UBC lecture 1

Calabi - Yau geometry

– basic ingredient of string compactification

het : $CY_3 \times \mathbb{R}^4$ $\mathcal{N}=1$ SUSY

(Candelas, et al)

IIA/B : $CY_3 \times \mathbb{R}^4$ $\mathcal{N}=2$

$\Rightarrow$ flux / brane $\mathcal{N}=1$

(GVW, Sethi, KKLT, ...)

M : S^1 on $I \times CY_3 \times \mathbb{R}^4$

w/ warping

F : $CY_4 \times \mathbb{R}^4$

$\uparrow$ elliptic fibrations

Today, I will discuss

- CY_3
- moduli space of CY_3
- mirror symmetry
- SYZ

CY_3 : Ricci-flat Kähler mfd

preserves $\frac{1}{4}$ of SUSY

10d spinor $\quad \epsilon_{10} = \epsilon_6 \times \epsilon_4$

ϵ_6 : 4 chiral comp's

$\frac{1}{4}$ SUSY $\Longleftrightarrow$ $\exists 1$ solution to

$$\delta \psi_\mu = D_\mu \epsilon_6 = 0$$

- Kähler

$z^i \quad i = 1, 2, 3$: complex coord's

$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \qquad K$: Kähler pot.

- Ricci flat

$$R_{i\bar{j}} \sim \partial_i \partial_{\bar{j}} \log(\det g_{i\bar{j}})$$

$$\Rightarrow \quad \det g_{i\bar{j}} = \Omega(z)\, \bar{\Omega}(\bar{z})$$

Ω_{123} : holomorphic (3,0) form

(1) Ricci flat Kähler requires

$\exists\, \Omega_{(3,0)}, \quad \bar{\partial}\Omega = 0$

$\Omega \neq 0$ everywhere

(2) $\exists 1\ \epsilon_6 \ (D_\mu \epsilon_6 = 0)$ requires $h^{3,0} = 1$

(3) Yau's theorem states that the converse of (1) is also true.

Hodge diamond

$$
\begin{array}{ccccc}
 & & h^{00} & & \\
h^{3,0} & & & & h^{0,3} \\
 & & h^{3,3} & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & n_K & & 0 & \\
1 & & n_C & & n_C & & 1 \\
 & 0 & & n_K & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

$$n_K = h^{1,1}$$
$$n_C = h^{2,1}$$

- moduli space of CY_3

$\mathcal{M}_K$: "complexified" Kähler moduli space

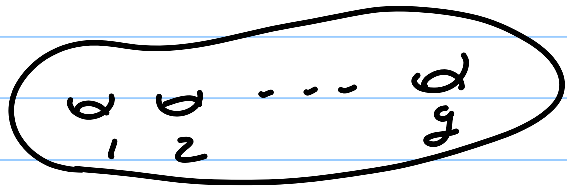
$\mathcal{M}_C$: complex str moduli space

$$\dim_{\mathbb{C}} \mathcal{M}_K = n_K = h^{1,1}$$
$$\dim_{\mathbb{C}} \mathcal{M}_C = n_C = h^{2,1}$$

① Complex structure of CY_3

exercise : complex structure of Riemann surface

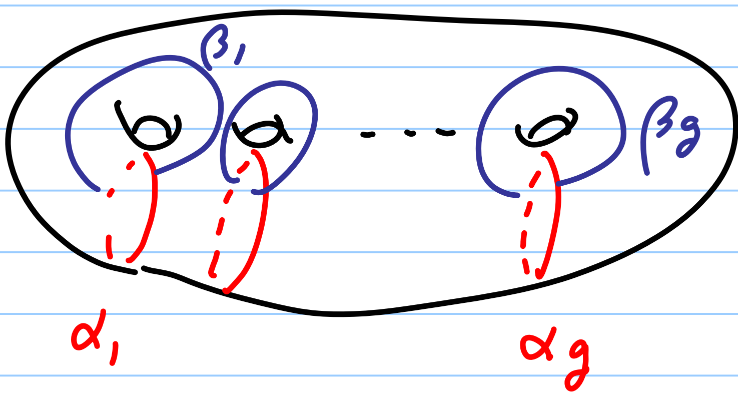
Σ : genus g



$$\omega_a = \omega_{az}\, dz \qquad a = 1, \cdots, g$$

holomorphic 1 forms $\in H^{1,0}(\Sigma)$

homology 1-cycles α_a , β^a



normalize ω's as $\displaystyle\oint_{\alpha_a} \omega_b = \delta_a{}^b$

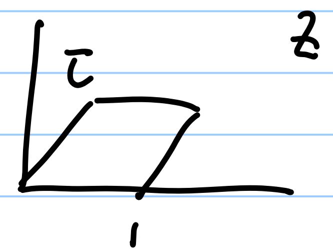
period matrix

$$\tau_{ab} = \oint_{\beta_a} \omega_b$$

Torelli's theorem:

