String Theory Compactification with/without Torsion

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Outline: Part I

- Geometry and Holomony
- Supersymmetry, Spinors, and Calabi-Yau
- Flux and Backreaction
- Energetics of Heterotic Flux Compactification
- Strominger System and Heterotic Flux as a Torsion
- A Supersymmetric Solution to Heterotic Flux Compactification
- Global Issues: Index Counting, Smoothness, etc

references

Geometry

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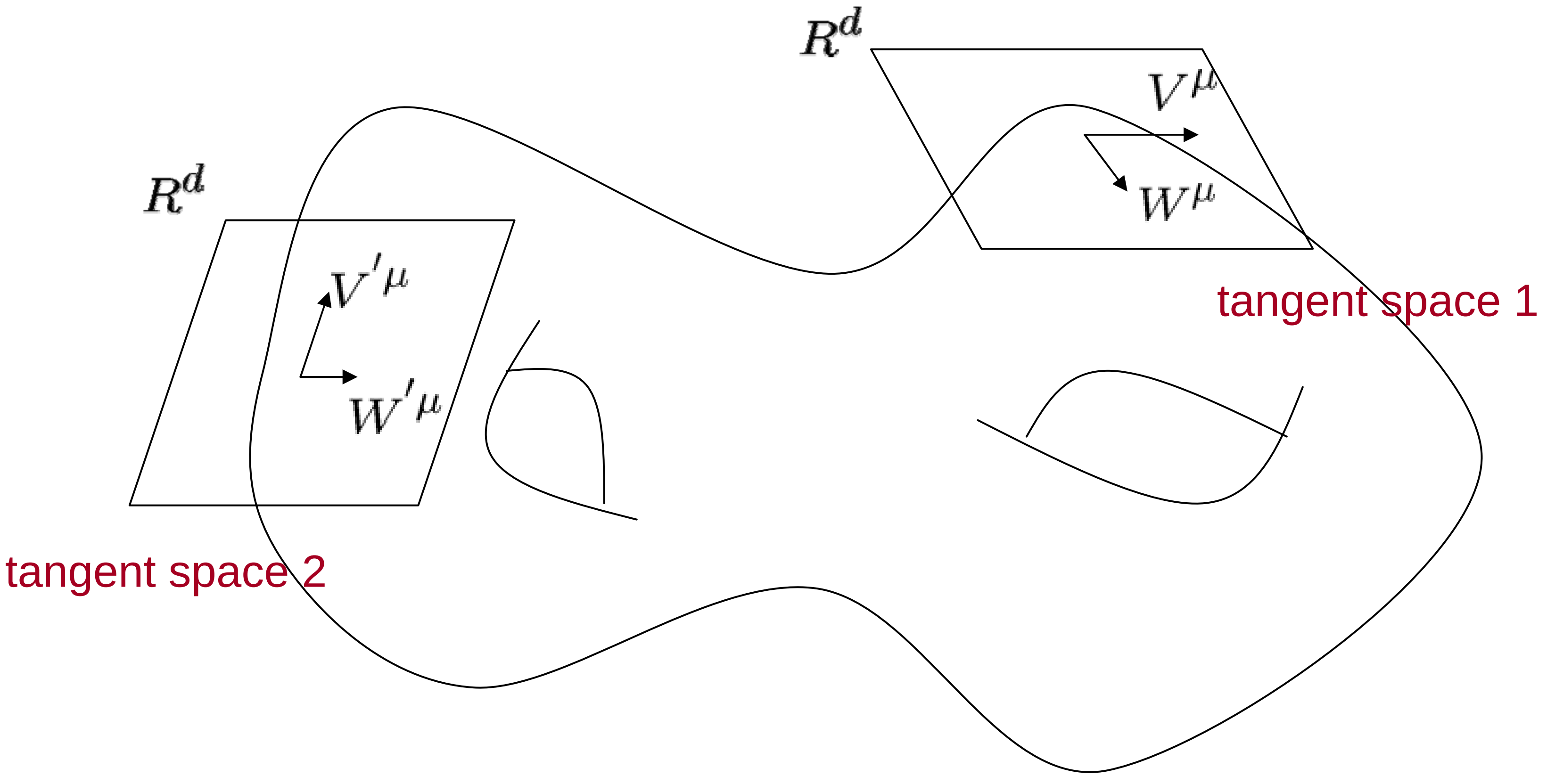
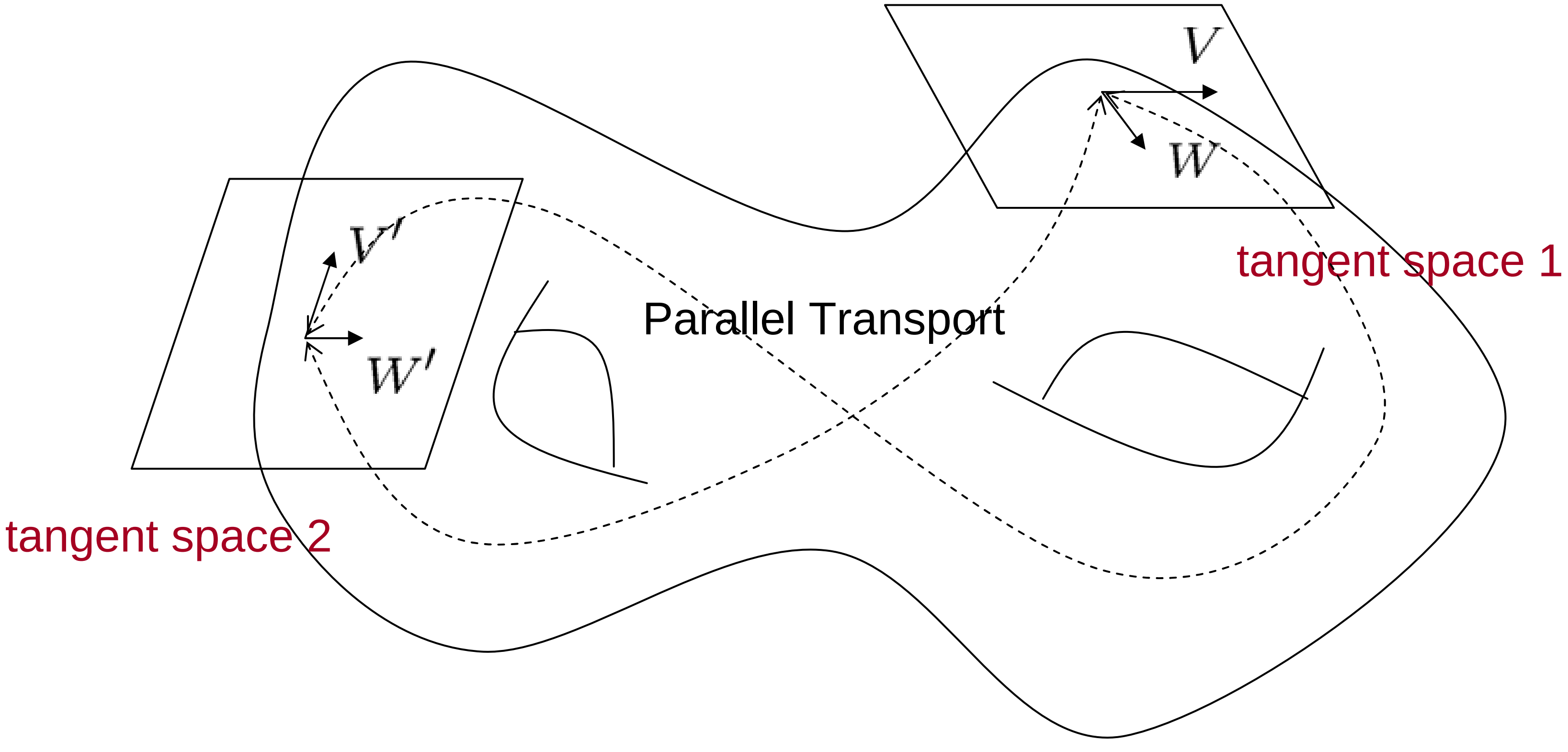
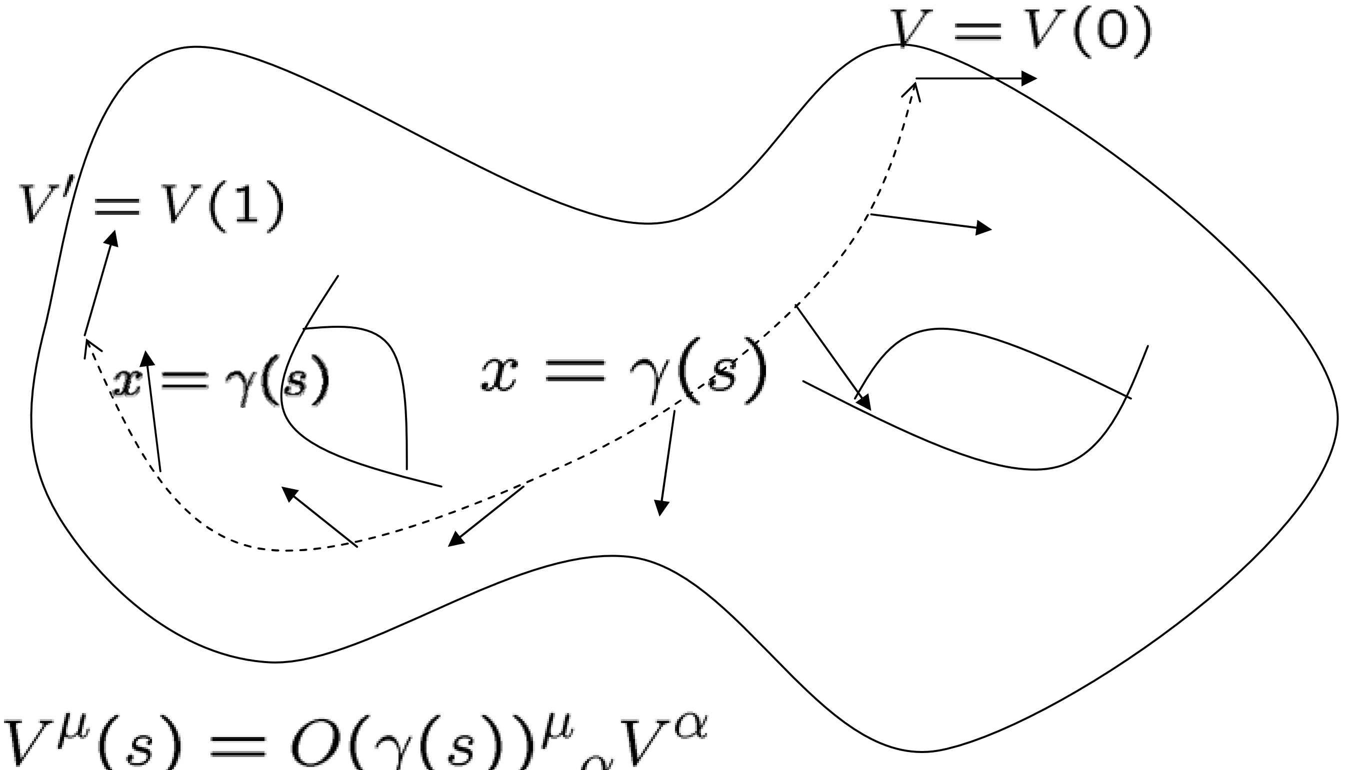
Geometry and Holonomy





