Theory of Equatorially trapped waves

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Basic References

A. J. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, (Chapter 9) 2005, Courant Lecture Series #8 Publ. Amer. Math. Soc.

Special Issues

#1) Theoretical Developments in Tropical Meteorology
Vol. 20, #5-6, November 2006
Theoretical & Computational Fluid Dynamics

#2) Atmospheric Convection and Wave Interactions: Convective Life Cycles and Scale Interactions in Tropical Waves
Dynamics of Atmospheres & Oceans
Vol. 42 2006

Waves and PDEs for the Equatorial Atmosphere and Ocean

9.1. Introduction to Equatorial Waves for Rotating Shallow Water

We begin by recalling the shallow water equations

$$\frac{D\vec{v}}{Dt} + f\vec{v}^{\perp} + g\nabla h = 0\,, \quad \frac{Dh}{Dt} + (H+h)\operatorname{div}\vec{v} = 0\,, \tag{9.1}$$

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LINEARIZED EQUATORIAL SHALLOW WATER EQUATIONS:

$$\frac{\partial u}{\partial t} - \beta y v = -g\frac{\partial h}{\partial x}\,, \quad \frac{\partial v}{\partial t} + \beta y u = -g\frac{\partial h}{\partial y}\,, \quad \frac{\partial h}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,. \tag{9.9}$$

9.1.1. Kelvin Waves. These waves will be in geostrophic balance in the meridional (north-south or y) direction, but will move at gravity wave speeds, so they have this mixture of the properties of what in the midlatitudes are separate classes of waves. They will be trapped in the equatorial waveguide. They are an important part of the observational record of the equatorial atmosphere and ocean. We seek

flows parallel to the equator ($v = 0$), so that the v-component of the momentum equation yields the geostrophic balance,

$$\beta y u = -g \frac{\partial h}{\partial y} . \tag{9.10}$$

The remaining evolution equations become

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x} , \quad \frac{\partial h}{\partial t} = -H \frac{\partial u}{\partial x} . \tag{9.11}$$

Introducing characteristic variables

$$q = \left(\frac{g}{H}\right)^{1/2} h + u \quad \text{and} \quad r = \left(\frac{g}{H}\right)^{1/2} h - u \tag{9.12}$$

allows us to isolate the eastward and westward moving waves

$$\frac{\partial q}{\partial t} + c \frac{\partial q}{\partial x} = 0 \ \text{eastward} \quad \text{and} \quad \frac{\partial r}{\partial t} - c \frac{\partial r}{\partial x} = 0 \ \text{westward} \tag{9.13}$$

where $c = \sqrt{gH}$ is the gravity wave speed. It looks like a general solution will be an arbitrary superposition of these two waves, but we need to inquire into their structure in y.

Eastward Group: $r \equiv 0$ (so $u = ch/H$). Solving (9.13) is naturally accomplished by separation of variables

$$q = G(y)q(x - ct)$$

where $G(y)$ is determined by the requirement of geostrophic balance (9.10), which leads to an equation for G

$$\frac{\partial G}{\partial y} + \frac{\beta}{c} y G(y) = 0\,,$$

which immediately integrates to gives

$$G(y) = G_0 e^{-\frac{\beta}{c}\frac{y^2}{2}}\,, \tag{9.14}$$

and we conclude that the equator acts as a waveguide, trapping the wave.

Westward Group: $q \equiv 0$. Again, separation of variables leads to an equation for the structure in y, but the sign is changed from the eastward group, so

$$\frac{\partial G}{\partial y} - \frac{\beta y}{c} G(y) = 0 \quad \text{gives} \quad G(y) = G_0 e^{\frac{\beta}{c}\frac{y^2}{2}}\,.$$

This solution violates finite energy and is clearly unphysical with large growth as y increases.

In summary, Kelvin waves only propagate in the eastward direction, do not disperse, and travel with the gravity wave speed $c = \sqrt{gH}$. They are in geostrophic balance and are trapped in the equatorial waveguide.

Inquiring into units, we see that there is a length scale that determines the trapping distance in y,

$$L_e = \sqrt{\frac{c}{\beta}} \quad \text{equatorial deformation radius.}$$

9.1.3. Weakly Nonlinear Kelvin Waves. We consider the nonlinear equatorial shallow water equations,

$$
\begin{aligned}
&\eta_t + [(1+\eta)u]_x + [(1+\eta)v]_y = 0\,, \\
(9.25) \qquad &u_t + uu_x + vu_y + \eta_x - yv = 0\,, \\
&v_t + uv_x + vv_y + \eta_y + yu = 0\,,
\end{aligned}
$$

written in nondimensional form through the length scale $L_e = (\frac{c}{\beta})^{1/2}$, time scale $T_e = (c\beta)^{-1/2}$, velocity scale c, and unit for height $\frac{c^2}{g}$. As shown in the formula from (9.15) above, the linear Kelvin waves are nondispersive but equatorially

We consider solutions to these equations in the form of power series in small parameter ϵ (weakly nonlinear expansion),

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon K(x-t,\tau)e^{-\frac{y^2}{2}} \\ \epsilon K(x-t,\tau)e^{-\frac{y^2}{2}} \\ 0 \end{pmatrix} + \epsilon^2 \vec{u}_2 \,, \tag{9.26}$$

where the parameter $\tau = \epsilon t$ is a long-time scale. We seek solutions that are valid uniformly for long times with τ of order 1. This particular form of the first-order solution is drawn from the linear theory of Kelvin waves in Section 9.1.1 (recall that $q = u + \eta$, $r = -u + \eta$, and $v = 0$ for a linear Kelvin wave). The first-order solution $\vec{u}_1$ is the Kelvin wave solution of the following linear system:

$$\begin{aligned} \frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} + \frac{\partial v}{\partial y} - yv &= 0\,, \\ \frac{\partial r}{\partial x} - \frac{\partial r}{\partial x} + \frac{\partial v}{\partial y} + yv &= 0\,, \\ \frac{\partial v}{\partial t} + y\frac{q-r}{2} + \frac{\partial}{\partial y}\frac{q+r}{2} &= 0\,. \end{aligned} \tag{9.27}$$

As discussed in Sections 5.7 and 8.3 above, in order for the asymptotic expansion in (9.26) to be valid formally, we must require that solutions at the next order grow sublinearly in t, i.e.,

$$|\vec{u}_2| = o(t)\,. \tag{9.28}$$

Now consider the forced linear system for $\vec{u}_2$, which appears at the next order:

$$\frac{\partial \eta_2}{\partial t} + \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = A \,, \tag{9.29a}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial \eta_2}{\partial x} - y v_2 = B \,, \tag{9.29b}$$

$$\frac{\partial v_2}{\partial t} + \frac{\partial \eta_2}{\partial x} + y u_2 = 0 \,. \tag{9.29c}$$

