pang's article (pages 12–14 of this issue). The little formula in gray comes from applying the Pythagorean Theorem twice and the identity $x^2 - y^2 = (x+y)(x-y)$ once -- and is left as an exercise (no trig, please).
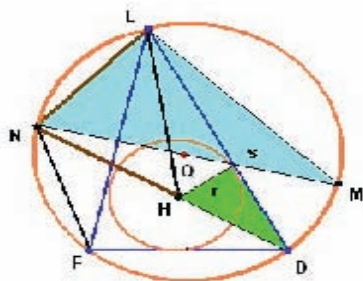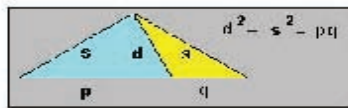
Figure 6

Of course, Bill assumes that the reader knows trigonometry, while the present warm-up tries to be as low-key as possible. The most subtle mathematical fact used here is the equality $2sr = |NE|\,|HD|$ deduced from a similarity of triangles. Usually ratios, in the form $2s : |NE| = |HD| : r$, are invoked at this stage -- but they are not the simple tools they are often thought to be, unless all quantities involved are mutually commensurable (dream on!). Otherwise they require much faith or some insight into post-secondary notions like continuity or axiomatic geometry (as has been indicated in earlier issues of this magazine: #4, p.16 and #5, p.25, as well as #7, pp. 17–19).

Quickie Problem #1:

Find all integers

X, Y, & Z such that

$$X^2 + Y^2 = \sqrt{Z^2 + 12}$$

FROM THE ORTHIC TRIANGLE

Bill Pang

Editing By: Gary Miller

Bill Pang, Sir Winston Churchill Secondary, received the Canadian Mathematical Society Award at the Canada-Wide Science Fair 2007 and Gold Medal in his category. Gary Miller is Associate Professor Emeritus, University of Victoria

Area Problem. Since each triangle has a unique orthic triangle [Fig.1] and *vice versa*, a problem arises:

> How can the area of the original triangle be expressed in terms of the side lengths (a,b,c) of the orthic triangle?

Since the right triangles △HEC and △HDC share HC, we have H,D,C,E lie on a circle S which has HC as a chord. In other words, HDCE is a cyclic quadrilateral. ∠EDH = ∠ECH since they subtend the same chord HE of circle S. Similarly, BFHD is a cyclic quadrilateral, and we have ∠FDH = ∠FBH. Since the right triangles △ABE and △ACF share a common angle at A, we have ∠FBH = ∠ECH. In summary, ∠EDH = ∠ECH = ∠FDH = ∠FBH. Since AD bisects ∠FDE and, similarly, BE bisects ∠FDE, we conclude H is the incenter of △DEF, i.e., the intersection of the angle bisectors of that triangle:

> H, the orthocenter of △ABC, is also the incenter of its orthic triangle △DEF.

As illustrated in [Fig.2], circumscribe △ABC, and let O be the center of the circumcircle of △ABC. Extend HD, HE, HF to meet that circumcircle at A′, B′, and C′, respectively. Since the two chords BC and AA′ intersect within the circle,

Figure 2. The expanded orthic △A′B′C′ of △ABC. Both have the same circumcircle with center O and radius R. The side lengths of the orthic triangle are denoted a = |EF|, b = |DF|, and c = |DE|.

the Chord–Chord theorem, used for the definition of "power of a point", states |BD| · |DC| = |AD| · |DA′|. The right triangles △ABD and △CBF share an angle at B and are similar. Since, for the same reason, △CBF ~ △CDH, we have △ABD ~ △CBF whence |BD| · |DC| = |AD| · |HD|. Thus, HD = DA′. For similar reasons, we have HE = EB′ and HF = FC′. Thus, A′B′ = 2DE, B′C′ = 2FE, A′C′ = 2FD. Notice H is the incenter of both the orthic △DEF and the expanded orthic △A′B′C′. The side lengths of the expanded orthic △A′B′C′ are twice that of the orthic △DEF, and the area, 4 times:

> The expanded orthic and the orthic are homothetic with center of dilation H and factor of dilation = 2.

Hexagon Method.

Consider the hexagon AB′CA′BC′, [Fig.3]. Since DA′ = DH and HA′ = BC, we have △BA′C ≅ △BHC. Similarly, we have △AB′C ≅ △AHC and △AC′B ≅ △AHB. Therefore, area[AB′CA′BC′] = area[△AHB] + area[△CHA] + area[△BHC] + area[△ABC] = 2 · area[△ABC]:

> The hexagon's area is twice that of the original triangle

Figure 1. The orthic triangle of △ABC is △DEF where D,E,F are the feet of the altitudes of △ABC. H is called the orthocenter of △ABC. The circumcircle K of the orthic triangle is also called the Feuerbach or nine point circle of △ABC. It turns out K amazingly passes through not only E,F,D but also the midpoints of AH, BH, CH and the midpoints the sides of △ABC as illustrated.
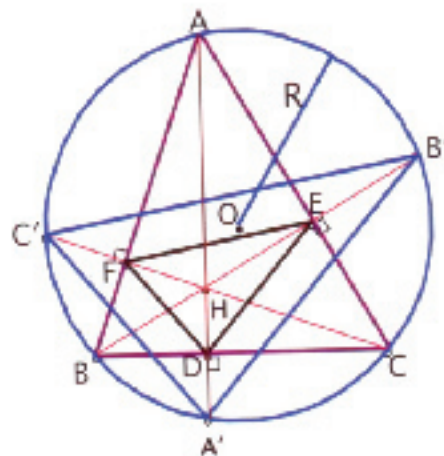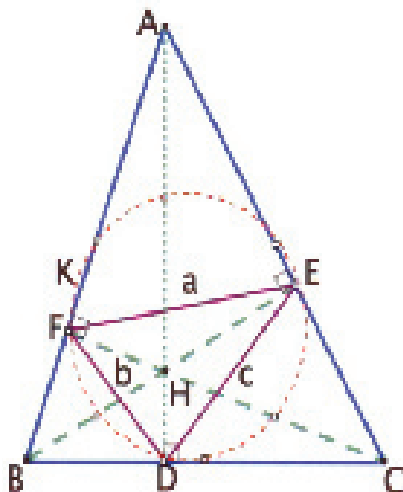
Since H is the incenter of the extended orthic, ∠AA'C'= ∠AA'B', the chords AC'and AB'are equal, and we have that △B'AC' is isosceles. Thus, the perpendicular bisector of B'C'goes through O. Therefore, the quadrilateral OC'AB' has C'B'and OA as perpendicular diagonals. Thus, we have

$$\text{area}[OC'AB'] = \frac{|C'B| \cdot OA}{2} = \frac{2 \cdot |FE| \cdot R}{2} = R \cdot a.$$

Symmetrically, area[OA'CB'] = R · c, and area[OC'BA'] = R · b. Therefore:

$$\text{area}[AB'CA'BC']=\text{area}[OC'AB'] + \text{area}[OC'BA'] + \text{area}[OA'CB']$$

$$= R \cdot (a+b+c).$$
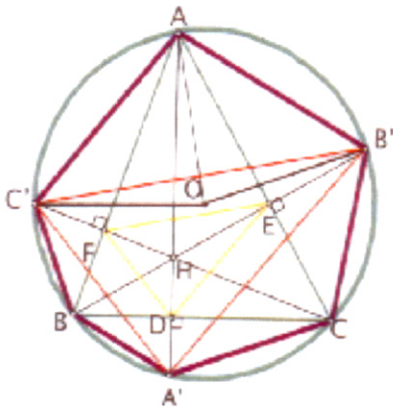
Figure 3. The hexagon AB'CA'BC'and quadrilateral OC'AB'.
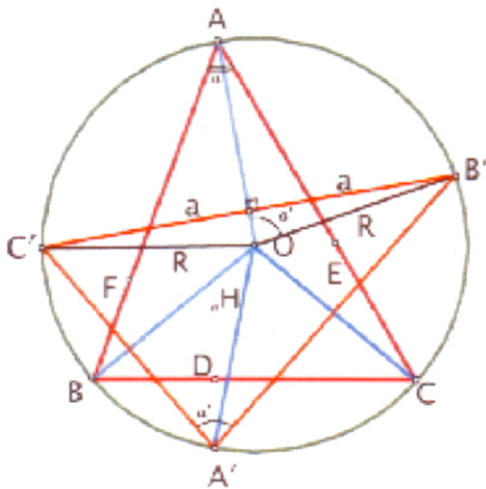
Figure 4: Thus, we have determined the hexagon's area can in terms of a, b, c, and R, by dividing it into three quadrilaterals with perpendicular diagonals. Since the original triangle has half that area,

$$\text{area } [\triangle ABC] = \frac{R \cdot (a+b+c)}{2}.$$

To express R in terms of a, b, c, recall △ABC and △A'B'C'share the same circumcircle, its radius is R, and |A'B'|= 2c,|B' C'|= 2a, |A'C'|= 2b. From
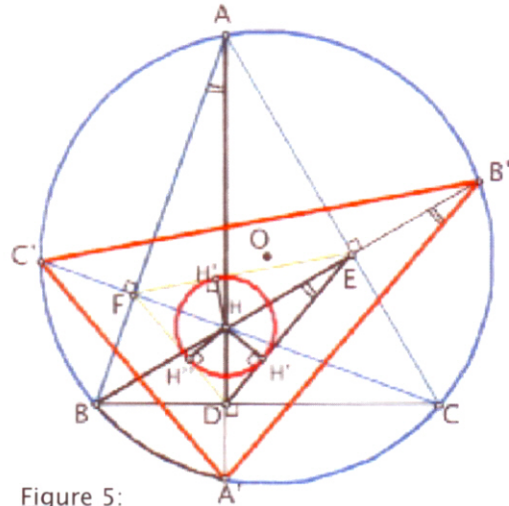
Figure 5:

Heron's formula, which is actually due to Archimedes, we then have:

$$\text{area}[\triangle A'B'C'] = \sqrt{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}.$$

Let α = ∠BAC and α '= ∠B'A'C', [Fig.4]. Since the angle at the center ∠C'OB' is double the angle α '= ∠C'A'B' at A', the two congruent right triangle halves of △OC'B' each have an angle α 'at O, whence R sin α '= a. Thus,

$$\text{Area } [\triangle A'B'C']= \frac{1}{2} bc \sin\alpha '= \frac{abc}{2R}$$

Upon substituting this into the preceding Heron formula and solving for R, we get:

$$R = \frac{2abc}{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}$$

Then, upon substituting this into the preceding expression for area[△ABC] in terms of R, a, b, c, we get the area solely in terms of a,b,c:

$$\text{area}[\triangle ABC] = \frac{abc(a+b+c)}{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}} \quad [*]$$

where a,b,c are the side lengths of the orthic triangle of area[△ABC] as desired.