If Σ and Σ' have periodmatrix $\tau = \tau'$, then $\Sigma \sim \Sigma'$.

e.g. torus



$$z = x + \tau y$$

$$\oint_{y = \text{fixed}} dz = 1 \qquad \oint_{x = \text{fixed}} dz = \tau$$

modular transf $SL(2, \mathbb{Z})$

For Σ_g, there is

the Schottky problem:

- moduli space of complex str: $(3g-3)$ dim
- $\dim \{\tau_{ij}\} = \frac{1}{2} g(g+1)$

for	$3g-3$:	3	6	9	...
	$\frac{1}{2}g(g+1)$:	3	6	10	...
	g :	2	3	4	...

So,

$$\mathcal{M}_C \to \{\tau_{ij}\}$$

Where is the image?

(Solved by M. Sato.)

Coming back to CY_3, its complex str.

We have 1 $\Omega \in H^3(CY_3)$

Choose homology 3 cycles

$$\alpha_I \,,\ \beta^I \qquad I = 1, \cdots, n_c + 1$$

$$\left(\text{note } \dim H_3 = 2\left(h^{3,0} + h^{2,1}\right) \right.$$
$$\left. = 2\,(1 + n_c) \right)$$

$$X^I \equiv \int_{\alpha_I} \Omega \,, \quad F_I \equiv \int_{\beta^I} \Omega$$

Ω is determine up to an overall const.

$\Rightarrow$ ratios of X^I determine complex str of CY_3

$$F_I = F_I(X)$$

$$F_I(cX) = c\, F_I(X).$$

moreover $\partial_I F_J = \partial_J F_I$

$$\Rightarrow \quad F_I(X) = \frac{\partial}{\partial X^I} F(X)$$

$F(X)$: prepotential

--- In type IIB string on $CY_3 \times \mathbb{R}^4$, the low energy effective action for the vector multiplets and its coupling to the gravity multiplet are completely determined, by 2 derivatives, by $F(X)$.

IIB string

10d $g_{\mu\nu}$, $B_{\mu\nu}$, Φ — dilaton

A, $A_{\mu\nu}$, $A_{\mu\nu\rho\sigma}$ + dual

+ fermions

$\downarrow$

$CY_3 \times \mathbb{R}^4$

$b_0 = 1$
$b_1 = 0$
$b_2 = 2h^{11}$
$b_3 = 2h^{21} + 2$

for each 3 cycle $\gamma = \alpha_I, \beta^J$

$\Rightarrow$ A^I_μ : massless vector in $\mathbb{R}^4$

$\tilde{A}_{I\mu}$ $\qquad dA^I = * dA_I$

$\underbrace{h^{2,1}}_{\text{superpartners of massless scalars = complex moduli}} + \underbrace{1}_{\text{graviphoton} \in \mathcal{N}=2 \text{ SUGRA}}$ massless vector

superpartners of massless scalars = complex moduli $\Leftrightarrow H^{0,1}_{\bar\partial}(T^{1,0}CY_3)$

$\sim H^{2,1}(CY_3)$

The complex structure moduli space $\mathcal{M}_c$

- $\dim_{\mathbb{C}} \mathcal{M}_c = h^{2,1} = \dim_{\mathbb{C}} H^{(0,1)}_{\bar\partial}(T^{1,0})$

- Kähler

not to be confused w/ K for CY_3

$$e^{-K} = i \int_{CY_3} \Omega \wedge \bar\Omega$$

Riemann bilinear id.