The last equation in the system above has no forcing, since the y-component of first-order velocity satisfies $v_1 = 0$. The components of forcing, A and B, have terms arising from two different sources: from a slow time derivative and from nonlinearity. They have the following structure:

$$\begin{aligned} A &= -e^{-\frac{y^2}{2}} K_\tau - e^{-y^2} (K^2)_\theta \,, \\ B &= -e^{-\frac{y^2}{2}} K_r - e^{-y^2} \left(\frac{1}{2} K^2\right)_\theta \,, \end{aligned} \tag{9.30}$$

where θ is the phase variable, $\theta = x - t$. Now, we add the equations in (9.29a) and (9.29b) to get

$$\frac{\partial(u_2+\eta_2)}{\partial t} + \frac{\partial(u_2+\eta_2)}{\partial x} + \frac{\partial v_2}{\partial y} - y v_2 = (A+B)\,. \tag{9.31}$$

In order to compute the nonlinear Kelvin wave response to forcing, we project the above equation onto the Kelvin mode by multiplying it by $e^{-y^2/2}$ and noticing that

$$e^{-\frac{y^2}{2}}\left(\frac{\partial v_2}{\partial y} - y v_2\right) = \frac{\partial}{\partial y}(e^{-\frac{y^2}{2}} v_2) \tag{9.32}$$

is a perfect derivative. If we assume that v_2 does not grow too much in the y-direction in order to have a valid asymptotic expansion (which is a natural requirement for equatorial waves), then

$$\lim_{L\to\infty}\int_{-L}^{L}\frac{\partial}{\partial y}(e^{-\frac{y^2}{2}} v_2)dy = \lim_{L\to\infty}(e^{-\frac{y^2}{2}} v_2)\Big|_{-L}^{L} = 0\,. \tag{9.33}$$

In order to formulate the nonresonance condition and to shorten the notation, we introduce the Kelvin wave projection of the second-order solution $\tilde{q}_2$ by

$$\tilde{q}_2 = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}(u_2+\eta_2)dy\,. \tag{9.34}$$

The resulting projected equation for the Kelvin wave response to forcing has the form

$$\frac{\partial \tilde{q}_2}{\partial t} + \frac{\partial \tilde{q}_2}{\partial x} = -2\int_{-\infty}^{\infty} e^{-y^2} K_\tau(x-t,\tau)dy - \frac{3}{2}\int_{-\infty}^{\infty} e^{-\frac{3y^2}{2}} K_\theta^2(x-t,\tau)dy\,. \tag{9.35}$$

This equation presents a specific example of an "auxiliary problem" discussed earlier in Section 5.7 and arising in a number of forced dispersive systems,

$$\frac{\partial \tilde{q}}{\partial t} + \frac{\partial \tilde{q}}{\partial x} = f(x - t, \tau)\,, \tag{9.36}$$

where the forcing can depend on a long-time scale. The equation above has an exact solution, vanishing at $t = 0$,

$$\tilde{q} = t f(x - t). \tag{9.37}$$

This solution is obviously secular for long times of order $\tau = \epsilon t$. In order for a solution to be valid, we must require a sublinear growth of the solution $\tilde{q}$. The only way to satisfy this requirement is to set $f \equiv 0$. Imposing this condition in (9.35) yields the

NONLINEAR KELVIN WAVE EQUATION:

$$c_1 K_\tau + c_2 K_\theta^2 = 0\,, \quad c_1 = 2\int_{-\infty}^{\infty} e^{-y^2} dy\,, \quad c_2 = \frac{3}{2}\int_{-\infty}^{\infty} e^{-\frac{3}{2}y^2}\, dy\,. \tag{9.38}$$

This is the famous inviscid Burgers equation, which has the typical feature of breaking, shock formation, and dissipation of energy stored in the propagated wave. It arises in many contexts as the canonical asymptotic equation for nondispersive waves; see [**16**, chap. 1].

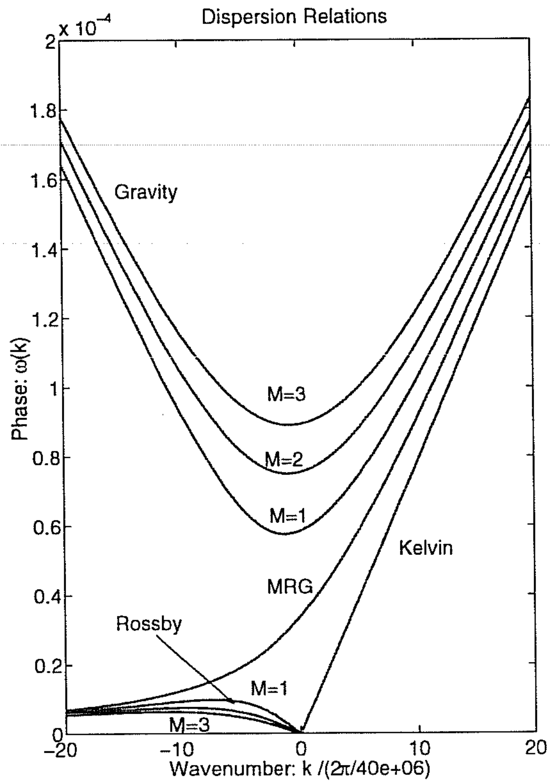


FIGURE 9.1. Phase versus the wavenumber normalized by the wavenumber 1 for the first baroclinic $c_1 = 50\ \mathrm{ms}^{-1}$.

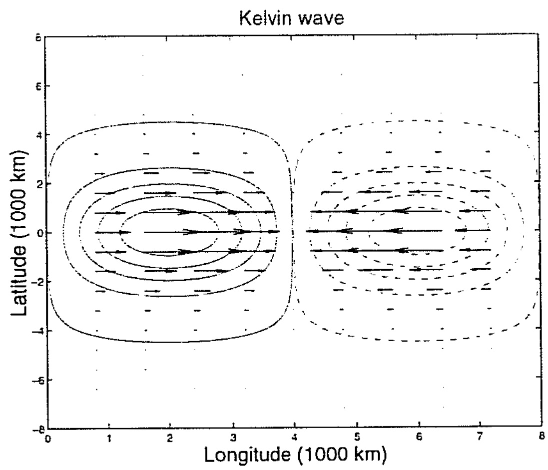


FIGURE 9.2. Filled contours of the pressure and velocity profile (arrows). Here and below, dotted contours are low pressure.

The Equatorial Primitive Equations

$$\frac{\partial \vec{v}_H}{\partial t} + \beta y \vec{v}_H^{\perp} = -\nabla_H p\,,$$
$$-p_z + B = 0\,,$$
$$\frac{\partial B}{\partial t} + N^2 w = \mathcal{S}(x, y, z, t)\,,$$
$$\mathrm{div}_H\, \vec{v}_H + w_z = 0\,.$$

Here $\vec{v}_H = (u(x, y, z, t), v(x, y, z, t))$ is the horizontal velocity field with $\vec{v}_H^{\perp} = (-v, u)$, w is the vertical velocity, B is the buoyancy with N the constant buoyancy frequency, β is the Coriolis parameter coefficient, $\beta = 2\Omega/R$ with $\Omega = (2\pi)/24$ hours the angular velocity of the earth, $R = 6378$ km its radius, and $\mathcal{S}$ a forcing term that may represent heating from evaporation or cooling by radiation effects, etc. We assume rigid lid boundary conditions at top and bottom of the troposphere, namely,

$$w(x, y, z, t)\big|_{z=0,H} = 0\,, \tag{9.40}$$

where H is the height of the troposphere, and $z = 0$ represents the surface of the earth.