Alternative Method. Instead of the hexagon, consider the points H', H'', H''' obtained by dropping perpendiculars from H to the sides of △DEF as indicated in

[Fig.5]. Since H is the incenter, we get $|HH'| = |HH''| = |HH'''| = r$, the inradius of $\triangle DEF$. Both $\angle HED = \angle BB'A'$ and $\angle HAB = \angle BAA'$ subtend the same arc BA', and we have $\angle HED = \angle HAB$. Similarly, $\angle EDH = \angle ABH$. Thus, due to angle equality, $\triangle EDH \sim \triangle ABH$. Therefore, the ratio of their areas is the square of the ratio of corresponding altitudes:

$$\frac{\text{area}[\triangle ABH]}{\text{area}[\triangle EDH]} = \frac{|FH|^2}{|HH'|^2}$$

Since $\triangle DEF$ is assumed to have known side lengths a, b, c, let us focus on that triangle to find FH and HH'. Since $\text{area}[\triangle DEF] = \text{area}[\triangle HEF] + \text{area}[\triangle DHF] + \text{area}[\triangle DHE]$,

$$\text{area}[\triangle DEF] = \frac{ra + rb + rc}{2} \quad [1]$$

Thus,

$$|HH'| = r = \frac{2 \cdot \text{area}[\triangle DEF]}{a + b + c}$$

which by Heron's formula for $\triangle DEF$ yields

$$r = \frac{\sqrt{(a + b + c)(a - b + c)(a + b - c)(-a + b + c)}}{2 \cdot (a + b + c)} \quad [2]$$

By the Pythagorean theorem,

$$|FH|^2 = r^2 + [FH']^2. \quad [3]$$

Since H is the incenter of $\triangle DEF$, we have $\triangle FHH'' \sim = \triangle FHH'''$ and, thus, $FH''' \cong FH''$. Similarly, $DH''' \cong DH'$ and $EH' \cong EH''$. Thus, $a + b = 2 \cdot |FH''| + |EH''| + |DH'''|$, and we get

$$FH'' = \frac{a + b - c}{2} \quad [4]$$

By substituting [4], [3], [2], and $|HH''| = r$ into [1],

$$\text{area}[\triangle ABH] = \text{area}[\triangle EDH] \frac{|FH|^2}{|HH'|^2} = \frac{rc}{2} + \frac{c(a + b - c)^2}{8r}$$

By symmetry, area $[\triangle BCH]$ and area $[\triangle ACH]$ are found to yield a corresponding result. Summing up the results for area $[\triangle ABH]$, area $[\triangle BCH]$, and area $[\triangle ACH]$,

$$\text{area}[\triangle ABC] = \frac{r(a + b + c)}{2} + \frac{c(a + b - c)^2 + c(a - b + c)^2 + a(-a + b + c)^2}{8r}$$

which, upon eliminating r with [2] and simplifying, gives the desired result [*].

Further Observations: At the outset, we proved H is also the incenter of $\triangle DEF$. This gives rise to observations regarding similarity and concurrency based on the orthic triangle. Notice, [Fig.2], the quadrilateral CDHE is cyclic. Moreover, $\angle HCD = \angle HED$ and $\angle HCE = \angle HDE$. Summing up,

$\angle ACB = \angle HCD + \angle HCE = \angle HED + \angle HDE = 180° - \angle DHE = \angle AHE.$

Since AEHF is also a cyclic quadrilateral, $\angle ACB = \angle AHE = \angle AFE$. Therefore, $\triangle ABC \sim \triangle AEF$. *Each of the three triangles formed by removing the orthic triangle from the original triangle is similar to that original triangle* since other cases of the similarity of the following triangles hold by relabeling vertices:

$$\triangle ABC \sim \triangle AEF \sim \triangle BDF \sim \triangle CDE$$

To obtain a concurrency result, drop perpendiculars from A, B, C to the sides of the orthic triangle $\triangle DEF$, with feet at H_1, H_2, H_3, respectively. Using the fact that CDHE is cyclic, we have that $\angle CHD = \angle CED$. Thus,

$$\angle HCD = 90° - \angle CHD = 90° - \angle CED = \angle ECH_3.$$

Lines, such as CF and CH_3, that go into each other when reflected about the angle bisector at a common point, C, are called isogonal conjugates. Similarly, BE is isogonally conjugate to BH_2, and AD is isogonally conjugate to AH_1. The lines CF, BE, AD are concurrent at H. The fundamental theorem on isogonal lines, which follows simply from Ceva's famous theorem, gives that their isogonal conjugate lines CH_3, BH_2, AH_1 are also concurrent at a point H, called the *isogonal conjugate of the point* H*. It is known that the isogonal conjugate of the orthocenter of any triangle is that triangle's circumcenter, and thus:

CH_3, BH_2, Ah_1 must meet at H* = 0, the circumcenter of $\triangle ABC$.

Euler has observed an interesting relationship: $d^2 = R'^2 - 2rR''$, [Fig.6]. To obtain this, return to the nine point circle theorem indicated in [Fig.1]. Recall M, the midpoint of BC, and N, the midpoint of AH are on the nine point circle, say K, which is the circumcircle of the orthic triangle $\triangle DEF$. Since $\angle ADC$ is a right angle, MN is the diameter of K. Therefore, $\angle NEM$ is a right angle. Since they share the same arc NE of K, $0 \angle HDH0 = \angle NME$. Thus, $\triangle HH0D \sim \triangle NEM$.

Therefore, $\frac{|HH'|}{|HD|} = \frac{|NE|}{|NM|}$ where $|HH'| = r$ and $|NM| = 2rR'$. Since N is the midpoint of the hypotenuse of right triangle $\triangle AEH$, $|NE| = |NH|$. Thus,

$$|NH| \cdot |HD| = 2rR'$$

To incorporate d, we apply the law of cosines to $\triangle HO'N$:

$d^2 = |HO'|^2 = |NO'|^2 + |NH|^2 - 2|NO'| \cdot |NH| \cdot \cos \angle HNO'$

$$=R'^2 + |NH|^2 - 2R' \cdot |NH| \cdot \frac{|ND|}{|NM|}$$

Since $|NM| = 2R'$,
$$d^2 = R'^2 + |NH|^2 - |NH| \cdot |ND|$$
$$= R'^2 - |NH| \cdot (|ND| - |NH|)$$
$$= R'^2 - |NH| \cdot |HD|.$$

Thus, we have obtained Euler's relationship:
$$d^2 = R'^2 - 2rR'$$

Cardinal Sins of the Infinite
by Keith F. Taylor, Dalhousie University

PART II

The Genie was stunned to see an exhausted ape collapsed in his cave with not a single coconut. Meanwhile, the monkey was a bit tired as well, but she was safely back in her cave with all of the coconuts. This certainly puzzled the Genie because, before he blinked, the ape had many times more coconuts than the monkey and seemed to be increasing his lead each trip.

Had the monkey broken the laws of logic and mathematics? The Genie decided to investigate, the way a mathematically oriented Genie does, by thinking it through. What about the coconut with the number k on it – when did it arrive in the monkey's cave? It was clear to the Genie that the monkey moved it on her k^{th} trip and that trip occurred during the time period from time $\frac{2^k - 1}{2^k}$ to time $\frac{2^{k+1} - 1}{2^{k+1}}$ hours after the ape started with his first load. So the monkey did not sin in any way, nor did she use any magic. No matter what numbered coconut the Genie considered, he could figure out when she deposited it in her cave. Satisfied with this conclusion, he went to bed.

At 3:00am the Genie sat up. Something was nagging at him. The coconuts were all the same except for the numbers. What if the monkey stole the highest numbered coconut from the ape's cave each trip? Then she would end up with those coconuts numbered as #10, #110, #1110, etc.

and the ape would end up with all the others (and probably not notice the missing ones). The Genie decided that would have been a better strategy for the monkey. She would have ended up with just as many coconuts and would not have had to deal with a furious ape when he recovered. He laid his head back down contented that he would be able to figure out the result no matter what strategy the monkey could have used in selecting her coconuts.

At 4:00am he was awake again. Things were getting stranger the more he thought about it and he could not stop thinking. What if the monkey could not read numbers and she just grabbed a coconut at random from the ape's cave on each trip? What would she have in the end? She certainly would have infinitely many coconuts, because she made infinitely many trips. What would the ape have? What if the coconuts had not been numbered at all? Could the ape actually move them all with his strategy then? Hmmmm.

The Genie decided thinking about the paradoxical properties of infinite processes was a lot more entertaining than owning pets. He returned the ape and monkey to the magic market and devoted himself to illuminating the infinite using his deductive powers. Although he never ran out of questions to ponder and lived happily ever after, he was often plagued with popping awake in the wee hours of the night.