$$= i \left(\bar X^I F_I(X) - X^I \bar F_I(\bar X) \right)$$

$$G_{a\bar b} = \partial_a \partial_{\bar b} K \qquad a, b = 1 \sim h^{2,1}$$

Hodge bundle : $\mathcal{L}$

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathcal{M}_{\mathbb{C}} \end{array}$$

transforms like Ω

e^{-K} : metric

- $\mathcal{M}_C$ is special Kähler

$$R_{a\bar{b}c\bar{d}} = G_{a\bar{b}} G_{c\bar{d}} + G_{a\bar{d}} G_{c\bar{b}} - e^{2K} C_{acm} \bar{C}_{\bar{b}\bar{d}\bar{n}} G^{m\bar{n}}$$

where
$$C_{abc} = \partial_a \partial_b \partial_c F(X) = -\int \Omega \wedge \partial_a \partial_b \partial_c \Omega$$

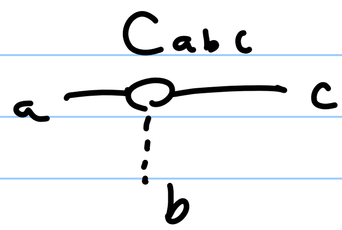
"Yukawa coupling"

(In the heterotic string compactifications,

$E_8 \times E_8 \rightarrow E_6 \times E_8$

complex str : 27

Kähler moduli : $\overline{27}$



)

This relation can be derived from

$$e^{-K} = i\,(\bar{X} F - X \bar{F})$$

(homework)

So far we looked at $\mathcal{M}_C$.

- Kähler moduli

not to be confused w/ K for $\mathcal{M}_C$

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \quad : \quad CY_3 \text{ metric}$$

$$k \equiv i\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}} \quad : \quad \text{Kähler form}$$

$$H_2 : \; C_A \qquad A = 1 \sim h^{1,1}$$

$$r^A \equiv \int_{C_A} k \quad : \quad \text{Kähler class.}$$

For given Kähler class and complex str,

there is a unique CY_3 metric.

$\Rightarrow$ massless scalar fields in 4d.

Combine this with

$$\theta^A = \int_{C_A} B \qquad dB = 0$$

$$t^A = r^A + i\theta^A$$

t^A : complexified Kähler moduli

$$t^A = \frac{X^A}{X^0} \qquad \mathcal{M}_K.$$

II B on $CY_3 \times \mathbb{R}^4$

$\boxed{\text{vector multiplet} \quad A^a_\mu \,. \quad \text{complex moduli}}$

hyper multiplet $\left[\text{Kähler} + \int_C A_2 + \int_{\tilde{C}} A_4 \right]$

topological string

[dilaton , axion]

IIA on $CY_3 \times \mathbb{R}^4$

vector multiplet Kähler

hyper multiplet complex

$\mathcal{M}_C$: complex str moduli

$\mathcal{M}_K$: complexified Kähler

Since there is no neutral coupling between vector & hyper, the metrics on $\mathcal{M}_C$ and $\mathcal{M}_K$ are not corrected by string loop effects. But

$\mathcal{M}_C$: classical geometry

$$e^{-K} = i \int \Omega \wedge \bar{\Omega}$$

$\mathcal{M}_K$: worldsheet instanton

$$e^{-K} = i\,(\bar{X} F - X \bar{F})$$

$$C_{abc} = \int_{CY_3} \omega_a \wedge \omega_b \wedge \omega_c$$

$$+ \quad \text{instantons}$$

more precisely

$$C_{abc} = \int_{CY_3} \omega_a \wedge \omega_b \wedge \omega_c$$

$$+ \sum_{n} n_a n_b n_c N_{\vec{n}} \frac{e^{-\vec{n}\cdot\vec{t}}}{1 - e^{-\vec{n}\cdot\vec{t}}}$$

$$\vec{n} = (n_1 \cdots n_{h^{1,1}})$$

$$\vec{t} = (t^1, \dots, t^{h^{1,1}})$$

$N_{\vec{n}}$: # of holomorphic maps

$$f : S^2 \longrightarrow CY_3$$

with $\int_{S^2} f^* \omega_a = \vec{n}_a$

$\frac{1}{1-e^{-\vec{n}\cdot\vec{t}}}$: effects of multicoverings mirror symmetry :

$(M, \tilde{M})$: mirror pair of CY_3

$\mathrm{II}A$ on $M \times \mathbb{R}^4$

$= \mathrm{II}B$ on $\tilde{M} \times \mathbb{R}^4$.