Vertical Separation Variables

$$\mathcal{S}(x, y, z, t) = \sum_{q=1}^{+\infty} \mathcal{S}_q(x, y, t)\frac{dG_q(z)}{dz},$$

$$\vec{v}_H(x, y, z, t) = \sum_{q=1}^{+\infty} \vec{v}_H^q(x, y, t)G_q(z),$$

$$p(x, y, z, t) = \sum_{q=1}^{+\infty} p^q(x, y, t)G_q(z),$$

$$B(x, y, z, t) = \sum_{q=1}^{+\infty} B^q(x, y, t)\frac{dG_q(z)}{dz},$$

$$w(x, y, z, t) = \sum_{q=1}^{+\infty} w^q(x, y, t)\frac{dG_q(z)}{dz}.$$

$$0 = w\big|_{z=0,H} \iff \frac{dG_q}{dz} = 0$$

Equate Coefficients in Expansion

$$\left\{ \frac{\partial \vec{v}_H^q}{\partial t} + \beta y \vec{v^q}_H^{\perp} = -\nabla_H p^q , \right\}$$

Momentum part for Linear Shallow Water

$$p^q = B^q ,$$

$$\frac{\partial B^q}{\partial t} + N^2 w^q = \mathcal{S}_q(x, y, t) ,$$

$$\mathrm{div}_H \, \vec{v}_H^q G_q(z) + w^q \frac{d^2 G}{dz^2} = 0 .$$

$$w^q = \frac{1}{N^2}\left(-\frac{\partial p^q}{\partial t} + \mathcal{S}_q(x, y, t)\right)$$

(*) $$\mathrm{div}_H\, \vec{v}_H^q G_q(z) + \left(-\frac{\partial p^q}{\partial t} + \mathcal{S}_q(x, y, t)\right)\frac{1}{N^2}\frac{d^2 G}{dz^2} = 0\,.$$

(*) $\Longleftrightarrow$ $$\frac{d^2 G_q}{dz^2} + \lambda_q^2 G_q = 0\,, \quad \frac{dG_q}{dz} = 0\,, \quad z = 0, H, \;\; q = 1, 2, \ldots,$$

Pressure eqn. for Shallow Water

Vertical Structure $$G_q(z) = \cos\left(\frac{q\pi z}{H}\right) \quad \text{with } \lambda_q = \frac{q\pi}{H}\,.$$

<u>Barotropic Mode</u> $q = 0$, $\lambda_q = 0$, $G_q \equiv 1$

Each Vertical Mode, $q \geq 1$

Shallow Water Eqn.

$$\frac{\partial u}{\partial t} - \beta y v + c_q \frac{\partial p}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} + \beta y u + c_q \frac{\partial p}{\partial y} = 0,$$
$$\frac{\partial p}{\partial t} + c_q \frac{\partial u}{\partial x} + c_q \frac{\partial v}{\partial y} = \frac{1}{c_q} S_q(x, y, t)$$