V
W
tangent space 1
V′
W′
Parallel Transport
tangent space 2

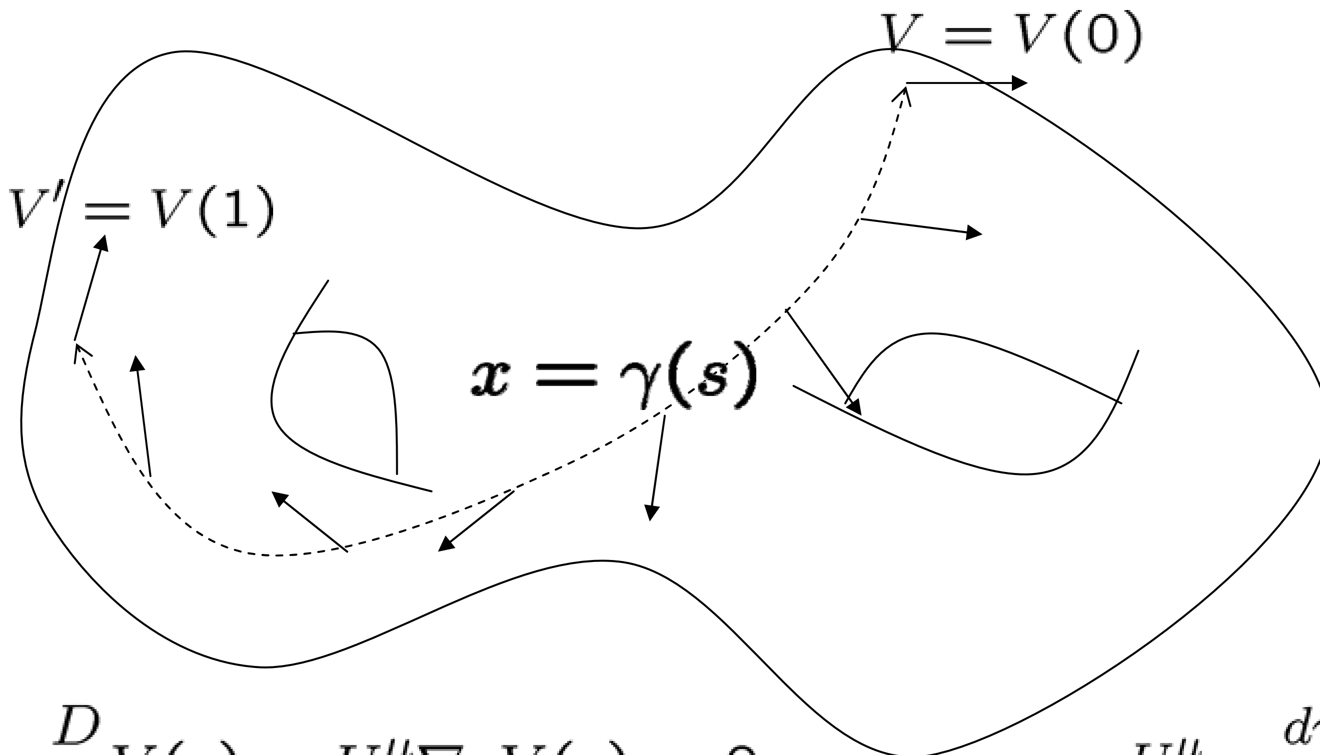
Parallel Transport = Covariant Derivative = Connection