A Pile of Shoes

by Ian Blokland

Interesting mathematics has a way of turning up in unexpected places. In September 2006, there was an orientation day for new students to the Augustana Campus of the University of Alberta. I was accompanying a group of about forty students as they toured the campus, attended short information sessions, and participated in some fun activities. In one activity, the students were asked to remove their shoes and put them in a large pile at the center of a room. The pile was then mixed and each student was asked to take two shoes, randomly chosen, from the pile. Next, the students were directed to locate the shoes which matched the ones they had chosen so that they would "link together" into chains. Eventually, it was expected that the students would form one giant loop.

As a chaotic mingling began to unfold, I started to think about what was going to happen. Eventually I turned to the volunteer standing beside me and told him that I expected that there would probably be more than one loop of students at the end and that my prediction was three loops. A few moments later, the students finished the task of pairing up all the shoes and, sure enough, there were three separate loops. Is there a predictable structure to this phenomenon or was I just lucky? As we'll soon see, it was a little bit of both.

Figure 1: An illustration of the basic problem. On the left we start with five pairs of shoes. On the right, these shoes are mixed into one big pile.

Figure 2: On the left, five students have each selected, at random, two shoes from the pile. On the right, the students link up into a single loop by matching shoes together.

Figure 3: With a different set of shoe selections from the pile, the students form two distinct loops.

1 The Model

We will start by working out the average number of loops, which we'll call L, that we can expect for a particular number of students, N. It is often a good idea to consider very simple cases first with the hope of recognizing a pattern which will apply to more complicated cases. For example, suppose $N = 1$. In this case, our lonely student puts her shoes into an otherwise empty pile, picks her own shoes out of the pile, and finds that the shoe in her left hand matches the one in her right hand. She then touches the shoes together, thereby forming one very simple loop. In other words,

$$L(N=1) = 1 \qquad (1)$$

Next, consider $N = 2$. No matter which shoe a student chooses with her left hand, there is one chance in three that she will find a matching shoe in her right hand and two chances in three that the second shoe will belong to the other pair. If each student has chosen a matching pair, they each touch the two shoes together to form two separate loops. If the students have each chosen mismatched shoes, they need to join together to link up the matching shoes, thereby forming one loop. We therefore calculate the average number of loops to be

$$L(2) = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot 1 = \frac{4}{3}. \qquad (2)$$

Another way to think about this is that we will always have at least one loop and that there is a one in three chance of getting an additional loop by virtue of each student selecting a matching pair of shoes:

$$L(2) = 1 + \frac{1}{3} = \frac{4}{3}. \qquad (3)$$

Moving up to $N = 3$, we again focus on one particular student. Regardless of which shoe is in the student's left hand, only one of the remaining five shoes will match it. If the shoe in the student's right hand produces such a match, this student forms a loop and the remaining two students will form either one or two loops according to the $N = 2$ case. If the original student has two mismatched shoes, she can link the shoe in her left hand with the matching shoe for another student, in which case we will have one chain of two students and one isolated student. The chain has an unpaired shoe at each end, therefore it behaves just like one of the students in the $N = 2$ situation. We conclude that the $N = 3$ case is just like the $N = 2$ case, except for the one-in-five chance that a specific student has chosen matching shoes in which case we will get an additional loop:

$$L(3) = L(2) + \frac{1}{5} = 1 + \frac{1}{3} + \frac{1}{5}. \qquad (4)$$

At this stage, we see a pattern beginning to emerge:

$$L(N) = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2N-1}. \qquad (5)$$

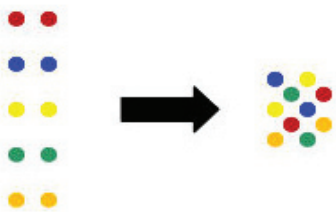We can justify it in the same way as the $N = 3$ case, namely by singling out a particular student and checking if she selected a matching pair of shoes. If so, and this happens with a probability of $1/(2N-1)$, we get an immediate loop along with a group of $N-1$ students. If not, this student will have to link up with someone else, thereby producing one double-student chain that acts like one student in a group of $N-1$ students. On average, we get $1/(2N-1)$ more loops with N students than we would expect with $N-1$ students.

2 An Approximation

The exact formula in (5) for the average number of loops $L(N)$ is quite cumbersome when N is not a single-digit number. For example, for the forty students I mentioned at the beginning, it takes quite a bit of time to compute the forty terms of (5) in order to determine that $L(40) \cong 2.826$. If only there was a button on a calculator that could add up lots of reciprocals in one step. It turns out that there already is such a button on most calculators: the "ln" button which calculates natural logarithms.

The specific connection between sums of reciprocals and the natural logarithm is the formula

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \cong \ln(n) + \gamma. \qquad (6)$$

The symbol γ refers to the Euler–Mascheroni constant and it has a numerical value of

$$\gamma = 0.5772156649\ldots \qquad (7)$$

Euler discovered this constant in 1735 and was able to compute it to 16 decimal places. He correctly anticipated its significance, as we now know that γ shows up in all sorts of different branches of mathematics. The $\cong$ in (6) indicates that the equation is only an approximation, but it is an approximation that gets better and better as the number of terms in the series, n, increases. Since it is much faster to compute the right-hand side of (6) than the left, we will be able to simplify our previous expression for $L(N)$. There is one minor problem, though: the approximation (6) computes the reciprocals of all whole numbers between 1 and n, whereas in (5), we only want the reciprocals of the odd numbers between 1 and $2N-1$. With a little bit of ingenuity, we can connect the two:

$$L(N) = 1 + \frac{1}{3} + \ldots + \frac{1}{2N-1}$$

$$= \left(1 + \frac{1}{3} + \ldots + \frac{1}{2N-1}\right) + \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2N}\right) - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2N}\right)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2N-1} + \frac{1}{2N}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{N}\right)$$

$$\cong \left(\ln(2N) + \gamma\right) - \frac{1}{2}\left(\ln N + \gamma\right)$$

$$\cong \ln 2 + \ln N + \gamma - \frac{\ln N}{2} - \frac{\gamma}{2} \qquad (8)$$

$$\cong \frac{\ln N}{2} + \ln 2 + \frac{\gamma}{2}.$$

Since we are already approximating and since

$$\ln 2 + \frac{\gamma}{2} \cong 0.98$$

we might as well round this to one and write

$$L(N) \cong \frac{\ln N}{2} + 1. \qquad (9)$$

Unlike the exact expression in (5), this approximation can be computed quickly on a calculator and, perhaps for some, estimated mentally. When applied to the $N = 40$ case, we obtain $L(40) \cong 2.844$, less than one percent away from the exact result. This compu-

tation explains why I was able to guess that the forty students in front of me might settle into three loops.

Dr. Ian Blokland is an Assistant Professor in the Physics and Mathematics Department of Science at the Augustana Campus of the University of Alberta.

Quickie Problem 2

Given X as

$$X = \sqrt{3\sqrt{3\sqrt{3\sqrt{2\sqrt{2\sqrt{3\sqrt{3\sqrt{3\sqrt{2\sqrt{2}}}}}}}}}\ldots}$$

$x + 2 = 5$
$x = 3$

"Just a darn minute! — Yesterday you said that X equals **two**!"

Prove X < 3

The Euler–Mascheroni constant (also called the Euler constant) is a mathematical constant recurring in analysis and number theory, usually denoted by the lowercase Greek letter γ (gamma). It is defined as the limiting difference between the harmonic series and the natural logarithm. The constant first appeared in a 1735 paper by the Swiss mathematician Leonhard Euler, titled De Progressionibus harmonicis observationes (Eneström Index 43). Euler used the notation C and O for the constant. In 1790, Italian mathematician Lorenzo Mascheroni introduced the notation A for the constant. The notation A appears nowhere in the writings of either Euler or Mascheroni, and was chosen at a later time because of the constant's connection to the gamma function.
(From Wikipedia, the free encyclopedia)

Euler-Mascheroni Constant

$$\gamma = \lim_{n \to \infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right]$$

From Our Readers

by David Leeming

An Inequality for Cyclic Quadrilaterals

In Issue # 10 of Pi in the Sky, it was shown that for any triangle, the inequality $A \leq \frac{1}{12\sqrt{3}} P^2$ holds, where A is the area and P is the perimeter of the triangle. The $\frac{1}{12\sqrt{3}}$ constant is also the best possible in the sense that it cannot be replaced by a smaller constant.

Two of our readers, Danesh Forouhari and Paddy Ganti observed that this inequality can be extended to cyclic quadrilaterals, that is, quadrilaterals whose four vertices lie on a circle. To do this we need Brahmagupta's formula for cyclic quadrilaterals. A good reference for this formula is Wikipedia.

Let A and P be the area and perimeter of a cyclic quadralteral (with sides a, b, c, and d) respectively. Brahmagupta's formula states:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad \text{where } s = \frac{P}{2} = \frac{a+b+c+d}{2} \quad \text{so}$$

using the Arithmetic-Geometric Inequality we have

$$[(s-a)(s-b)(s-c)(s-d)]^{\frac{1}{4}} \leq \frac{(s-a)+(s-b)+(s-c)+(s-d)}{4} = \frac{4s-2s}{4} = \frac{2s}{4} = \frac{s}{2} = \frac{P}{4}.$$

Squaring both sides, we have

$$A \leq \left(\frac{P}{4}\right)^2 = \frac{P^2}{16} \tag{1}$$

Note that equality holds in (1) if and only if $a = b = c = d$.