If $M = T^6$, we can choose $\tilde{M}$ to be T dual along odd # of cycles.

more non-trivial example (Greene-Plesser)

Fermat Quintic :

M : $(x_1)^5 + \cdots + (x_5)^5 = 0$ in $\mathbb{C}P^4$

$\tilde{M} = M/(\mathbb{Z}_5)^3$: mirror pair

Candelas et al used this to count $N_{\vec{n}}$.

Decoupling gravity

$$M_{Pl}^2 \int d^4x \sqrt{g}\, R \Rightarrow M_{Pl}^2 \propto \mathrm{vol}\, CY_3$$

$$M_{Pl} \to \infty \Leftarrow \mathrm{vol}\, CY_3 \to \infty$$

One can retain non-trivial dynamics if we take CY_3 geometry singular simultaneously.

(e.g. small 2 or 3 cycles
--- conifold and ADE generalization)

$\Rightarrow$ geometric engineering

For finite M_{pl}:

$$e^{-K} = i\left(\bar{X}^I \partial_I F - X^I \bar{\partial}_I \bar{F}\right)$$

$$= i\left(2F - 2\bar{F} + \bar{t}^A \partial_A F - t^A \bar{\partial}_A \bar{F}\right)$$

$$t^A = \frac{X^A}{X^0}$$

$M_{pl} \to \infty$ limit:

$$F = i M_{pl}^2 + f$$

$$K \simeq -\log M_{pl}^2 + \frac{1}{i M_{pl}^2}\left(\bar{t}^A \partial_A f - t^A \bar{\partial}_A \bar{f}\right) + \cdots$$

$$X^I = \int_{\alpha_I} \Omega \;, \quad F_I = \int_{\beta^I} \Omega$$

$$\Downarrow$$

$$F_I(X) = \partial_I F$$

$$F = i M_{pl}^2 + f$$

$$M_{pl} \to \infty$$

$$K^{\text{rigid}}_{N=2} \sim \bar{t}^A \partial_A f - t^A \bar{\partial}_A \bar{f}$$

Reproduces the Seiberg-Witten solution.

SYZ

Claim:

Calabi-Yau M can always expressed as a T^3 fibration over some B.

mirror symmetry $\approx$ T-duality on T^3.

Reasoning:

IIA on M consider D0 brane.

moduli space $\simeq M$.

(not metrically!)

$\Updownarrow$

IIB on $\tilde{M}$ D3 $\gamma \subset \tilde{M}$

moduli space of D3 on γ

$= M$ by mirror symmetry

D3 on $\gamma \times$ time

$U(1)$ $\mathcal{N}=4$ SYM   6 scalars

($3 \to \mathbb{R}^3$, $3 \to$ transverse $\gamma \subset CY_3$)

moduli space W

= flat connection
$\downarrow$
deformation space of γ

$$\dim W = b_1 + b_1 \qquad b_1(\gamma)$$

holonomy

$\because$) special Lagrangian

By mirror symmetry

$$\dim W = \dim M = 6$$

$$\Rightarrow \quad b_1 = 3 \qquad \{\text{flat connection}\} = T^3$$

$$\therefore \quad W = \begin{array}{c} T^3 \\ \downarrow \\ B \end{array} .$$

Thus $$M \sim W = \begin{array}{c} T^3 \\ \downarrow \\ B \end{array}$$

Similarly
$$\tilde{M} \sim \tilde{W} \sim \begin{array}{c} T^3 \\ \downarrow \\ \tilde{B} \end{array}$$

By mirror symmetry $B = \tilde{B}$.

Lecture 2

Topological String

- definition
- relation to type II string
- OSV
- derivation?