, $$c_q \equiv \frac{NH}{q\pi}$$

Introduce as for Kelvin – MRG

$$q = \frac{1}{\sqrt{2}}(p+u), \qquad r = \frac{1}{\sqrt{2}}(p-u)$$

Typical Troposphere Values

$N \approx 10^{-2} s^{-1}$, $H = 16 km$ ⇒ Prominent Observations

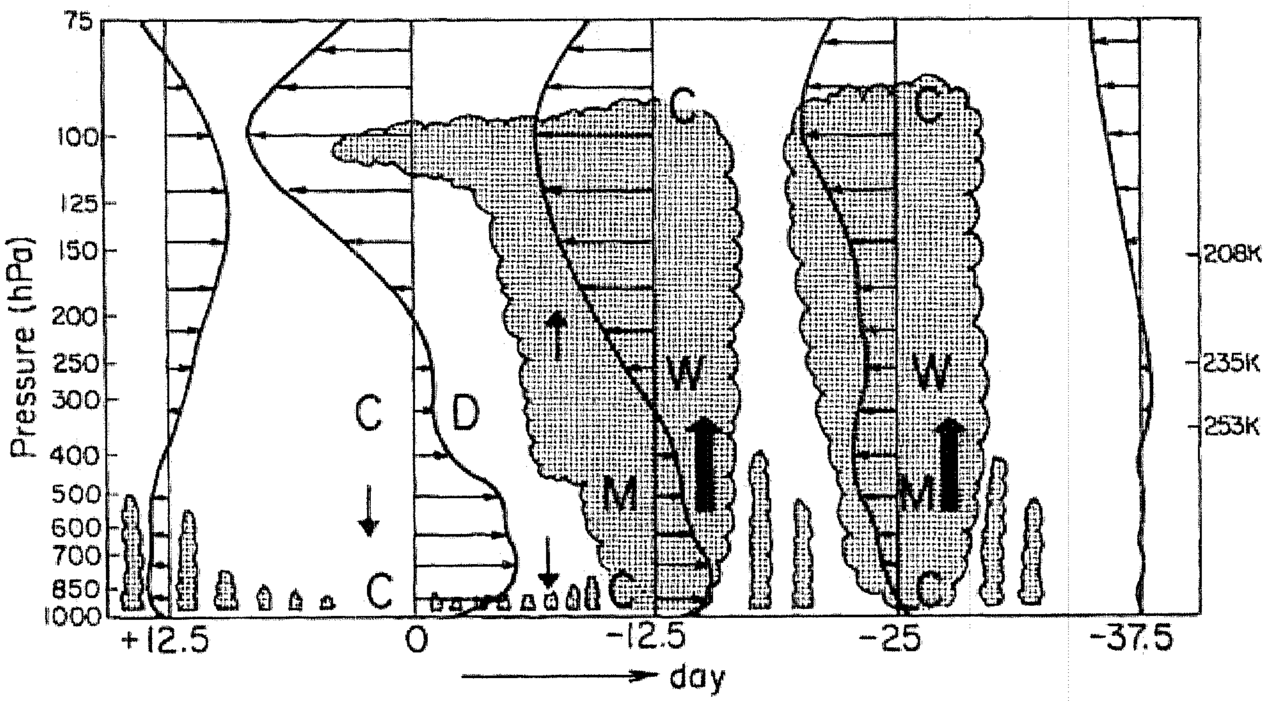
$c_1 = 50\ ms^{-1}$ — 1st Baroclinic

$c_2 = 25\ ms^{-1}$ — 2nd Baroclinic

Dry wave speeds

Kinematics of tropical convection: A flavor

1 March 1996 LIN AND JOHNSON 711



Self-similarity in tropical convection

Squall lines

Zipser 1969

Zipser et al 1981

Two-day waves

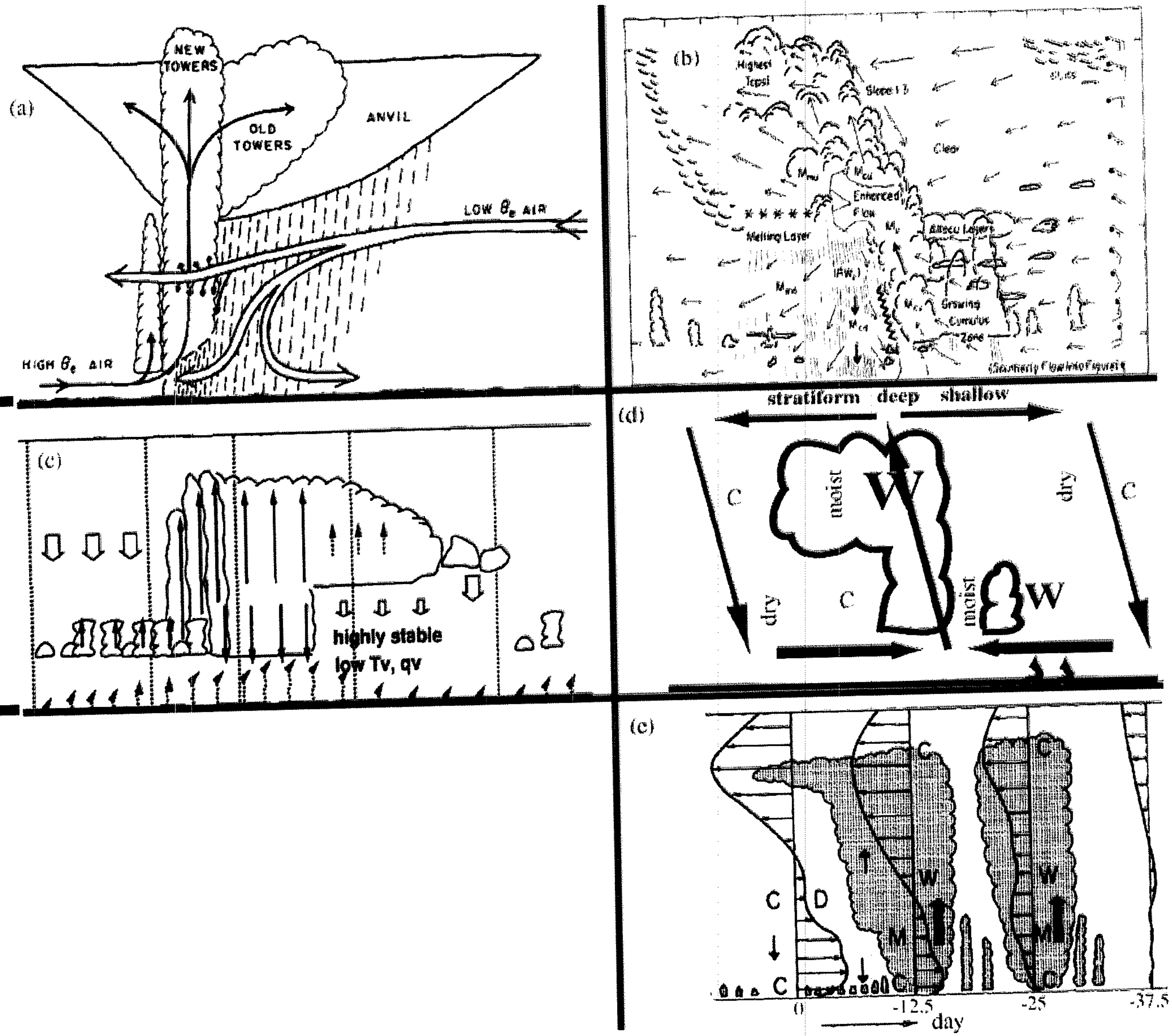
Takayabu et al 1996

C. C. Kelvin waves

Straub and Kiladis 2003

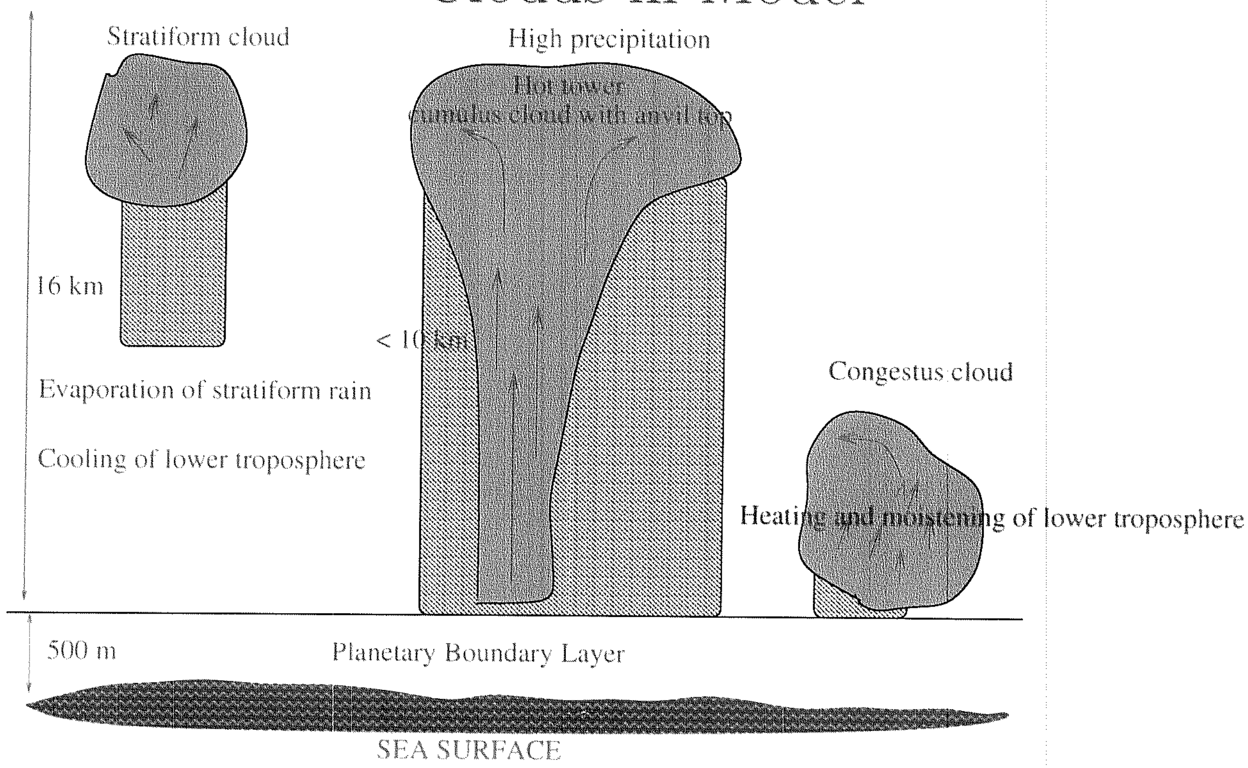
Madden–Julian Osc.