A MATHEMATICAL FOUNTAIN

By A. Clausing & T. Melkert

The Roman Fountain

High climbs the jet and, falling, fills
up to the brim the marble round
which overflows in veils and frills,
into a second basin's ground;
the second, now too rich, forsakes
its waves and on the third one spills and
equally it gives and takes
and stir and stills.

Conrad Ferinand Meyer (1825-1898)
Translation by Walter A. Aue[1]
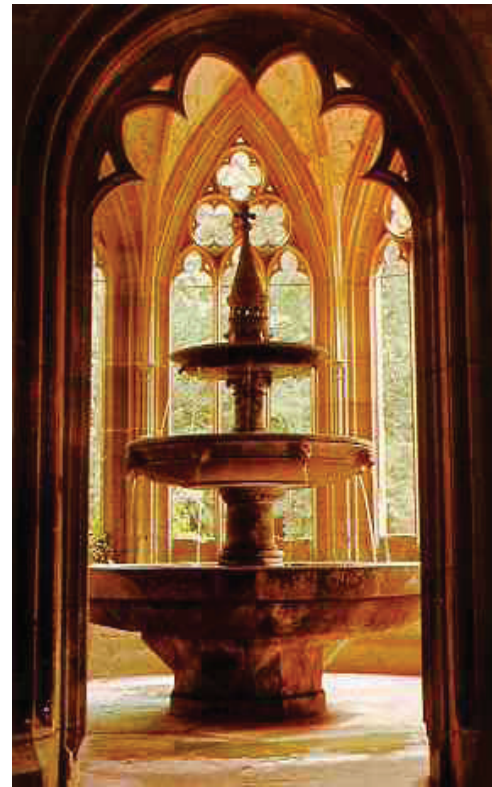
A curious observation

Even if you have no faible for poetry, please bear with us. This article is about a about a simple property of numbers. So simple that it could have been found a long time ago, but apparently it has been overlooked so far.

Write down four real numbers x, y, z, u and let $a = \frac{1}{4}(x + y + z + u)$ be their arithmetical mean. Below them, write the distances they have to a, that is, the four numbers $|x - a|, |y - a|, |z - a|, |u - a|$. If you repeat this process a few times, you arrive at four zeros (almost always, there is one exception).

Here is a fairly typical example:

x	y	z	u		a
6	1	7	9		$\frac{23}{4}$
$\frac{1}{4}$	$\frac{19}{4}$	$\frac{5}{4}$	$\frac{13}{4}$		$\frac{19}{8}$
$\frac{17}{8}$	$\frac{19}{8}$	$\frac{9}{8}$	$\frac{7}{8}$		$\frac{13}{8}$
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$		$\frac{5}{8}$
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$		$\frac{1}{8}$
0	0	0	0		

To make the calculation easier, you may replace a by the sum $s = x + y + z + u$ and then continue with the numbers $|4x - s|, |4y - s|, |4z - s|, |4u - s|$ as your next line. That way, if you start with four integers, you do not leave the set of integers. Obviously, you may multiply each line with any constant $c > 0$ without altering the number of steps it takes to reach the zero tuple.

It is, however, not very helpful to stick to the integers. The phenomenon holds for all real numbers. Look at the following example, it is chosen more or less arbitrarily: *(see next page)*
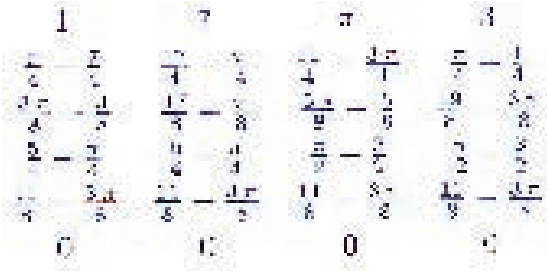
Sometimes, it takes a little longer to reach zero:

10	9	11	31
$\frac{21}{4}$	$\frac{25}{4}$	$\frac{17}{4}$	$\frac{63}{4}$
$\frac{21}{8}$	$\frac{13}{8}$	$\frac{29}{8}$	$\frac{63}{8}$
$\frac{21}{16}$	$\frac{37}{16}$	$\frac{5}{16}$	$\frac{63}{16}$
$\frac{21}{32}$	$\frac{11}{32}$	$\frac{53}{32}$	$\frac{63}{32}$
$\frac{1}{2}$	$\frac{13}{16}$	$\frac{1}{2}$	$\frac{13}{16}$
$\frac{5}{32}$	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{5}{32}$
0	0	0	0

But in general, the number of steps is between three and six: We ran a simple computer program to see how many steps a random tuple of numbers in [0,1] would need to reach zero. On average, only about 4.5 steps were needed. And it is particularly remarkable that this holds *independently of the size* of the numbers we start with – we could as well choose them in $[0,10^6]$ or in whatever range we like.

There is one notable exception to our obversation: If three, but not all four, of the numbers we start with are equal, then you never reach zero. Let us assume $x = y = z \neq u$. Then $s = 3x + u$, so we have $|4x - s| = |x - u|$ and $|4u - s| = 3|u - x|$. Thus the next line is, up to the constant $c = |x - u| > 0$, equal to

$$1 \quad 1 \quad 1 \quad 3$$

and these four numbers are now repeated forever.

Ducci sequences

Before we proceed, let us say a few words about the origin of this topic. Some of the older readers perhaps will remember that in 2004 there was an article by one of us in this magazine about a similar phenomenon dating back to the 1930s: If, instead of considering the map $(x,y,z,u) \mapsto (|x - a|, |y - a|, |z - a|, |u - a|)$ discussed in this article, one considers $(x, y, z, u) \mapsto (|x - y|, |y - z|, |z - u|, |u - x|)$, then essentially the same phenomenon holds: After finitely many steps, usually very few, the zero tuple appears.

The sequence of quadruples that is created by repeatedly applying this latter map is known as a Ducci sequence, after the Italian mathematician Enrico Ducci who first noticed this phenomenon. Apparently, he did not publish his discovery. Nevertheless, Ducci sequences have, in the 70 years since their appearance in the literature, found a considerable amount of attention. A complete bibliography of the subject would probably contain close to 100 papers.

Let $\mathbb{R}_+^4$ denote the set of all quadruples of nonnegative numbers, and let $f : \mathbb{R}_+^4 \to \mathbb{R}_+^4$ be the map taking $t = (x,y,z,u)$ to $(|x - a|, |y - a|, |z - a|, |u - a|)$, with $a = (x+y+z+u)/4$. Also, we use the notation $f^2(t)$ for $f(f(t))$, etc., for the iterated application of f.

We are interested in the sequence $t,\ f(t),\ f^2(t),\ f^3(t),\ \dots$ which we call the Fontana sequence of t, after the nonexisting Italian mathematician Felice Fontana. We call the length of the Fontana sequence of t up to, but excluding, the first zero tuple the height $h(t)$ of t. For example: $h(1,7,\pi,3) = 5$, $h(0,0,0,0) = 0$

$$h(x,x,x,x) = 1 \text{ if } x \neq 0$$
$$\text{if } x \neq y, \text{ then } h(x,x,x,y) = \infty$$

The Fontana and Ducci sequences of t are defined in a very similar way, but there is one clear difference: The length of the Ducci sequence of t in general depends on the order of the elements of t, whereas the length of the Fontana sequence is the same for every ordering of the tuple as the arithmetic mean does not depend on this order.

There are two different kinds of tuples $t \in \mathbb{R}_+^4$: For some, two of the numbers lie on either side of their arithmetic mean a. Let us call the set of these tuples B (for balanced). The other kind is the tuples where three of the numbers lie on one side of a and one on the other side. We let A (for *apart*) be the set of these. A and B are not disjoint, since some of the numbers $x,\ y,\ z$ or u may be equal to their average a, as for example in the tuple $(1,3,4,8)$, where $a = z = 4$.

A:
B:
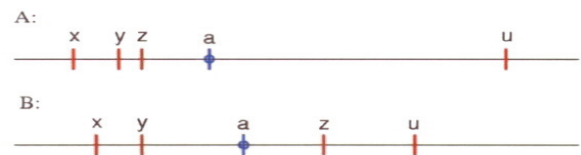
Balanced tuples cannot have a height greater than 4:

$$\text{If } t \in B, \text{ then } f^4(t) = (0,0,0,0).$$

Let us see why. In a balanced tuple $t = (x,y,z,u)$ with $x \leq y \leq a \leq z \leq u$, we have $f(t) = t' = (x',y',z',u')$ with $x' = a - x$, $y' = a - y$ and $z' = z - a$, $u' = u - a$. Hence $(x'+y')-(z'+u') = 2a-(x + y) + 2a-(z + u) = 4a-(x + y + z + u) = 0$ that is, $x'+y'=z'+u'$. Let now $a' = \frac{1}{4}(x' + y' + z' + u')$ be the arithmetic mean of t', then. Hence a' is between x' and y' and likewise between z' and u'. We may assume $x' \leq a' \leq y'$ and $z' \leq a' \leq u'$.

Then for

$$f(t') = t'' = (x'', y'', z'', u'')$$

we find

$$x'' = a' - x', \quad y'' = y' - a', \quad z'' = a' - z'$$

and $u'' = u' - a'$. Hence $x'' - y'' = 2a' - (x' + y') = 0$ or $x'' = y''$ and, similarly, $z'' = u''$. Thus $t'' = (x'', x'', z'', z'')$. It is now easy to see that $t''' = f(t'')$ is a constant tuple and $f(t''') = f^4(t) = (0,0,0,0)$.

Here is an example:

x	y	z	u		a
1	3	6	10		5
4	2	1	5		3
1	1	2	2		$\frac{3}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$
0	0	0	0		

Almost all unbalanced tuples have finite height:

If $t \in A$ and no three components of t are equal, then there exists $k \in \mathbb{N}$ such that $f^k(t) = B$.