$$V^{\mu}(s) = O(\gamma(s))^{\mu}{}_{\alpha} V^{\alpha}$$

$$\nabla_\mu = \partial_\mu + \cdots$$

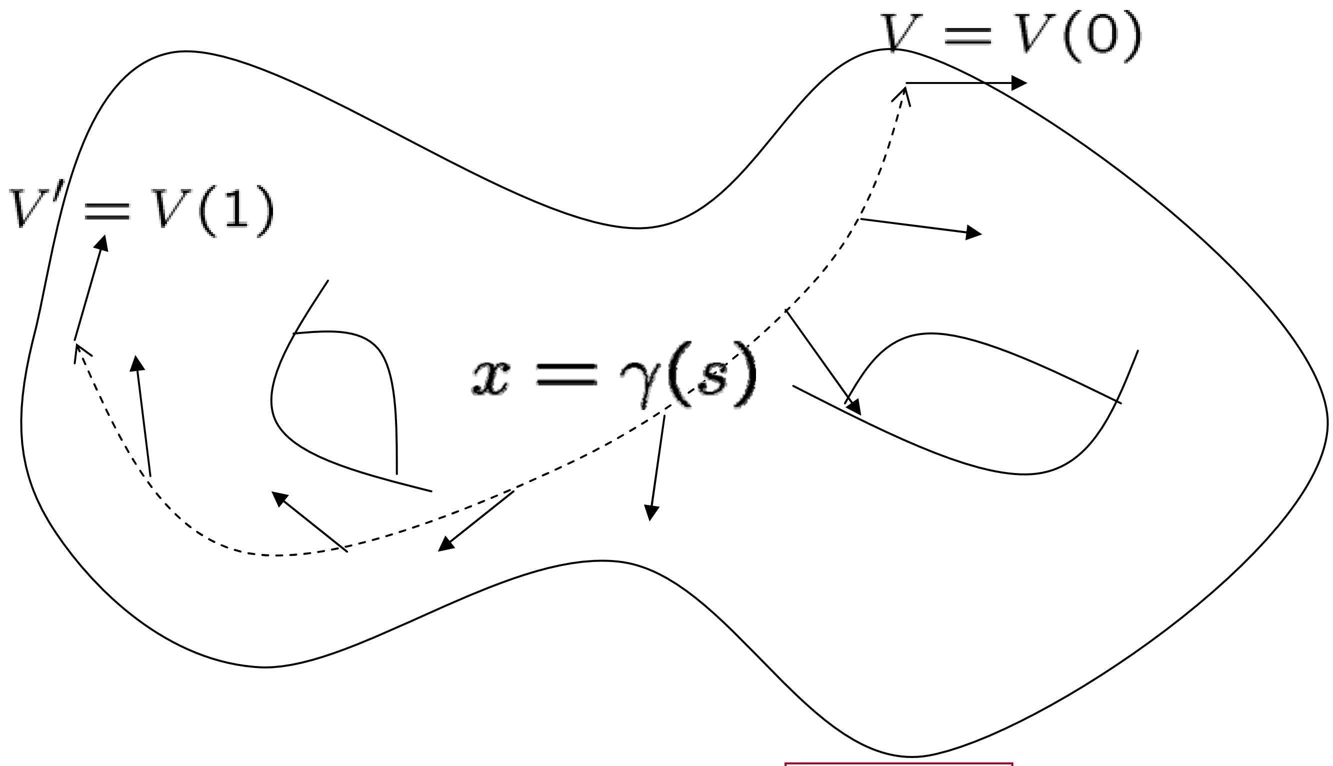
Parallel Transport = Covariant Derivative = Connection



$$\frac{D}{ds}V(s) = U^\mu \nabla_\mu V(s) = 0$$

$$U^\mu \sim \frac{d\gamma^\mu(s)}{ds}$$

Riemannian Geometry = Parallel Transport + Preservation of Lengths



$$U^{\mu}\nabla_{\mu}\left(g_{\alpha\beta}V(s)^{\alpha}V(s)^{\beta}\right) = U^{\mu}\boxed{\left(\nabla_{\mu}g_{\alpha\beta}\right)}V(s)^{\alpha}V(s)^{\beta}$$