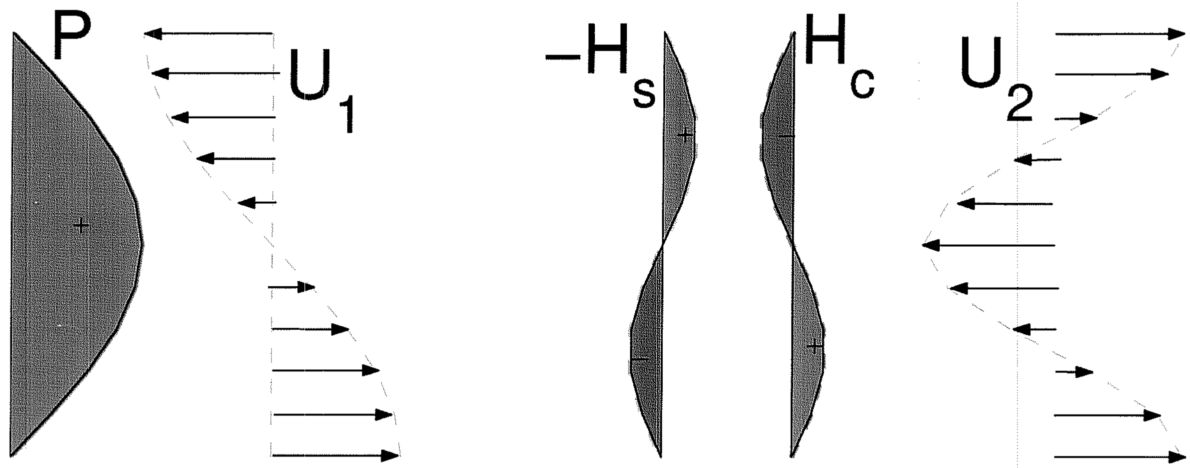
Lin and Johnson 1996



Source: Mapes et al. 2006 DAO

Clouds in Model





- Coupled two shallow water systems

$$\frac{\partial \mathbf{v}_j}{\partial t} + \bar{\mathbf{U}} \cdot \nabla \mathbf{v}_j + \beta y \mathbf{v}_j^{\perp} - \nabla \theta_j = -C_d(u_0)\mathbf{v}_j - \frac{1}{\tau_R}\mathbf{v}_j, \; j = 1, 2$$

$$\frac{\partial \theta_1}{\partial t} + \bar{\mathbf{U}} \cdot \nabla \theta_1 - \operatorname{div} \mathbf{v}_1 = \frac{\pi}{2\sqrt{2}} P + S_1$$

$$\frac{\partial \theta_2}{\partial t} + \bar{\mathbf{U}} \cdot \nabla \theta_2 - \frac{1}{4} \operatorname{div} \mathbf{v}_2 = \frac{\pi}{2\sqrt{2}} (-H_s + H_c) + S_2.$$

- Dry gravity waves: 50 m/s 1st baroclinic; 25 m/s 2st baroclinic

Characteristic Invariant Form

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + \frac{c}{\sqrt{2}}\left(\frac{\partial v}{\partial y} - \frac{\beta}{c}yv\right) = \frac{1}{\sqrt{2}c}\mathcal{S}(x, y, t),$$

$$\frac{\partial r}{\partial t} - c\frac{\partial r}{\partial x} + \frac{c}{\sqrt{2}}\left(\frac{\partial v}{\partial y} + \frac{\beta}{c}yv\right) = \frac{1}{\sqrt{2}c}\mathcal{S}(x, y, t),$$

$$\frac{\partial v}{\partial t} + \frac{c}{\sqrt{2}}\left(\frac{\partial q}{\partial y} + \frac{\beta}{c}yq\right) + \frac{c}{\sqrt{2}}\left(\frac{\partial r}{\partial y} - \frac{\beta}{c}yr\right) = 0$$

Parabolic Cylinder Functions, Hermite Polynomials, and Recursive

$$D_m(\eta) = 2^{-m/2} H_m\left(\frac{\eta}{\sqrt{2}}\right) e^{-\eta^2/4} \quad H_m(\xi) = (-1)^m e^{\xi^2} \frac{\partial^m e^{-\xi^2}}{\partial \xi^m}$$

where H_m, $m \geq 0$, are the Hermite polynomials with the first few given by

$$H_0(\xi) = 1\,, \quad H_1(\xi) = 2\xi\,, \quad H_2(\xi) = 4\xi^2 - 2\,,$$
$$H_3(\xi) = 8\xi^3 - 12\xi\,, \quad H_4(\xi) = 16\xi^4 - 48\xi^2 + 12\,.$$

$$f''(\eta) + \frac{1}{2}\left(2m + 1 - \frac{\eta^2}{2}\right) f(\eta) = 0\,.$$

$$\phi_m^q(y) = \left(m!\sqrt{\frac{\pi c}{\beta}}\right)^{-1/2} D_m\left(\left(\frac{2\beta}{c}\right)^{1/2} y\right).$$

$$\mathcal{L}_\pm = \frac{\partial}{\partial\eta} \pm \frac{1}{2}\eta\,.$$

$$\mathcal{L}_+ D_m(\eta) = m D_{m-1}(\eta)\,,$$
$$\mathcal{L}_- D_m(\eta) = -D_{m+1}(\eta)\,.$$

$$\eta = \left(\frac{2\beta}{c}\right)^{1/2} y$$

$$\mathcal{L}_\pm = \left(\frac{2\beta}{c}\right)^{-1/2} \frac{d}{dy} \pm \left(\frac{\beta}{2c}\right)^{1/2} y = \left(\frac{2\beta}{c}\right)^{-1/2} \left(\frac{d}{dy} \pm \frac{\beta}{c} y\right),$$

$$L_\pm^q = \frac{d}{dy} \pm \frac{\beta}{c} y$$

$$L_-^q \phi_m^q(y) = -\left(\frac{2\beta}{c}\right)^{1/2} (m+1)^{1/2} \phi_{m+1}^q(y),$$

$$L_+^q \phi_m^q(y) = \left(\frac{2\beta}{c}\right)^{1/2} (m)^{1/2} \phi_{m-1}^q(y).$$

The Complete Expansion for Equatorial Shallow Water.