Let $t = (x, y, z, u) \in A$ and $a' = \frac{1}{4}(x' + y' + z' + u')$
We assume $x \le y \le z \le a < u$ (hence also $x < a$) and not $x = y = z$. Then

$$f(t) = (a - x, a - y, a - z, u - a)$$

Clearly, the height of t is not altered, and the resulting tuple is still in A, if we add an arbitrary constant c to every component of t.

Thus by adding $c = \frac{1}{2}(u - x - y - z)$ we can assume that $x + y + z = u$ which implies $a = \frac{1}{2}(x + y + z) = \frac{1}{2}u$.

Let $d_x = x - \frac{u}{3}$, $d_y = y - \frac{u}{3}$, and $d_z = z - \frac{u}{3}$. Note that $d_x \le d_y \le d_z$ and $d_x + d_y + d_z = x + y + z - u = 0$ Hence $d_x \le 0$ and $d_z \ge 0$.

(In fact, $d_x < 0$ since $x < a$).

Then $t = (x, y, z, u) =$

$$(\frac{u}{3} + d_x, \frac{u}{3} + d_y, \frac{u}{3} + d_z, u)$$

is mapped to

$$f(t) = (\frac{u}{2} - x, \frac{u}{2} - y, \frac{u}{2} - z, \frac{u}{2}) =$$

$$(\frac{u}{6} - d_x, \frac{u}{6} - d_y, \frac{u}{6} - d_z, \frac{u}{2})$$

The arithmetic mean of this tuple is $\frac{u}{4}$. If $\frac{u}{6} - d_x \le \frac{u}{4}$ this tuple is again in A, otherwise it is in B. If it is in A, then it is mapped by f to

$$\left(\frac{u}{12} + d_x, \frac{u}{12} + d_y, \frac{u}{12} + d_z, \frac{u}{4} \right)$$

If $\frac{u}{12} + d_z \le \frac{u}{8}$ this is again in A. As long as the Fontana sequence of t remains in A, we have

$$f^k(t) = \left(\frac{u + (-1)^k d_x}{3 \cdot 2^k}, \frac{u + (-1)^k d_y}{3 \cdot 2^k}, \frac{u + (-1)^k d_z}{3 \cdot 2^k}, \frac{u}{2^k} \right)$$

with the arithmetic mean $\frac{u}{2^{k+1}}$ by the same argument. The condition for this tuple to be in A is $-d_x \le \frac{u}{2^{k+1}} - \frac{u}{3 \cdot 2^k} = \frac{u}{6 \cdot 2^k}$ for even k and $d_z \le \frac{u}{6 \cdot 2^k}$ for odd k. If k is large enough, then this condition is certainly violated since $d_x < 0$.

Note that this argument only works since not $x = y = z$ and hence not $d_x = d_y = 0$. Otherwise the Fontana sequence of t indeed remains in A as we have seen.

The Fountain

There is a nice geometric interpretation of our sequences. For this, we note that we can confine ourselves to tuples $t = (x, y, z, u)$ with $u = x + y + z$. If t is not a constant tuple, we have $u > 0$, hence by dividing with u we can also assume $u = 1$ without altering the height of the tuples.

For $(x, y, z, 1)$ with $x \ge 0$, $y \ge 0$, $z \ge 0$ and $x + y + z = 1$ we write $[x, y, z]$. The set of all tuples of this form can be visualized as a triangle T:
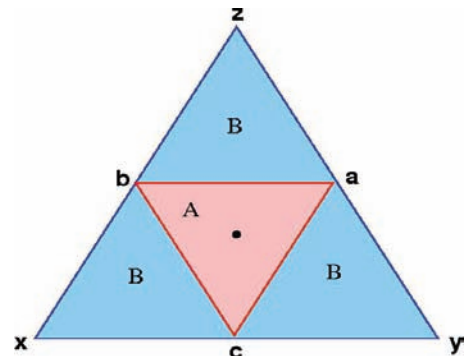
In this representation, if x, y, z are the corners of T (somewhere in $\mathbb{R}^2$), then $[x, y, z]$ corresponds to the point

$$p = x \cdot \mathsf{x} + y \cdot \mathsf{y} + z \cdot \mathsf{z}.$$

In particular, the exceptional tuple $(1,1,1,3)$ is represented by $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$, corresponding to the center point of T.

All tuples $(x,y,z,1) = [x,y,z]$ in the triangle have the arithmetic mean $a = \frac{1}{2}$, thus the red triangle in the center contains the points $[x, y, z]$ for which $x \le a$, $y \le a$, $z \le a$ holds, that is, the points in the set A. The blue triangles contain the balanced points.

The Fontana map f maps the corners a,b,c of A to the corners x, y, z of T, and is affine on A. Explicitly, $[x, y, z] \in A$ is mapped to $[1 - 2x, 1-2y, 1-2z]$, in the preceding section we had shown that:

$$f(x,y,z,1) = \left(\frac{1-x}{2}, \frac{1-y}{2}, \frac{1-z}{2}, \frac{1}{2}\right) = \frac{1}{2}(1-2x, 1-2y, 1-2z, 1)$$

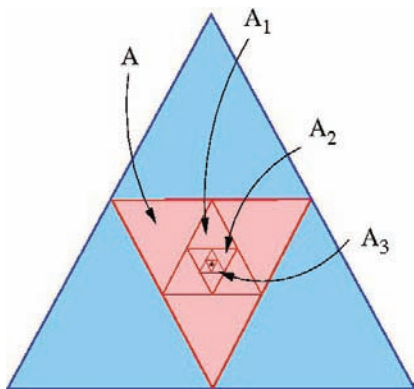It just turns A around and stretches it by a factor of 2.

If you trace the map f backwards, you see that the red triangle $A = A_0$ contains a smaller copy A_1 of itself, which again contains a smaller copy A_2 of itself, and so on. These copies converge to the center point $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$.

A point in A_k $(k > 0)$ is mapped by f to a point in A_{k-1}. Thus with increasing k, the heights of the points in A_k increase. The points with really large height are found close to $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$.

If you imagine the A_k as the basins of a fountain similar to the one pictured at the beginning of this article, then you can see the Fontana sequence of a point in A_k: The point "flows down" to the basins with lower heights until it reaches the lowest basin consisting of the blue corner triangles of T.

What happens then? From the previous section you can easily deduce that points in the interior of the blue basin are mapped to it's boundary, from here to one
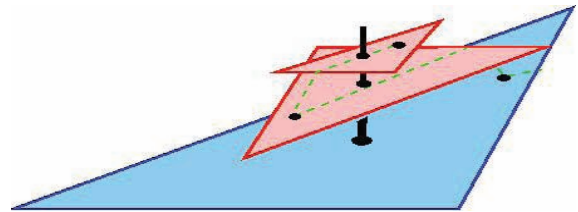
of the corners of T, and finally to a constant tuple (which has no representing point in T). It sort of vanishes into the ground.

We hope that you now can see the fountain: The basins are triangular, and there are infinitely many of them, but we find that this even underlines the special beauty of this structure. Carl Friedrich Gauss once wrote: "You have no idea how much poetry there is in the calculation of logarithms" -- and likewise, as he certainly would have agreed to, in the calculation of some sequences.

1 References

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Challenge for our readers:

Students are encouraged to write a program that, for a positive integer n produces a 4-tuple whose Fontana sequence is n steps long. The programming involved would not be difficult, and the basic idea for a solution is contained in the paper.

Beg, Steal or Borrow?
Making Better Decisions in the Library

By Jon Warwick, London South Bank University

If you have ever wandered into your college library to look for a particular book only to discover that the book is not on the shelf as it has been borrowed, then you'll know that this can be an exasperating experience. Having discovered that there isn't a copy of the book in the library you have, perhaps, five possible actions open to you. First, you could look for a substitute book among the shelves in the local vicinity hoping that you will come across one that covers the material you wanted; second, you could just shrug your shoulders and make a mental note to come back again in a couple of weeks and check again; third, you could make your way to the service counter (or nearest computer terminal in many cases) and place a reservation for the book; fourth, you could just give up and resolve to try getting the book from another source (another library, sub-lend from a colleague, photocopy relevant parts from somebody else's copy, or just buy it yourself); fifth, you could just give up altogether!

These actions are not mutually exclusive in the sense that you might try again next week and if still unsuccessful then place a reservation or just give up. Whatever your sequence of actions is, though, it is likely that it will be influenced by your expectation as to what will happen in response to your actions otherwise (and assuming that you are not acting totally irrationally!) you would have no basis for choosing the 'best' action. You would need to ask yourself questions such as "Is the book likely to be available next week if I come back?" or "How long would I wait if I make a reservation for the book?" Since making a reservation simply places you in a queue to receive the book, this last question is akin to asking "How long can I expect to wait in a queue before I can be served?" where service here corresponds to borrowing the book.

Since the 1960's mathematical modellers have been using this analogy with queuing theory to try and answer questions just such as these and we will take a brief look at some models that will, perhaps, help you answer some of these questions for yourself. Details of the classic text by Phillip Morse describing the use of queuing theory in libraries are given in the short bibliography.

A Context for Queuing Theory

Let us consider a simple example of a single copy of a title in demand by customers. In queuing theory models, it is assumed that customers arrive at a facility requiring a particular service (perhaps at a check-out to pay for grocery shopping or at a bank to withdraw money from a cash machine) and if the server (the checkout person or the cash machine) is already busy, then an orderly queue forms. The customers are assumed to be arriving individually and at random but at a known average rate of λ per unit time. Similarly the average service rate of customers per unit time is denoted by μ but the amount of time serving each customer varies according to some statistical distribution with a known mean of $\frac{1}{\mu}$. The standard deviation of service times is σ and measures how variable service times are from one customer to the next so that, for example, low values of σ would indicate that all customers get roughly the same service time. For the purposes of our discussion, the customers will be students arriving at the library to try and borrow the book, and the service time will be the time a successful borrower actually has the book in his or her possession. Once the service is complete (the book is returned) then the next student in the queue may borrow it. The queue corresponds to those students who have placed a reservation for the book or, if the queue is empty, a student coming in to borrow it.