Riemannian Geometry = Parallel Transport + Preservation of Lengths

$$\frac{d}{ds}\left(g_{\alpha\beta}V(s)^{\alpha}V(s)^{\beta}\right) = U^{\mu}\boxed{\left(\nabla_{\mu}g_{\alpha\beta}\right)}V(s)^{\alpha}V(s)^{\beta} = 0$$

$= 0$ everywhere

↑ ↓ *

$$\Gamma^{\alpha}_{\mu\beta} = \frac{g^{\alpha\lambda}}{2}\left(\partial_{\beta}g_{\lambda\mu} + \partial_{\mu}g_{\lambda\beta} - \partial_{\lambda}g_{\mu\beta}\right)$$

$$\nabla_{\mu}V^{\alpha} = \partial_{\mu}V^{\alpha} + \Gamma^{\alpha}_{\mu\beta}V^{\beta}$$

$$\nabla_{\mu}W_{\beta} = \partial_{\mu}W_{\beta} - \Gamma^{\alpha}_{\mu\beta}W_{\alpha}$$

* must assume that Γ is symmetric under the exchange of the two lower indices.

Riemannian Geometry = Parallel Transport + Preservation of Lengths

$$\frac{d}{ds}\left(g_{\alpha\beta}V(s)^{\alpha}V(s)^{\beta}\right) = U^{\mu}\boxed{\left(\nabla_{\mu}g_{\alpha\beta}\right)}V(s)^{\alpha}V(s)^{\beta} = 0$$

$= 0$ everywhere

$$V^{\mu}(s) = O(\gamma(s))^{\mu}{}_{\alpha}V^{\alpha}$$

$$O^{T}gO = g$$

?

$$O^{T}I_{d}O = I_{d}$$

d x d Identity matrix

Riemannian Geometry = Parallel Transport + Preservation of Lengths

$$\frac{d}{ds}\left(g_{\alpha\beta}V(s)^\alpha V(s)^\beta\right) = U^\mu \left(\nabla_\mu g_{\alpha\beta}\right) V(s)^\alpha V(s)^\beta = 0$$

$= 0$ everywhere

$$V^\mu(s) = O(\gamma(s))^\mu{}_\alpha V^\alpha$$

$$V^\mu = \sum_{a=1}^{d} V^a e_a^\mu$$

orthonormal frame = "vielbein"

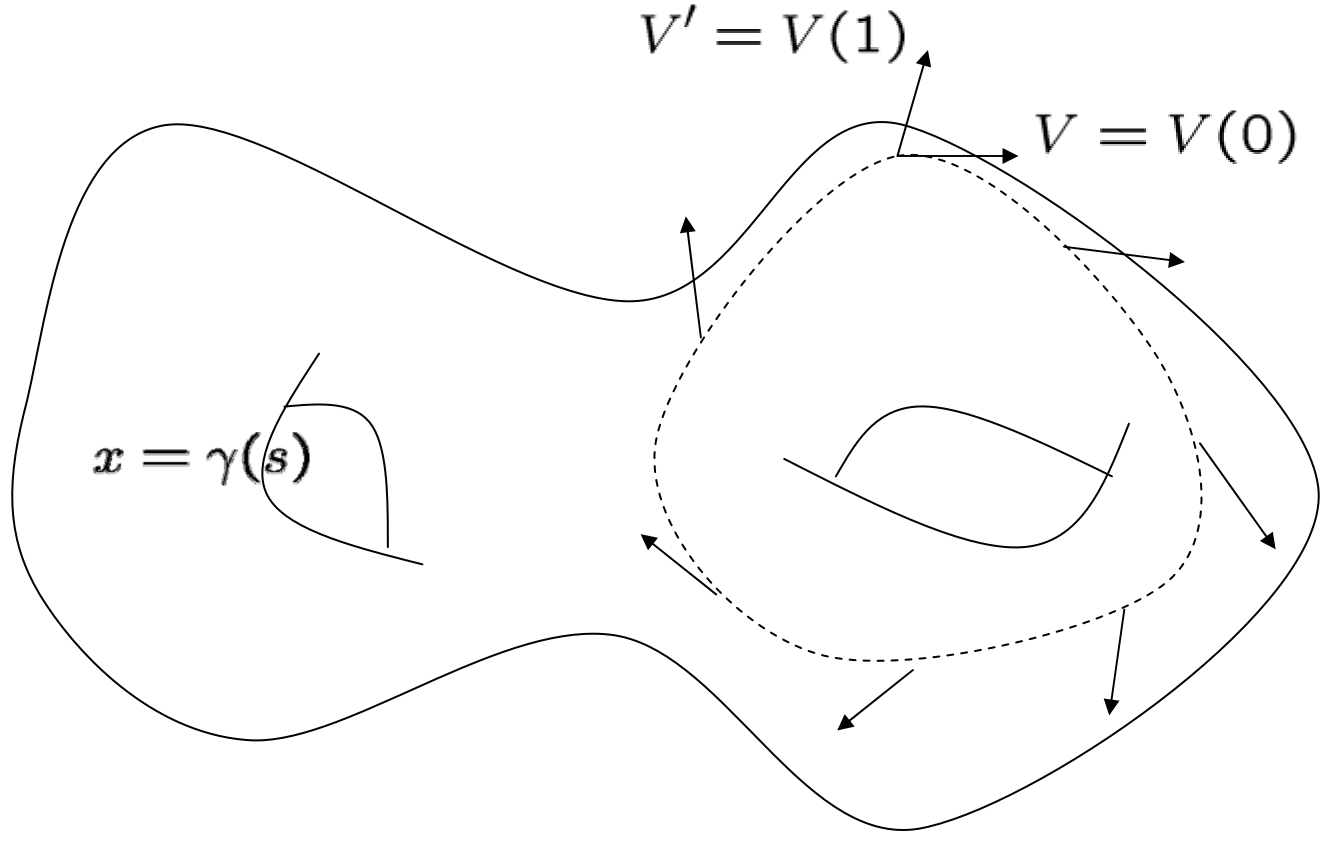
$$g_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b$$