- Definition:

NSR $\quad X : \Sigma \longrightarrow CY_3 \times \mathbb{R}^4$

$\quad \psi : \Sigma \longrightarrow$ tangent space.

consider the CY_3 part only

$= \mathcal{N}=2$ SUSY NL σ-model

superconformal $\quad c = 9 = 6 \times \frac{3}{2}$

$\hat{c} = c/3 = 3$

	Spin	3/2	1	2
$N=2$ SCA		$G_L^\pm$,	J_L,	T_L
		$G_R^\pm$,	J_R,	T_R

topological twisting

G_L^+ $(1,0)$, G_L^- $(2,0)$

$Q_L = \oint G_L^+$: topological BRST

$Q_L^2 = 0$, $\{Q_L, G_L^-\} = T_L$

$G_L^- \sim$ anti ghost

Physical states Q_L, Q_R

A model $\Rightarrow$ Kähler moduli

B model $\Rightarrow$ complex moduli

A-model

$$\omega_{a\, i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$$

$$\Rightarrow \quad \phi_a = \omega_{a\, i\bar{j}}\, \psi_L^i \psi_R^{\bar{j}}$$

$$[Q_L, \phi_a] = 0, \quad [Q_R, \phi_a] = 0$$

$$S \longrightarrow S + t^a \int G_L^- G_R^- \phi_a$$

: nontrivial deformation

$$+ \bar{t}^a \int G_L^+ G_R^+ \bar{\phi}_a$$

: trivial

Similarly for

the B-model

genus g amplitude

$$F_g \equiv \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} (\eta_i, G_i^-)(\bar{\eta}_i, G_{\bar{i}}^-) \right\rangle$$

This works only for $\hat{c} = 3$.

Holomorphic anomaly

$$\frac{\partial}{\partial \bar{t}^a} F_g = 0 \ ?$$

No, since $\{ Q_c , G_i^- \} = T_c$

$$\frac{\partial}{\partial \bar{t}^a} F_g = \frac{1}{2} \bar{C}_{\bar{a}\bar{b}\bar{c}} \, e^{2K} G^{a\bar{a}} G^{b\bar{b}}$$

$$\times \left\{ D_b D_c F_{g-1} + \sum_{r=1}^{g-1} D_b F_r \, D_c F_{g-r} \right\}$$

$g=1$ case is special:

$$\partial_a \partial_{\bar b} F_1 = \frac{1}{2} C_{amn} \bar{C}_{\bar b \bar m \bar n} e^{2K} G^{m\bar m} G^{n\bar n} - \left(\frac{\chi}{24} - 1\right) G_{a\bar b}$$

$$F_1 = \frac{1}{2} \log\left[e^{(n+3-\frac{\chi}{24})K} \det G \times |f(t)|^2 \right]$$

$f(t)$: holomorphic in t.

$n = n_c$ or n_K.

geometric meaning :

F_1 : A-model

$$\frac{\partial}{\partial t^A} F_1 \Big|_{\bar{t}\to\infty} = -\frac{1}{24}\int \omega_a \wedge c_2 + \sum_{\vec{n}} n_a N^{(1)}_{\vec{n}} \sum_m \frac{e^{-m\, n\cdot t}}{1-e^{-m\, n\cdot t}} + \frac{1}{12}\sum n_a N^{(0)}_{\vec{n}} \frac{e^{-nt}}{1-e^{-nt}}$$

B-model

$$F_1 = \frac{1}{2}\log \prod_{p,q} \left(\det \Delta^{(p,q)}\right)^{p\cdot q\,(-1)^{p+q}}$$

$$\Delta^{(p,q)} : \Lambda^p T^{(0,1)}M \otimes \Lambda^q T^{(1,0)}M$$

Topological string partition function is a wave function.

Consider the B-model:

- $\psi_{top} = \exp\left(\sum_g F_g \right)$

  is a holomorphic function of $H^3(CY_3)$.

  $\dim_{\mathbb{C}} H^3 = h^{2,1} + 1$

  $h^{2,1}$: deformation of complex structure.

  1: string coupling

- ψ_{top} also depends on the initial choice of complex structure.