There is a variety of queuing situations that have been well studied by mathematical modellers and most management science textbooks will have sections relating to queues or waiting lines. Many will deal with well behaved first-in-first-out queues and the different models are generated by considering different arrival and service time patterns. We shall assume for our discussion that arrivals are random (with an aver-

age rate λ) but we will consider three different treatments of the service process since this relates to the loan period which is an important parameter in determining the effectiveness of any library service. In each case, though, the average service rate is denoted as μ and the standard deviation of service times as σ.

First, though, we need to note some formulae that have been derived in the theoretical analysis of queues (for a good treatment of basic theory see Quantitative Analysis for Business Decisions listed in the bibliography). First, we denote the traffic intensity, $\rho = \frac{\lambda}{\mu}$ and for a stable queue we need $\rho < 1$. This means that on average customers are served at a faster rate than the rate at which they arrive. If this is not the case then the queue would just grow without bound over time and attempting to calculate other queue statistics is meaningless.

Second, the probability of the queue being empty and the server idle (we call this an empty system and denote it as P_0) is $1 - \rho$.
Thirdly, the expected queuing time , $\bar{Q}$, is given by:

$$\bar{Q} = \frac{\rho \, (1 + \mu^2 \sigma^2)}{2\mu(1 - \rho)} \qquad (1)$$

and this denotes the average time that customers spend waiting in the queue before their service can start.

These three parameters are useful in our library context because they tell us whether the queue is in a stable situation (ρ) and if so then P_0 gives the likelihood of finding the book on the shelf if we return at a later date and $\bar{Q}$ gives us the likely waiting time if we make a reservation. It is worth noting here that P_0 and ρ depend only on the average arrival and service rate. The variation in service times, σ , only alters $\bar{Q}$. From equation 1 we see that the more variation there is in service times (larger values of σ) then the longer waiting times are likely to be. It is the variation in the arrival and service patterns that causes queues to grow and shrink.

Now, the most difficult modelling decision is that of how we assume students who have the book will behave. In other words, will they observe the correct loan period, return the book early or be tempted to keep it out overdue? To examine this, we will look at three cases.

Case 1

Let us assume that every student borrows the book for exactly the duration of the loan period, no less and

no longer. In this case, the average service time, $\frac{1}{\mu}$ will be equal to the loan period and as there is no variation in service times $\frac{1}{\mu}$ then $\sigma = 0$. This gives $\frac{1}{\mu}$ from (1):

$$\bar{Q} = \frac{\rho}{2\mu(1 - \rho)} \qquad (2)$$

with ρ and P_0 just as before.

Case 2

It is unlikely that every student will return their book exactly in accordance to the loan period so to make things more realistic we allow some variability in the return of the book by assuming that the service process is also random, just like the arrival process. In this case, the average service time (the loan period) will be, as before, $\frac{1}{\mu}$ but we will have $\sigma = \frac{1}{\mu}$ also. This is the classic queuing model[1] and although the formulae for ρ and P_0 remain the same, we now have from (1):

$$\bar{Q} = \frac{\rho}{\mu(1 - \rho)} \qquad (3)$$

Although better than case 1, this case is also somewhat unrealistic. Even though there is now variability in the return process (which is realistic), some students will return the book early and some will keep the book out way past the loan period since we have only assumed the average service time equals the loan period. This is not realistic if, as is the case in many libraries, a firm policy is in place to deter late returns using fines or restricting further borrowing etc.

Case 3

Finally, let us make the return process even more realistic by borrowing an idea from project planning. Very often, project planners need to estimate how long particular activities will take to run. They make use of a probability distribution known as the Beta distribution which is useful because it has two parameters that can alter its shape, and we can also specify limits on the variable, x. Below we have given two examples with different values of the two shape parameters a and b, and allowing x to range from zero to four.

After years of painstaking research I can finally reveal this equation. It proves conclusively that I am a chick magnet.
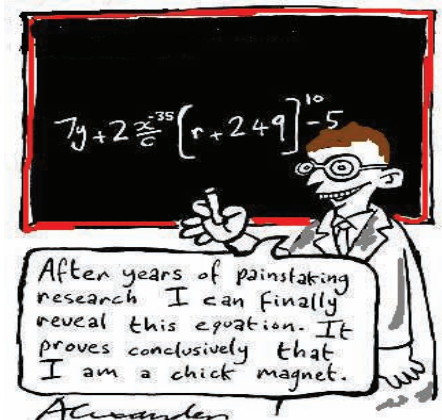
[1] Often referred to in text books as the M/M/1 model

Plot of the Beta Distribution
Shape parameters a = 2 and b = 4

Plot of the Beta Distribution
Shape parameters a = 4 and b = 3
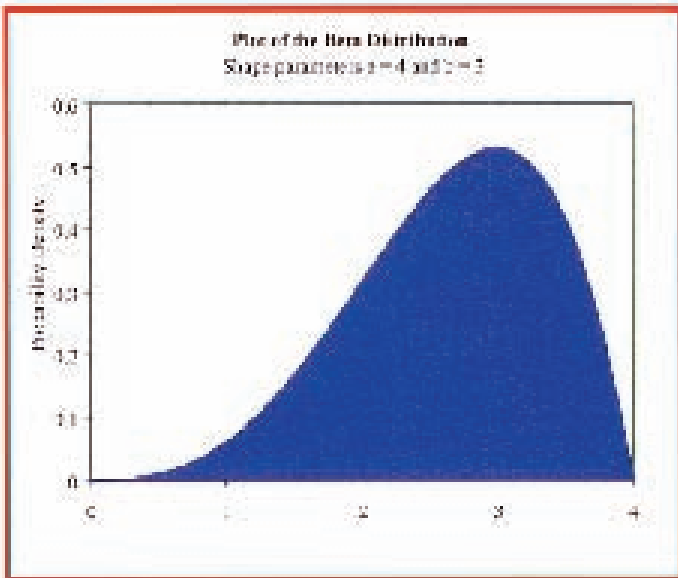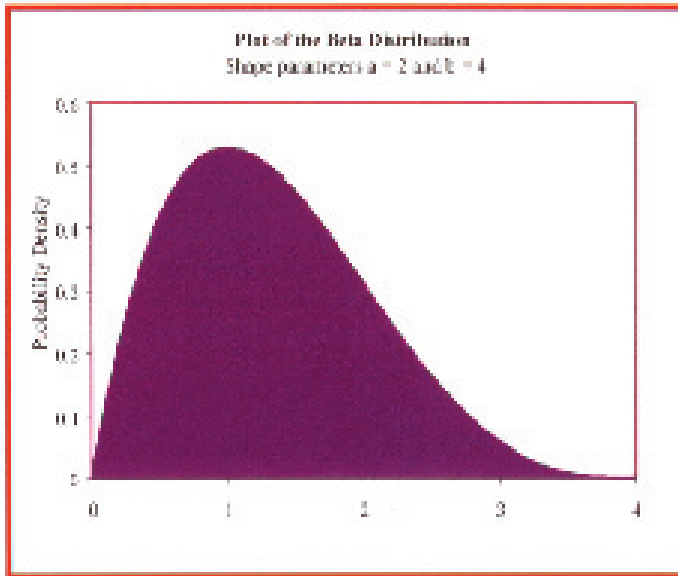
The horizontal scale denotes the variable (in our case the time a book is kept on loan) and the vertical scale the probability density. In the second case we can see that this is useful because it allows us to specify a book return pattern whereby students are more likely to return the book as the loan period nears its end (a four week loan period in the illustration) and if the library operates strong penalties for overdue books, nobody keeps the book out once the loan period has expired.

Project planners use the Beta distribution in the following way. They first estimate the minimum and maximum time that an activity will take and then specify the most likely time (these are sometimes called the optimistic, pessimistic and most likely times). These figures define the range of values for x and the approximate position of the distribution mode which is the highest point on the density function. Then the average activity time is estimated as:

$$\frac{T_{Min} + 4T_{Likely} + T_{Max}}{6}$$

and the standard deviation of activity time is estimated as:

$$\frac{T_{Max} - T_{Min}}{6}$$

To give an example, if the loan period is 4 weeks and there is a fine system in operation then we might choose $T_{Min} = 1$, $T_{Max} = 5$ and assuming the book has been strongly recommended students will want to keep it out so $T_{Likely} = 4.5$. The formulae above then give us the following:

Mean service time, $\frac{1}{\mu}$, is now

$$\frac{1 + 4 \times 4.5 + 5}{6} = \frac{24}{6} = 4 \text{ weeks.}$$

The standard deviation of service times, σ, is

$$\frac{5 - 1}{6} = \frac{4}{6} = 0.67 \text{ weeks.}$$

An Illustrative Example

Let us now take an illustrative example to compare all three cases. Let us suppose that the book attracts one demand every 5 weeks on average from students who are willing to place a reservation, and that the loan period in operation is 4 weeks. Let us also assume that the library operates a fairly harsh fine system for overdue books. The table on the next page gives the results for all three cases described above and we have used a week as the basic unit of time.

From the information presented above, we can see that the chance of finding a book on the shelf is the same for each case ($P_0 = 0.2$) since λ and μ remain the same. Under case 2 (the standard queuing model with purely random service times) the expected waiting time for a reserved book is an unreasonably long and unrealistic 16 weeks!