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + \frac{c}{\sqrt{2}}L_- v = \frac{1}{\sqrt{2}c}\mathcal{S}(x, y, t)\,,$$

$$\frac{\partial r}{\partial t} - c\frac{\partial r}{\partial x} + \frac{c}{\sqrt{2}}L_+ v = \frac{1}{\sqrt{2}c}\mathcal{S}(x, y, t)\,,$$

$$\frac{\partial v}{\partial t} + \frac{c}{\sqrt{2}}L_+ q + \frac{c}{\sqrt{2}}L_- r = 0\,.$$

$$\mathcal{S}(x, y, t) = \sum_{m=0}^{+\infty} \overline{\mathcal{S}}_m(x, t)\phi_m(y)$$

$$\begin{pmatrix} q \\ r \\ v \end{pmatrix}(x, y, t) = \sum_{m=0}^{+\infty} \begin{pmatrix} \overline{q}_m \\ \overline{r}_m \\ \overline{v}_m \end{pmatrix}(x, t)\phi_m(y)\,.$$

Expand in Hermite Polynomials

By plugging (9.70) and (9.71) into (9.67)–(9.69) and using the identities in (9.6: and (9.66), we have, after taking the inner product with ϕ_m, for each m, the co stant-coefficient PDEs

$$\frac{\partial \overline{q}_m}{\partial t} + c\frac{\partial \overline{q}_m}{\partial x} - \frac{c}{\sqrt{2}}\left(\frac{2\beta}{c}\right)^{1/2}(m)^{1/2}\overline{v}_{m-1} = \frac{1}{\sqrt{2}c}\overline{\mathcal{S}}_m\,,$$

$$\frac{\partial \overline{r}_m}{\partial t} + c\frac{\partial \overline{r}_m}{\partial x}\phi_m^q + \frac{c}{\sqrt{2}}\left(\frac{2\beta}{c}\right)^{1/2}(m+1)^{1/2}\overline{v}_{m+1}\phi_m^q = \frac{1}{\sqrt{2}c}\overline{\mathcal{S}}_m\,,$$

$$\frac{\partial \overline{v}_m}{\partial t} + \frac{c}{\sqrt{2}}\left(\frac{2\beta}{c}\right)^{1/2}(m+1)^{1/2}\overline{q}_{m+1} - \frac{c}{\sqrt{2}}\left(\frac{2\beta}{c}\right)^{1/2}(m)^{1/2}\overline{r}_{m-1} = 0\,,$$

where the coefficients of negative index are zero. This system of equations deco ples as follows:

- a single PDE for $\overline{q}_0$,

$$\frac{\partial \overline{q}_0}{\partial t} + c\frac{\partial \overline{q}_0}{\partial x} = \frac{1}{\sqrt{2}c}\overline{\mathcal{S}}_0(x,t)$$

- a 2×2 system coupling $\overline{v}_0$ and $\overline{q}_1$,

$$\frac{\partial \overline{v}_0}{\partial t} + (\beta c)^{1/2}\overline{q}_1 = 0\,, \qquad \frac{\partial \overline{q}_1}{\partial t} + c\frac{\partial \overline{q}_1}{\partial x} - (\beta c)^{1/2}\overline{v}_0 = \frac{1}{\sqrt{2}c}\overline{\mathcal{S}}_1(x,t)\,,$$

- three equations coupling $\overline{r}_{m-2}$, $\overline{v}_{m-1}$, and $\overline{q}_m$ for $m \geq 2$,

$$\frac{\partial \overline{q}_m}{\partial t} + c\frac{\partial \overline{q}_m}{\partial x} - (m\beta c)^{1/2}\overline{v}_{m-1} = \frac{1}{\sqrt{2}c}\overline{\mathcal{S}}_m(x,t)\,,$$

$$\frac{\partial \overline{r}_{m-2}}{\partial t} - c\frac{\partial \overline{r}_{m-2}}{\partial x} + (\beta c(m-1))^{1/2}\overline{v}_{m-1} = \frac{1}{\sqrt{2}c}\overline{\mathcal{S}}_{m-2}(x,t)\,,$$

$$\frac{\partial \overline{v}_{m-1}}{\partial t} + (\beta c m)^{1/2}\overline{q}_m - (\beta c(m-1))^{1/2}\overline{r}_{m-2} = 0\,.$$

Notice for the 3×3 systems that if we introduce

$$\vec{u}_m = \begin{pmatrix} \overline{q}_m \\ \overline{r}_{m-2} \\ \overline{v}_{m-1} \end{pmatrix} \qquad \text{for } m \geq 2\,,$$

we have

$$\frac{\partial \vec{u}_m}{\partial t} + A_m \frac{\partial \vec{u}}{\partial x} + B_m \vec{u}_m = \mathcal{S}_m$$

where

$$A_m = \begin{bmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_m = \begin{bmatrix} 0 & 0 & -(m\beta c)^{1/2} \\ 0 & 0 & ((m-1)\beta c)^{1/2} \\ (m\beta c)^{1/2} & -((m-1)\beta c)^{1/2} & 0 \end{bmatrix},$$

and

$$\mathcal{S}_m = \frac{1}{\sqrt{2}c} \begin{pmatrix} \overline{\mathcal{S}}_m(x,t) \\ \overline{\mathcal{S}}_{m-2}(x,t) \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} \overline{q}_m \\ \overline{r}_m \\ \overline{v}_m \end{pmatrix} = \mathrm{Re} \begin{pmatrix} q_m \\ r_m \\ v_m \end{pmatrix} e^{i(kx-\omega t)} .$$

Kelvin Waves. The first equation (9.72) gives

$$\omega = ck \tag{9.76}$$

and the associated solution in terms of the (c-rescaled) pressure p, and th zontal velocity field (u, v) is given by

$$\begin{pmatrix} p \\ u \\ v \end{pmatrix} (x, y, t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cos(k(x - ct))\phi_0^q(y) . \tag{9.77}$$

This solution is known as the *Kelvin wave* from Section 9.1.1. The Kelvi can also be expressed in a general fashion as

$$\begin{pmatrix} p \\ u \\ v \end{pmatrix} (x, y, t) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} G(x - ct)e^{-(\frac{\beta}{c})\frac{y^2}{2}} , \tag{9.78}$$

Mixed Rossby-Gravity Waves. When plugging (9.75) into the system in (9.73), as in Section 9.1, we obtain a 2×2 linear eigenvalue problem with constant coefficients; then, by substituting $v_0 = -i(\beta c)^{1/2} q_1/\omega$ in the second equation, we get

$$\omega_{\pm} = \frac{1}{2}ck \pm \frac{1}{2}\sqrt{(ck)^2 + 4\beta c}\,, \tag{9.79}$$

and the solution is

$$\begin{pmatrix} p \\ u \\ v \end{pmatrix}(x, y, t) = \begin{pmatrix} \frac{1}{\sqrt{2}}\cos(kx - \omega_{\pm}t)\phi_1(y) \\ \frac{1}{\sqrt{2}}\cos(kx - \omega_{\pm}t)\phi_1(y) \\ ((\beta c)^{1/2}/\omega_{\pm})\sin(kx - \omega_{\pm}t)\phi_0(y) \end{pmatrix}. \tag{9.80}$$

These are the two mixed Rossby-gravity (MGR) waves discussed in Section 9.1.