  $\psi_{top}(X^I ; z, \bar{z})$

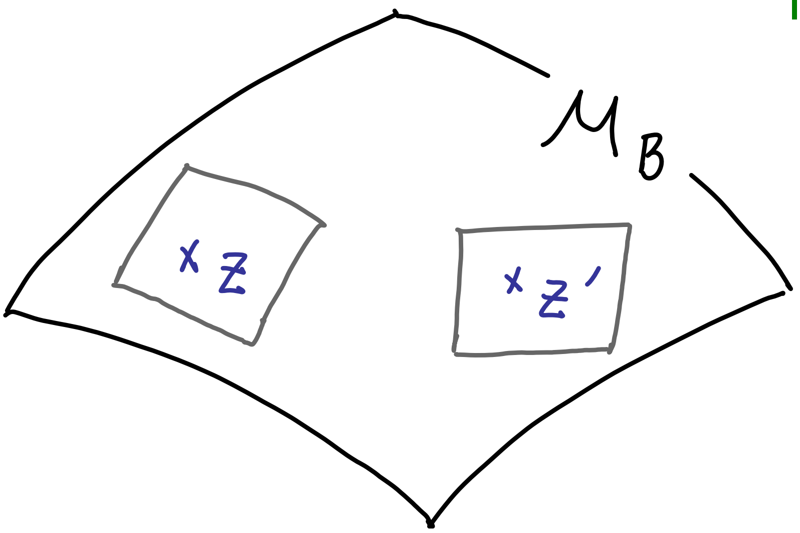
  - $X^I \in H^3$, $I = 0, 1, \cdots, h^{2,1}$
  - z : background

The holomorphic anomaly equations

BCOV 1993

$$\frac{\partial}{\partial z^I}\psi_{top}(x;z,\bar z) = \left(\frac{\partial}{\partial x^I}+\cdots\right)\psi_{top}$$

$$\frac{\partial}{\partial \bar z^I}\psi_{top}(x;z,\bar z) = \left(\bar C_I{}^{JK}\frac{\partial^2}{\partial x^J\partial x^K}+\cdots\right)\psi_{top}$$



Interptetation:

Witten 1993

(1) On each tangent space, there is a Hilbert space.

(2) The holomorphic anomaly equation is describing the paralell transport between tangent spaces at different points.

More on (1):

- $H^3(CY_3,\mathbb{R})$ has a symplectic structure.
- Topological string uses a holomorphic polarization.

$$\underbrace{H^{3,0}\oplus H^{2,1}}_{x^I}\oplus H^{2,1}\oplus H^{0,3}$$

More on (2):

The polarization depends on $(z,\bar z)$

The anomaly equations give parallel transportation.

Extremal Black Holes in Type II theory

Type II superstring complactified on CY3 has many charged extremal black holes constructed as D branes wrapping cycles in CY3.

The number of ground states of the black hole is computable using the gauge theory on the branes.

$$\Omega(p, q)$$

When the charges p, q are large, the classical gravity descrpition is good, and the number of states is given by the Bekenstein-Hawking formula:

$$\ln \Omega(p, q) \sim \frac{1}{4} A_{\text{HORIZON}}$$

$$p.q \gg 1$$

This has been successfully tested.

Can we make this more precise, by incorporating quantum corrections to the right-hand side?

To do this, we need to know string loop corrections to the relevant low energy effective theory terms.

Conjecture (Strominger, Vafa + H.O.)

$$|\psi_{top}(X)|^2 = \sum_q \Omega(p,q)\, e^{-q\phi}$$

where $X^I = p^I + \frac{i}{\pi}\phi^I$

(1) Calabi-Yau moduli X^I are determined by the magnetic charges p^I and the electric potentials ϕ^I.

attractor mechanism

(2) This identity is supposed to hold to all order in the string perturbation theory.

There are important non-perturbative corrections to the formula.

Extremal Black Hole in String Theory

Consider type IIB superstring on $CY_3 \times \mathbb{R}^{3,1}$

RR 4-form potential $\Rightarrow$ gauge field A_μ^I on $\mathbb{R}^{3,1}$

$I = 0, 1, \cdots, h^{2,1}$

Choose a symplectic basis of 3-cycles on CY_3.

$\{ \alpha_I , \beta^I \}$ $\quad I = 0, 1, \cdots, h^{2,1}$

$\alpha_I \cap \alpha_J = 0 = \beta^I \cap \beta^J$

$\alpha_I \cap \beta^J = \delta_I{}^J$

D3 branes wrapping on 3 cycles

q_I times on α_I , p^I times on β^I

= BPS black hole in four dimensions

with electric charges q_I , magnetic charges p^I

$\Omega(p, q)$ = number of BPS states (Witten index)

We do not have a derivation of this formula from the first principles. I will describe an argument for it, based on observations by:

Cardoso, de Wit, Mohaupt	1998
Strominger, Vafa, + H.O.	2004
Sen	2005

Classically, the near horizon geometry of the extremal black hole is AdS2 x S2 x CY3.

$$ds^2 \sim \underbrace{-r^2 dt^2 + \frac{dr^2}{r^2}}_{AdS_2} + \underbrace{d\theta^2 + \sin^2\theta \, d\phi^2}_{S^2} + \underbrace{ds_{CY}^2}_{CY_3}$$

The vector multiplet scalars (i.e., complex structure in the IIB case) are fixed by the attractor equations.