$$g_{ab} = \delta_{ab}$$

$$W_\mu = \sum_{a=1}^{d} W_a e_\mu^a \qquad e_\mu^a e_a^\nu = \delta_\mu^\nu$$

$$V^a(s) = O(\gamma(s))^a{}_b V^b$$

Loops

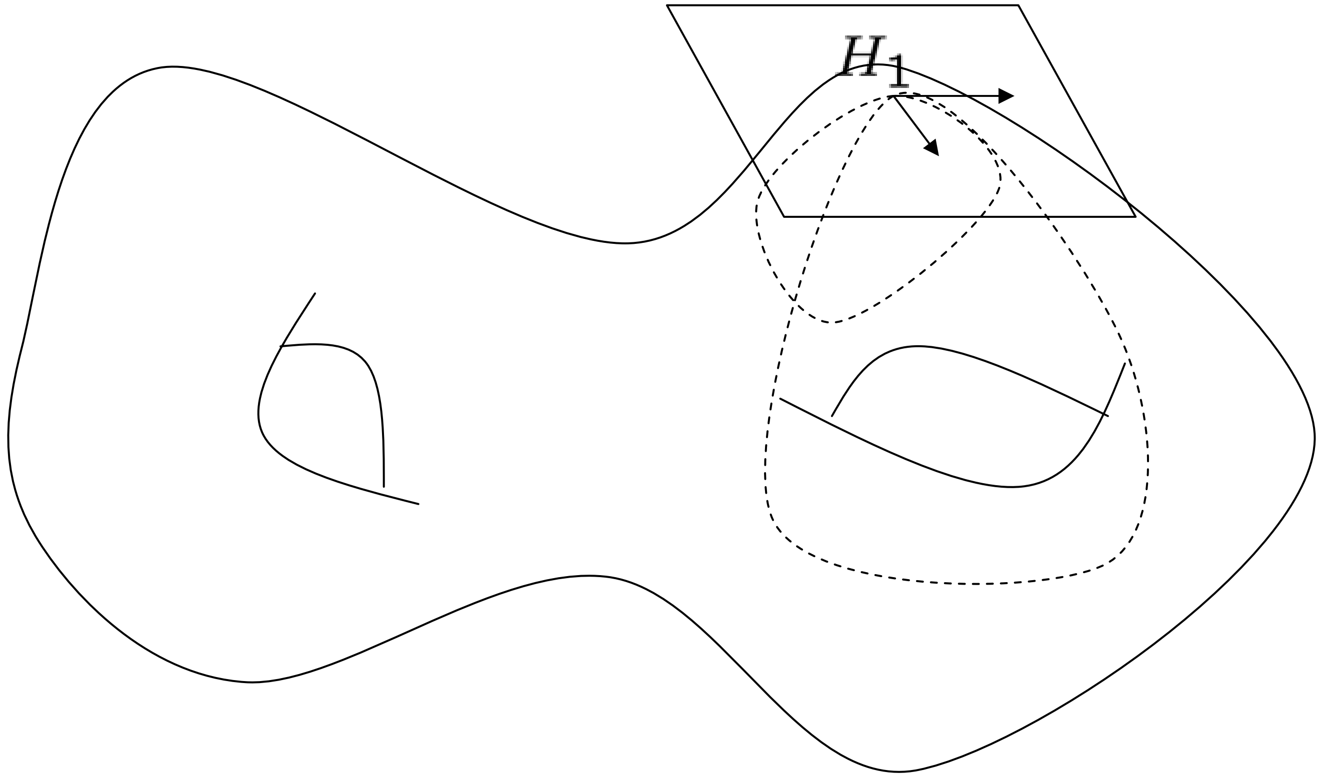


$$V'^a = O(\gamma(1))^a{}_b V^b$$

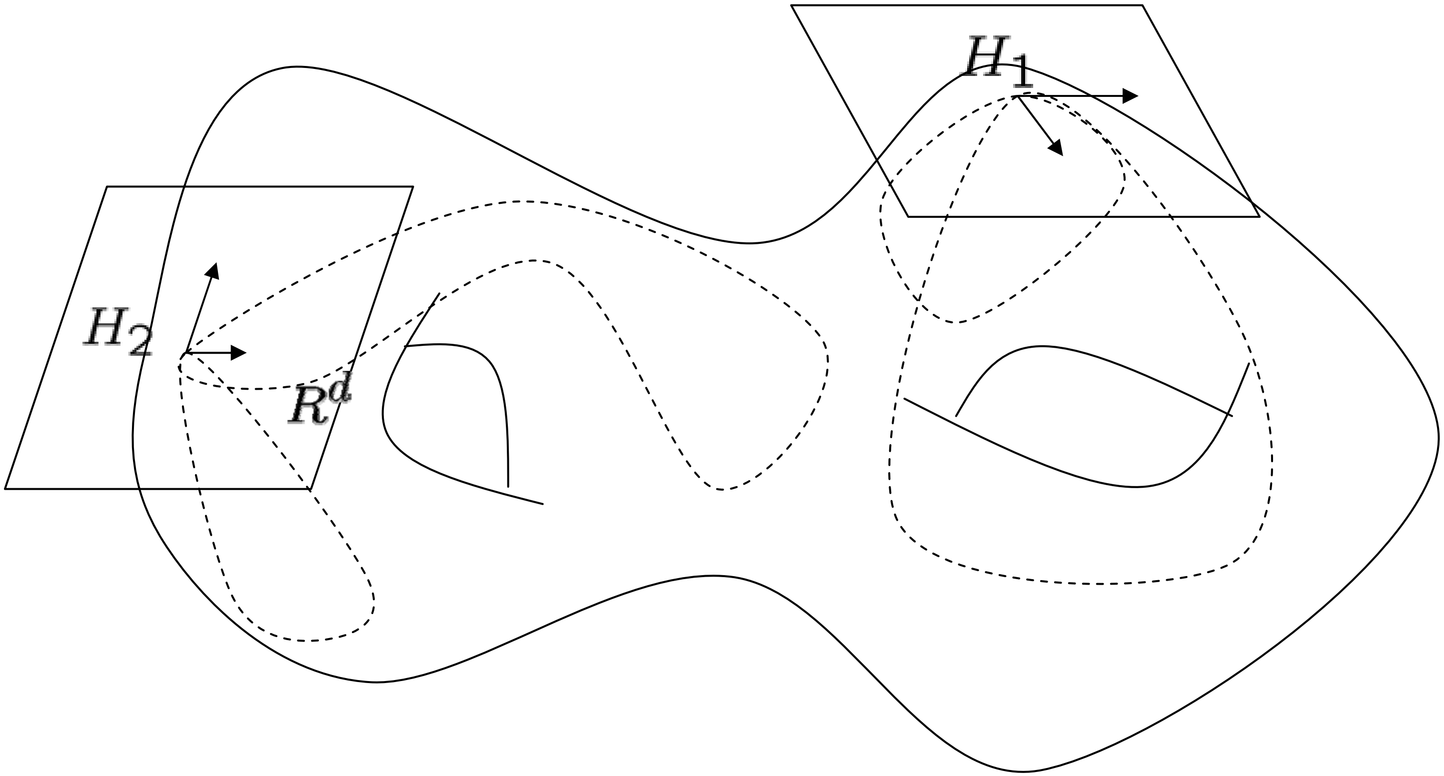
$$\sum_a g_{ab} O^a{}_g O^b{}_f = (O^T I O)_{gf} = I_{gf}$$

Collection of rotation elements $O(\gamma(1))$ along closed loops starting and ending at the same point form a group. Each loop rotates a vector and doing it twice in succession defines a multiplication

Holonomy Groups: Collection of all $O(\gamma(1))$'s from closed loops with a base point; different choice of the base point gives a conjugation so that the group remain isomorphic to each other.



Holonomy Groups: Collection of all $O(\gamma(1))$'s from closed loops with a base point; different choice of the base point gives a conjugation so that the group remain isomorphic to each other.



$$H_2 = O_{12}^T H_1 O_{12}$$

O_{12} depends on path chosen between 1 and 2 but the holonomy groups themselves are insensitive.