Equatorial Rossby and Gravity Waves. With (9.75), the system in (9.74) is equivalent to the 3×3 eigenvalue problem

$$\begin{bmatrix} -i\omega + ick & 0 & -(m\beta c)^{1/2} \\ 0 & -i\omega - ick & \big((m-1)\beta c\big)^{1/2} \\ (m\beta c)^{1/2} & -\big((m-1)\beta c\big)^{1/2} & -i\omega \end{bmatrix} \begin{pmatrix} q_m \\ r_{m-2} \\ v_{m-1} \end{pmatrix} = 0$$

where the characteristic equation is

$$\omega(\omega^2 - c^2k^2) - \beta c((2m-1)\omega + ck) = 0\,, \tag{9.81}$$

and the eigenvectors are given by

$$q_m = i\frac{(m\beta c)^{1/2}}{\omega - ck} v_{m-1}\,, \qquad r_{m-2} = -i\frac{((m-1)\beta c)^{1/2}}{\omega + ck} v_{m-1}\,,$$

where ω is any solution of (9.81). Note that the solution $\omega(k) \equiv 0$ is not possible in (9.81), so it can be rewritten as

$$\left(\frac{\omega}{c}\right)^2 - k^2 - \frac{\beta}{\omega}k = \frac{\beta}{c}(2M+1) \quad \text{with } M = m - 1\,.$$

This equation has three distinct solutions that can be approximated as follows: When $\frac{\omega}{c}$ is small, we have one single solution,

$$\omega_0(k) \approx -\frac{\beta k}{\frac{\beta}{c}(2M+1) + k^2}\,, \tag{9.82}$$

and when $\frac{\beta}{\omega}k$ is small, we get

$$\omega_\pm(k) \approx \pm\sqrt{c^2k^2 + c\beta(2M+1)} \quad \text{for } M = 1, 2, \ldots. \tag{9.83}$$

The maximum errors in these simplified approximations are only a few percent (see Gill [**11**, chap. 11]). From Section 5.1 and (9.2), recall that the dispersion relation for the midlatitude Rossby waves in the β-plane approximation, $f = f_0 + \beta y$ with $f_0 \neq 0$, is given by

$$\omega_r(k,l) = -\frac{\beta k}{k^2 + l^2 + L_R^{-2}} \, .$$

Also from Section 4.4 for the midlatitude Poincaré waves in an f-plane approximation $f = f_0$, it is given by

$$\omega_p(\vec{\mathbf{k}}) = \pm c\sqrt{|\vec{k}|^2 + L_R^{-2}} \, .$$

Recall from (9.2) that $L_R = \sqrt{gH}/f_0$ is the midlatitude Rossby radius. The waves associated with the dispersion relation in (9.82) are known as the equatorial *Rossby waves* or equatorial *planetary waves* because the form of the dispersion relation resembles the one of the midlatitude Rossby waves while the waves corre- sponding to (9.83) are called the equatorial *inertio-gravity waves* or simply equa- torial *gravity waves*; their dispersion relations are approximately the same as those of the midlatitude Poincaré waves. Recall that the equatorial deformation radius i $L_e = (\frac{c}{\beta})^{1/2}$ and in both (9.82) and (9.83), $L_e^{-2}(2M+1)$ plays the same role a L_R^{-2} for midlatitudes.

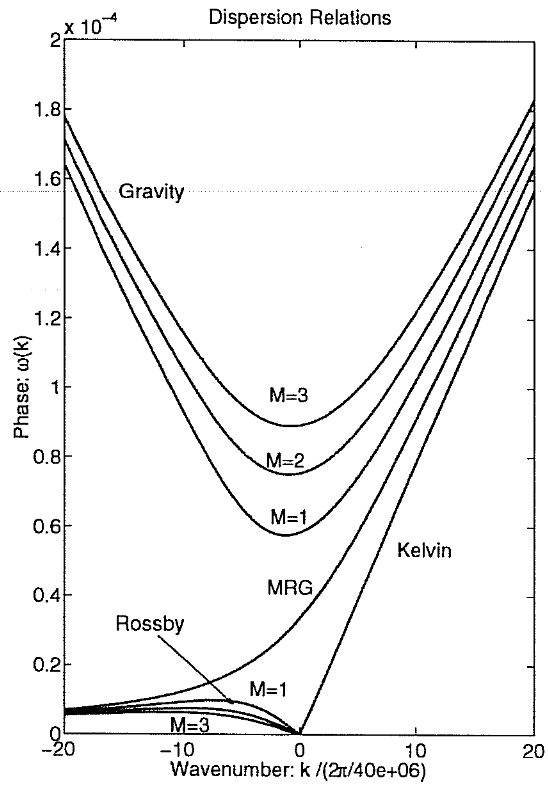


FIGURE 9.1. Phase versus the wavenumber normalized by the wavenumber 1 for the first baroclinic $c_1 = 50\ \text{ms}^{-1}$.

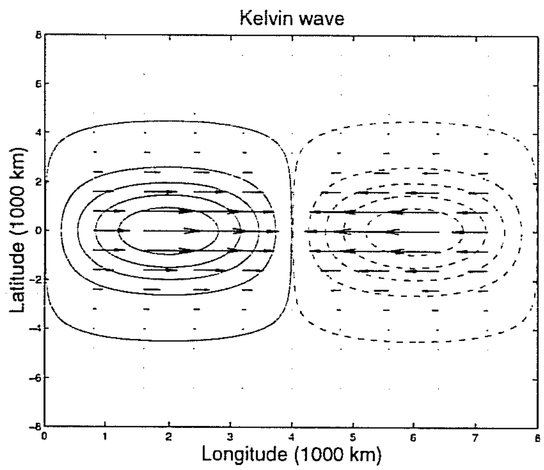


FIGURE 9.2. Filled contours of the pressure and velocity profile (arrows). Here and below, dotted contours are low pressure.