Note that the expected waiting time between cases 1 and 2 are quite different. As we mentioned earlier, it is the variation in arrival and service times that cause queues to grow and reduce and in case 1 we had (unrealistically) no variability at all in service times.

Finally it is worth pointing out that in case 3 we have the added modelling effect of the library fines policy which helps to govern T_{Max}. Selection of values for T_{Min} and T_{Likely} would depend on the perceived popularity of the book, whether students had been strongly recommended to read it, how much course material it covered etc. The model presented is relatively easy to use in a spreadsheet and you could play with the

parameters of case 3 to get a feel for the variation in output values.

We have looked at a simple queuing model under different assumptions about service times. Modellers would also be concerned about the arrival process (which we have not considered in detail here). For example, we have assumed that students arrive at random and individually, whereas in reality recommending a book to students would probably result in several students trying to borrow the book immediately and probably arriving together! Queuing theory does take account of cases where arrivals are grouped together (or 'batched' in queuing terminology) and also where λ changes over time (perhaps high initially and then reducing) but this is beyond the scope of this article.

Faced with a waiting time of a little over 8 weeks what would I do? I'd probably make a reservation for the book but whilst waiting I'd certainly investigate borrowing a copy from another library or beg a friend to lend me their copy. I might even consider buying it – but stealing it would be out of the question!
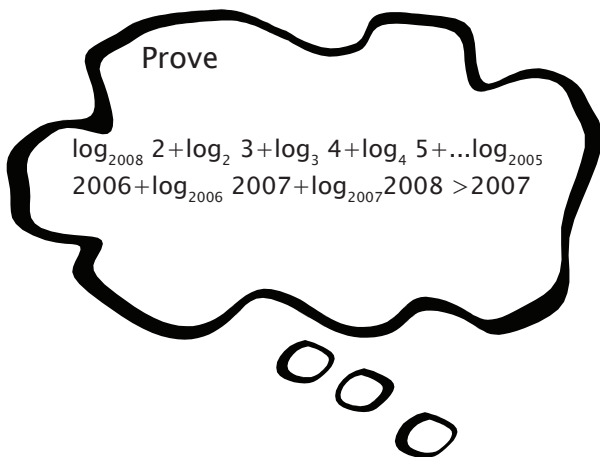
Bibliography
Bonini, C. P., Hausman, W., Bierman, H. (1997) *Quantitative Analysis for Business Decisions*, (9th Edition), McGraw–Hill.

Morse, P. M., (1968) *Library Effectiveness* – A Systems Approach, MIT Press, Cambridge, Mass.

	Case 1	Case 2	Case 3
	$\lambda = 0.2,\ \mu = 0.25,\ \sigma = 0$	$\lambda = 0.2,\ \mu = 0.25,\ \sigma = 4$	$T_{Min} = 1;\ T_{Max} = 5;\ T_{Likely} = 4.5$ $\lambda = 0.2,\ \mu = 0.25,\ \sigma = 0.67$
ρ	$\frac{0.2}{0.25} = 0.8$	$\frac{0.2}{0.25} = 0.8$	$\frac{0.2}{0.25} = 0.8$
P_0	$1 - 0.8 = 0.2$	$1 - 0.8 = 0.2$	$1 - 0.8 = 0.2$
$\overline{Q}$	$\frac{0.8}{(2 \times 0.25)(1-0.8)} = 8.0$	$\frac{0.8}{0.25(1-0.8)} = 16.0$	$\frac{0.8(1-0.2^2 \times 0.67^2)}{(2 \times 0.25)(1-0.8)} = 8.1$

Quickie Problem 3

> Prove
>
> $\log_{2008} 2 + \log_2 3 + \log_3 4 + \log_4 5 + \dots \log_{2005} 2006 + \log_{2006} 2007 + \log_{2007} 2008 > 2007$

Definition: Queueing theory
 is the mathematical study of waiting lines (or queues). The theory enables mathematical analysis of several related processes, including arriving at the (back of the) queue, waiting in the queue (essentially a storage process), and being served by the server(s) at the front of the queue. The theory permits the derivation and calculation of several performance measures including the average waiting time in the queue or the system, the expected number waiting or receiving service and the probability of encountering the system in certain states, such as empty, full, having an available server or having to wait a certain time to be served.

(From Wikipedia, the free encyclopedia)

Impossible? Surprising Solutions to Counterintuitive Conundrums

(Hardback – 264 pages (2008); Princeton University Press: ISBN 13:978–0691131313)

A book review by Gordon Hamilton

In his book "*Impossible?*" Julian Havil grabs the talented grade 12 student and recreational mathematician and bustles them through some counter-intuitive mathematics. He is not gentle, in that he uses a lot of equations, but the frequency that this book delivers those "Aha!" moments is unsurpassed by any book I've read.

Topic selection is brilliant. In the 18 chapters, Julian deals with both recreational mathematical topics like card tricks and hard core mathematics topics like the Banach-Tarski paradox. No self-respecting mathematician would even think of bundling such diverse topics into one book, but Julian pulls it off so effectively that the reader is not even conscious how unusual and inspired the choices are.

"*Impossible?*" also consistently targets the same audience. Julian accomplishes this by increasing the difficulty of the easier topics by exploring their extensions. For example, the Monty Hall three-door problem is a great venture into conditional probability, but it is too simple to base a chapter on, so Julian explores extensions in different directions – including what might unpoetically be called the Monty Hall four-door, two-option problem. That's refreshing.

The Monty Hall four-door, two-option problem: Behind three of the doors are goats, but behind the last door is a brand new car! Choose a door. Now Monty Hall opens one of the other three doors and shows you that it contains a goat. He then gives you your first option: "Do you want to stay with your initial choice – or switch to one of the other closed doors?" He then repeats the procedure – opening one of the other doors – revealing a goat – and giving you your second option: "Do you want to stay with your choice – or switch to the other closed door?" What is the best way to play this game so that you are most likely to drive away in a new car?

The chapters are not all equally long – two of the shortest are on Gamow-Stern Elevators and Wild-Card Poker. The behaviour of the elevators was the one part of the book that I found unconvincing, so I've written a mini article about them (to appear in the next issue). The Wild-Card Poker chapter is fun, and will give you a good flavour for the book as a whole, so I'm going to reproduce its essence – declaring a *spoiler alert* before I proceed.

Julian begins by calculating the frequency of the poker hands – in a deck without any wild cards:

This table matches our sense of justice. We know that the Straight flush beats four of a kind which beats a full house which beats a Flush which beats a Straight...and it is not a coincidence that the frequency

Poker Hand	Frequency	Probability
Straight flush	40	0.0000154
Four of a kind	624	0.000240
Full house	3 744	0.00144
Flush	5 108	0.00197
Straight	10 200	0.00392
Three of a kind	54 912	0.0211
Two pairs	123 552	0.0475
One pair	1 098 240	0.423
Odd card	1 302 540	0.501

of the poker hands decreases as one ascends this hierarchy.

Surely it is a simple matter to construct such a table after a wild card is added to the deck – well let's do it:

Poker Hand	Frequency	Probability
Five of a kind	13	0.0000045
Straight flush	204	0.000071
Four of a kind	3 120	0.001087
Full house	6 552	0.002283
Flush	7 804	0.00272
Straight	20 532	0.00715
Three of a kind	137 280	0.04783
Two pairs	123 552	0.04305
One pair	1 268 088	0.44189
Odd card	1 302 540	0.45390

Oh no! Our sense of justice is outraged to see that Three of a kind actually has a higher frequency than the lower ranked Two pairs. Now think about how to solve this problem. Think carefully.

It seems to be an easily solved problem – the Three of a kind has become more popular than Two pairs – so let's just swap them:

Two pairs	205 920	0.04783
Three of a kind	54 912	0.04305

Unfortunately, when the Two pair is made more valuable than the Three of a kind, all of the poker hands with a wild card, which we previously chose to elevate to Three of a kind, we now choose to elevate to Two pairs. Justice is impossibly elusive. Isn't that irritating!

**A final couple of points that indirectly reveal how much this book is valued by those who have read it:

1) When I received this book from Pi in the Sky's managing editor, David Leeming, it came with a curt little sticky note pasted to the front cover: "please return after reviewing".

FLATLAND: A JOURNEY OF MANY DIMENSIONS: DVD
A REVIEW OF THE SPECIAL EDUCATION EDITION

Review By Sharon Friesen, PhD

Flatland: A Journey of Many Dimensions is based on the extraordinary book by Edwin A. Abbott (1884). The Special Educational Edition DVD contains the movie (35 minutes), a copy in pdf format, an interview with Dr. Thomas Banchoff of Brown University and four math lesson samples.

There are two movie versions of Flatland, one by independent filmmaker Ladd Ehlinger Jr. and the other by Jeffrey Travis. This is a review of the Travis film to which actors such as Martin Sheen, Kristen Bell and Michael York lend their voices.

Travis has created strong visual effects. He has addressed many of the social issues that reading Abbott's book in today's classrooms raises. The film plot unfolds about two central characters a square, Arthur and his granddaughter, a hexagon named Hex. This film plays fairly loose with the book, but it holds together well. It has a compelling emotional component which resonates with today's youth.

This film is weak in its ability to remain true to the ideas of a 2-dimensional universe. The central characters are human-like in appearance. Things such as eye movement, toys, opening a briefcase do not make sense in a 2-dimensional universe. They only make sense within a 3-dimensional universe. Far too many of these instances cause me to wonder whether the animators caught on to the implications of living in a 2-dimensional universe.

This is an engaging film to be shown to middle and high school students but it really lacks the mathematical depth contained in Abbott's masterpiece; therefore, I would not recommend it for university math majors. That said, it is socially inoffensive and makes available geometrical ideas and concepts that occur to few nonmathematicians. It was a rare treat to be able to watch a film that entertained mathematical ideas. In that, it has the potential to inspire a new generation of students, teachers and the public to engage with and be fascinated by geometrical ideas.