The hypermultiplet scalars are not fixed.

These features are preserved when string loop corrections are included.

The magnetic charges of the black hole:

$$p^I = \int_{S^2} f^I_{\theta\phi} = \mathrm{Re}\, X^I$$

The electric charges of the black hole:

$$q_I = \int_{S^2} \frac{\delta S_{eff}}{\delta f^I_{rt}} = -\frac{\partial}{\partial \phi^I}\left(F + \bar{F}\right)$$

The electric charges are non-linear functions of $X = p + \frac{i}{\pi}\phi$.

Applying the macroscopic entropy formula for a higher derivative gravity action given by Wald, one obtains:

$$\text{Entropy} = F + \bar{F} + q_I \phi^I$$

Thus, the macroscopic entropy is the Legendre transform of the topological string partition function.

Knowing the thermodynamical relation

$$\text{Entropy} = F + \bar{F} + q_I \phi^I$$

and based on an analogy with the standard statistical interpretation of the free energy,

we conjectured:

$$|\psi_{top}(X)|^2 = \sum_q \Omega(p,q)\, e^{-q\phi}$$

where

$$\psi_{top} = \exp\left(\sum_g F_g\right),$$

$$X^I = p^I + \frac{i}{\pi}\phi^I.$$

Effective Action:

$$S_{eff} = \int \sqrt{g}\, \mathcal{L}_{F\text{-term}} + \text{hypermultiplet terms}$$
$$+ \text{ D-terms}$$

$$\mathcal{L}_{F\text{-term}} \sim \mathcal{R} + (F_{IJ} + \bar{F}_{IJ})\, DX^I D\bar{X}^J$$
$$+ \text{ 20 more terms .}$$

Evaluating the effective action near the horizon, one finds:

$$2\pi \int_{S^2} \sqrt{g}\, \mathcal{L}_{F\text{-term}} = F(X) + \bar{F}(\bar{X})$$

where $$F(X) = \sum_{g=0}^{\infty} F_g(X)$$

topological string partition function

Frequently asked questions and incomplete answers:

Question 1: Why are D-terms irrelevant?

It must have to do with the fact that what we are calling the entropy is the Witten index in the Hilbert space of the black hole.

Question 2: Why are hypermultiplet fields irrelevant?

To all order in the perturbation theory, hypermultiplet fields are not fixed. This means that the effective action, when evaluated near the horizon, does not depends on hypermultiplets.
There may be some non-perturbative dependence, as suggested by S-duality in topological string theory.

Question 3: Why is the Laplace transformation better than the Legendre transformation?

Higher loop corrections contribute to finite size effects, so thermodynamic arguments are not sufficient.
The Laplace transform works well in examples I know.
It may be possible to explain why the Laplace transformation works if we understand AdS_2/CFT_1.

Question 4: What about the holomorphic anomalies?

The entropy may have background dependence.
There is a suggestion that the dependence vanishes at least infinitesimally in an appropriate basis.

Derivation by Gaiotto - Strominger - Yin

$D_0 - D_2 - D_4$

II A $\quad AdS_2 \times S^2 \times CY_3$

M theory lift

$$\underbrace{AdS_3}_{\substack{\uparrow \\ \text{boundary } S^1 \times \mathbb{R}}} \times S^2 \times CY_3$$

S^1: "x^{11}" circle.

Thus, the dual CFT is

2d CFT on $S^1 \times \mathbb{R}$.

Claim

$$Z_{BH} = \mathrm{Tr}_{NS \times R} (-1)^F e^{-\frac{1}{\phi_0} L_0 - \frac{\phi^A}{\phi^0} J_A}$$

elliptic genus

This can be evaluated easily when $\phi^0 \to 0$.

... dilute gas.

On the other hand OSV is for $g_s \sim \frac{1}{\phi^0} \to 0$

related by modular invariance.

- $\phi^0 \to 0$: Gopakumar - Vafa

  M2 + KK momentum

  $\Rightarrow Z_{top} (q = e^{-g_s})$