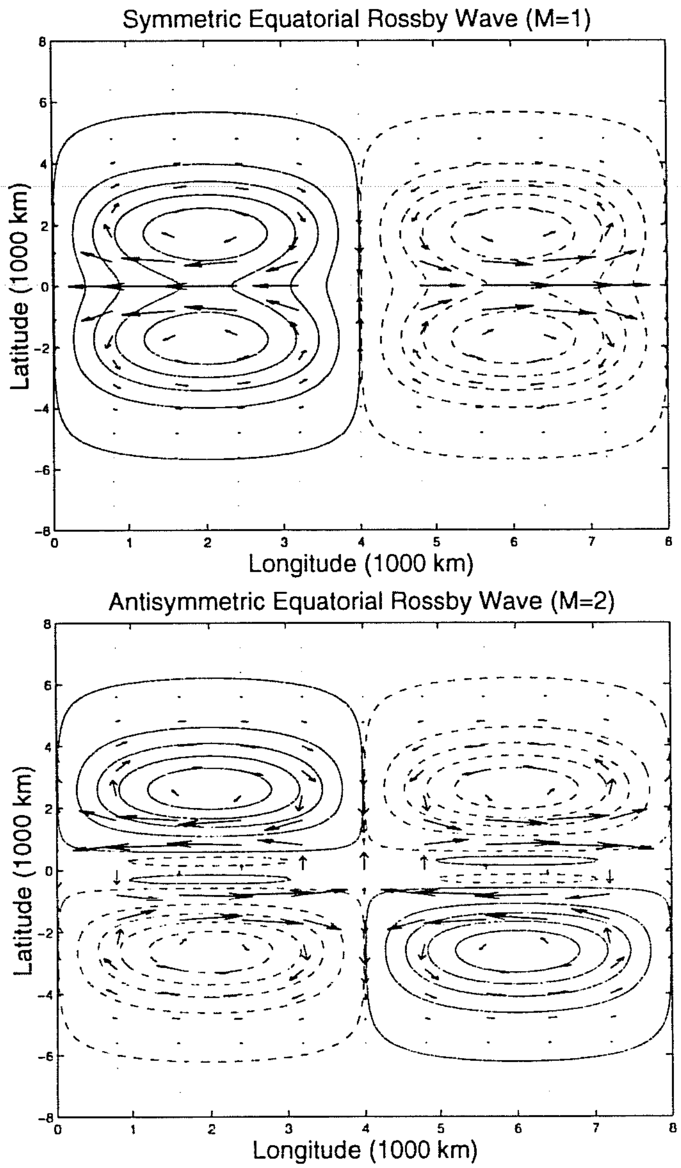


FIGURE 9.4. Filled contours of the pressure and velocity profile (arrows).

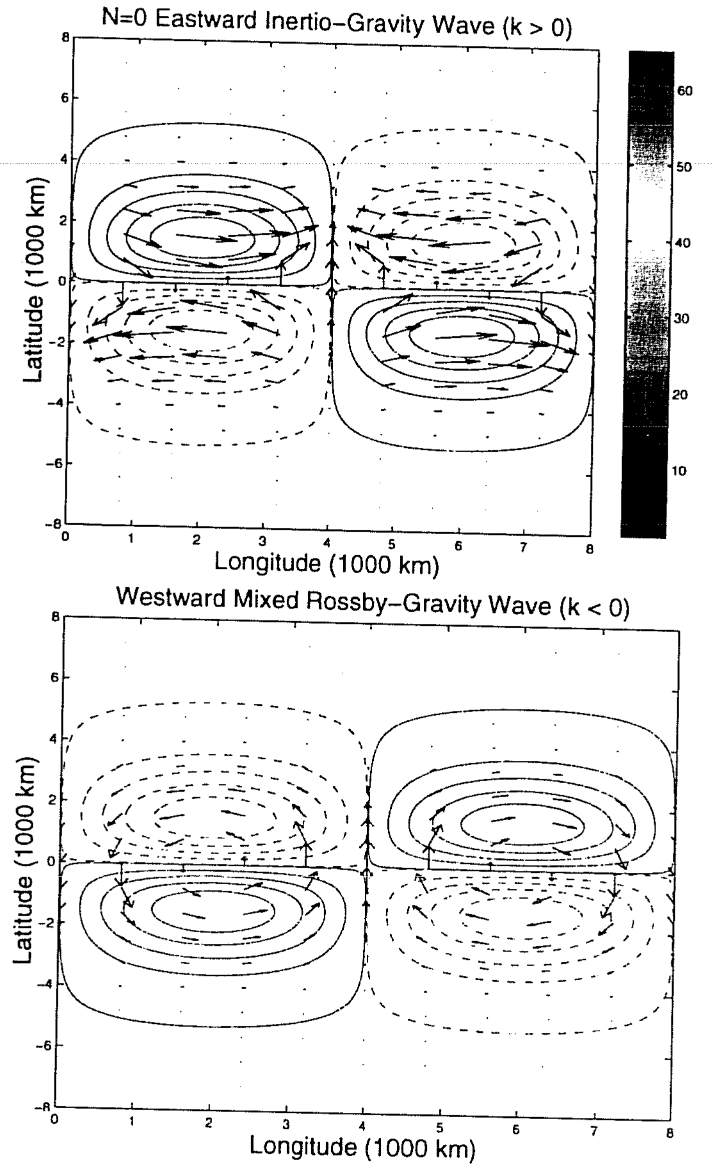


FIGURE 9.3. Filled contours of the pressure and velocity profile (arrows).

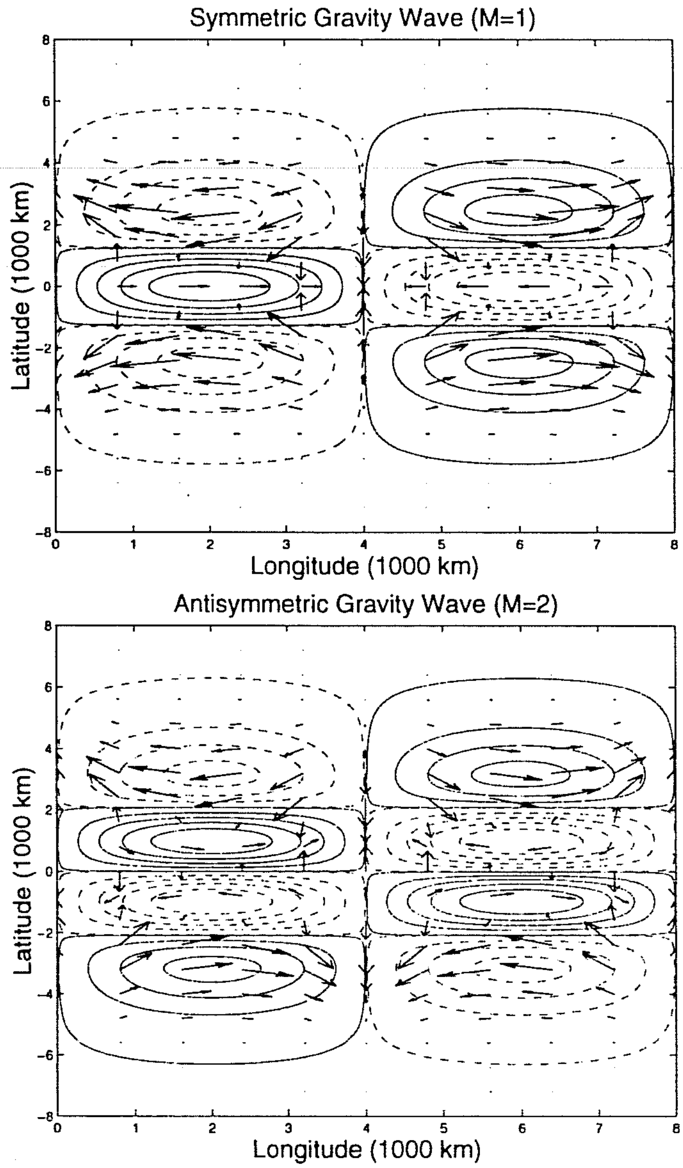


FIGURE 9.5. Filled contours of the pressure and velocity profile (arrows).