The interview with Dr. Tom Banchoff of Brown University is excellent. Dr. Banchoff makes many ideas related to dimensionality explicit through various animations and related commentary. His models of 4-dimensions are particularly good and he puts forward a challenge to the viewer, "What if we were visited by a sphere of the 4th-dimension?".

Dawson's Chess (Board Game)

Equipment: A 3 x n chessboard, n white pawns (or counters) and n black pawns (or counters)
Number of players: 2

The Set-up:
Players decide on the size of board. Our examples use a 3x8 board i.e. three rows of a chess board.
The set up is as follows:

A Move:
At your turn you move one of your pawns forward one square. However if you are in a position to capture one of your opponents pawns (one square diagonally forward to left or right as in normal chess) then you must capture that pawn. Capturing is mandatory in this game! White always moves first in a game.

A Win:

The last player to move wins. If you have no legal move then you lose.
Notes: In Dawson's Chess sequences of moves can occur where there is no choice for either player.
For example: suppose White on his first move moves one of his edge pawns:

Then Black is forced to capture this pawn:

White in reply is forced to capture this Black pawn:

The Black and White pawns at the right of the board cannot now make any further moves in the game. Who wins, if they play their best move at each turn, in a 3 x 3 game, in a 4 x 3 game and in a 5 x 3 game, White or Black?

Sprouts (a paper & pencil game)

The Game of Sprouts was invented in 1967 by Princeton mathematician John H. Conway and by Michael S. Paterson, when both were at the University of Cambridge in the UK. Here is a quote from Conway:

"The day after sprouts sprouted, it seemed that everyone was playing it, at coffee or tea times, there were little groups of people peering over ridiculous to fantastic sprout positions."

Equipment:

Paper and pencil

Number of players: 2

The Set-up: A smallish number of spots are drawn.

Your move: Join two spots, or one spot to itself, with a curve which cannot touch or cross any other curve or spot. Now place a new spot on this new curve. No spot can have more than three curves attached to it.

You win: If your opponent has no valid move.

Comments: Here is a complete sample game starting with 2 spots:
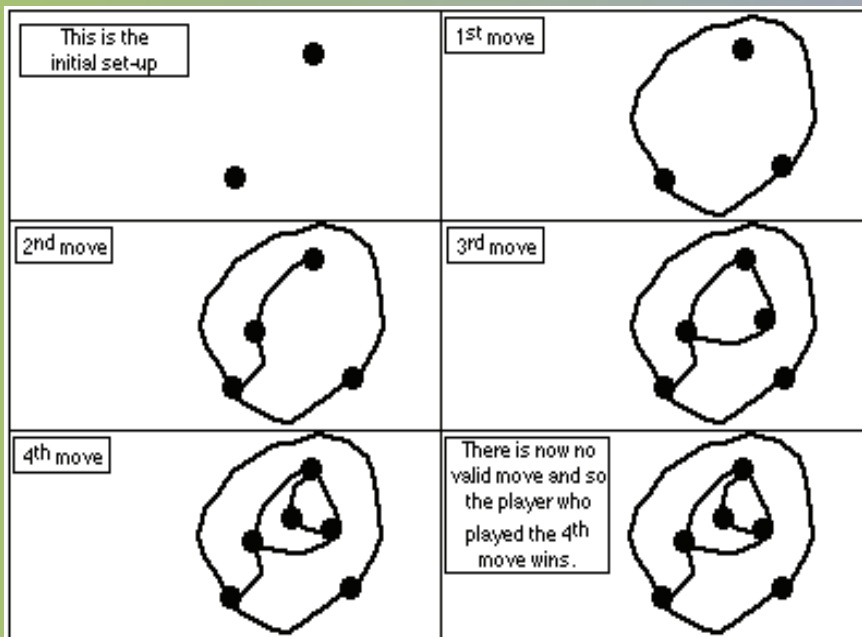
For these and other interesting mathematical games go to: http://www.madras.fife.sch.uk/maths/games

Other mathematical games available:

Dvonn Awards:

2002 International Gamers Award Winner

Mensa Select Mind Games 2002

YINSH Awards:

Board Game Award: Mensa Select Mind Game 2004

http://www.boardgameratings.com/award_winners.php?award_type=MS

For grades 1 through 5, "For Sale" by Stefan Dorra, www.boardgamegeek.com/game/172, and ThinkFun puzzles like Rush Hour and Hot Spot. These and other games are found at http://www.thinkfun.com.

Pi in the Sky Math Challenges

Problem 1:
Prove that the equation $x^5 + y^5 + 2 = (x + 1)^5 + (y + 2)^5$ does not have integral solutions

Problem 2
Prove that $p_n > 2n$ for every $n \geq 5$, where p_n denotes the nth prime number ($p_1 = 2$).

Problem 3
Inside a square with side length 1 there are 201 points. Prove that there exists a circle of radius 0.1 which contains at least three of these points.

Problem 4:
Let a, b, c be positive numbers such that $abc = 18$. Prove that
$$\frac{a^3 + b^3 + c^3}{3} \geq a\sqrt{b + c} + b\sqrt{a + c} + c\sqrt{a + b}$$

Problem 5
Let $ABCD$ be a convex quadrilateral, M the midpoint of BC and N the midpoint of CD. If $AM + AN = 1$ then the area of the quadrilateral is less than $1/2$.

Problem 6:
There are given five line segments having the property that any three of them can be the sides of a triangle. Prove that at least one of these triangles must be acute.

Solutions to the problems published in the Spring 2008 Issue of PI In The Sky

Problem 1:
Find all four digits numbers $abcd$ such that $abcd - a^4 - b^4 - c^4 - d^4$ has a maximum value.
Solution by Henry Ricardo, Medgar Evers College, New York:
We have
$$f(a,b,c,d) = abcd - a^4 - b^4 - c^4 - d^4 = a(10^3 - a^3) + b(10^2 - b^3) + d(1 - d^3)$$
Since each term of the last sum is nonnegative, the maximum value of f(a,b,c,d) is attained when each summance is maximized. It is an elementary calculus exercise to determine that for each $k \in \{0,1,2,3\}$ the function $f(x) = a(10^k - x^3)$ attains its maximum value when $x = 3\sqrt{10^k/4}$. Taking the nearest integer value of x for $k = 0, 1, 2$, and 3 yields $a = 6$, $b = 3$, $c = 1$ and $d = 1$. It is clear that $d = 0$ results in the same maximum. Thus the numbers we seek are 6310 and 6311, leading to a maximum value of
$f(6, 3, 1, 1) = f(6,3,1,0) = 4932$.

Problem 2:
Let $\{a_1 ... a_n\} \cup \{b_1 ... b_n\} = \{1, 2 ... 2n\}$ such that $a_1 < ... < a_n$, $b_1 > ... > b_n$. Prove that
$$|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = n^2$$
Solution:
For each $i \in \{1, 2, ..., n\}$ one of the two numbers a_i, b_i is less than or equal to n and the other is greater than n. Indeed, if by contradiction $a_i \leq n$ and $b_i \leq n$, then $a_1 < ... < a_i \leq n$ and $b_n < ... < b_i \leq n$. Hence we have found $n + 1$ distinct positive integers which are less then or equal to n, a contradiction. Similarily, we can not have $a_i > n$ and $b_i > n$. Thus:
$$|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = (n + 1) + ... + (n + n) - 1 - 2 - ... - n = n^2$$

Problem 3:
Let A be a set of postive integers such that for every $m, n, \in A$, $|m - n| \geq 1/20 (m + 1)(n + 1)$. What is the maximal number of elements that A could have?
Solution: Since $|m - n| \geq 1/20 (m + 1)(n + 1)$ we must have $\left| \frac{1}{m + 1} - \frac{1}{n + 1} \right| \geq \frac{1}{20}$ (*)

By the Pigeonhole Principle, we conclude that at most one element of A is greater than 19 and that the interval $(0, 1/5)$ contains at most 4 numbers of the form $\frac{1}{n + 1}$, $a \in A$.

Hence A contains at most 4 elements which are are greater than 4. Since $\{1, 2, 3, 4\}$ verifies (*), the largest set A whose elements satisfy (*) could be $A = \{1, 2, 3, 4, a, b, c, d\}$ where $4 < a < b < c < d$ are integers, with at most one greater than 19.

Call For Papers

Submissions are now open for the next issue of *Pi in the Sky*, Spring 2009

Pi in the Sky is aimed primarily at high school students and teachers, with the main goal of providing a cultural context–landscape for mathematics. It has a natural extension to junior high school students and undergraduates, and articles may also put curriculum topics in a different perspective.

Guidelines:

Papers should have a scientific content of original, unpublished material. Submissions should be no more than 5,000 words. Submissions can be sent in Latex, Adobe pdf, or Word format. If including graphics, please submit all graphics in either *png* or *tif* format. Contents should contain material suitable for students of high school.

Pi in the Sky accepts materials on any subject related to mathematics and its applications, including articles, problems, cartoons, statements, and jokes. Copyright of material submitted to the publisher and accepted for publication remains with the author, with the understanding that the publisher may reproduce it without royalty in print and electronic forms. Submissions are subject to editorial review and revision. We also welcome Letters to the Editor from teachers, students, parents, and anybody interested in math education (be sure to include your full name, phone number and e–mail address).

Deadline for submissions of articles for review is January 15, 2009.

For further information on the content and style of the journal please view previous issues on the following website:

http://www.pims.math.ca/media/publications/pi-sky

Submissions can be sent either by email: pi@math.uvic.ca

or

By mail if sending a disk:

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