Riemannian Geometry ~ Holonomy Group = SO(d)

“Vielbein” and “spin connections” play special roles
since they are the natural representation of SO holonomy

$$\nabla_\mu V^a = \partial_\mu V^a + \omega_\mu{}^a{}_b V^b$$

$$\nabla_\mu W_a = \partial_\mu W_a + \omega_{\mu a}{}^b W_b$$

SO(d) valued 1-form

$$V^\mu = \sum_{a=1}^{d} V^a \, e_a^\mu$$

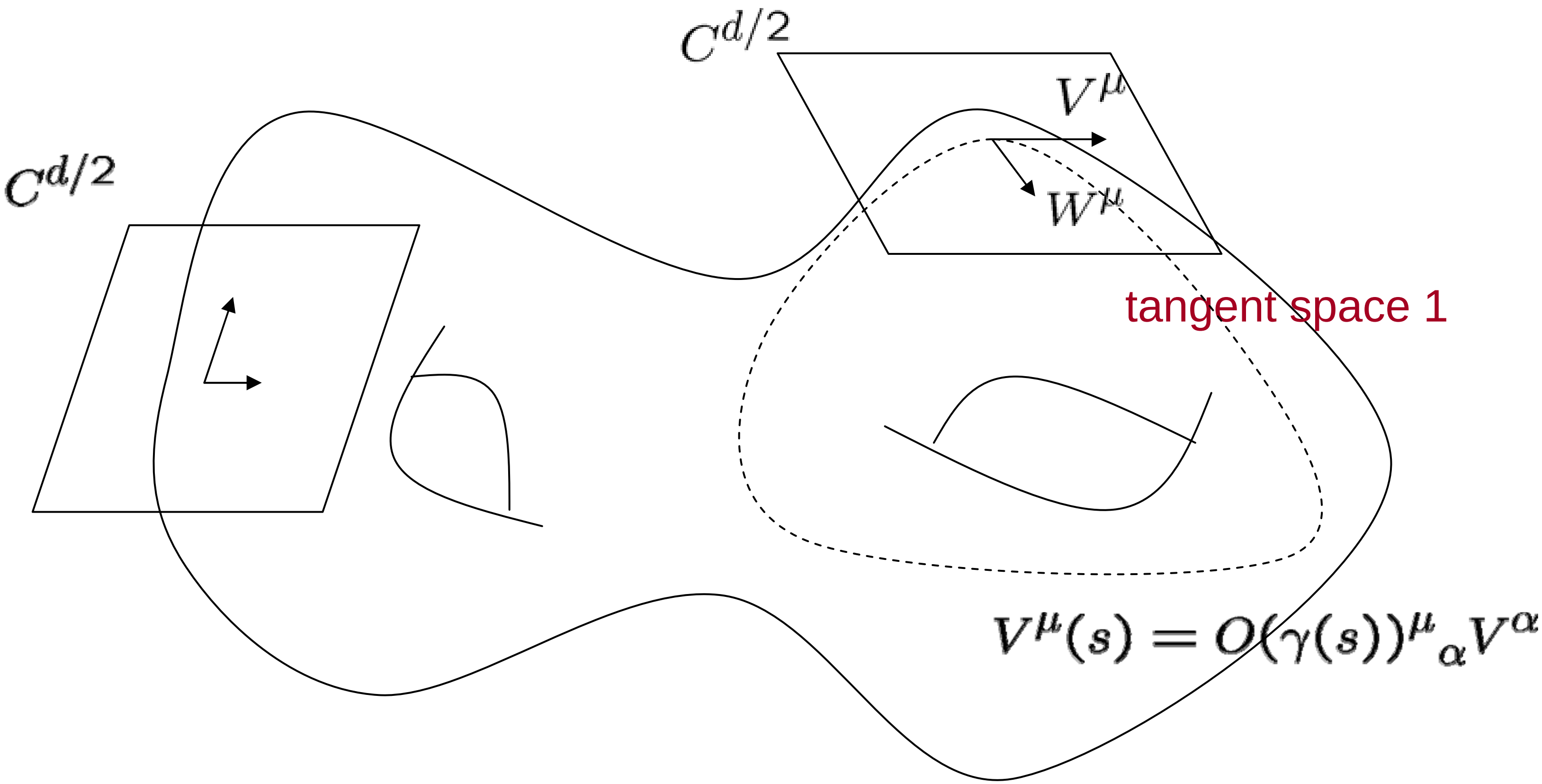
$$W_\mu = \sum_{a=1}^{d} W_a \, e_\mu^a$$

$$\omega_{\mu ab} + \omega_{\mu ba} = 0$$

$$de^a + \omega^a{}_b \wedge e^b = 0$$

$$d\omega^a{}_b + \omega^a{}_f \wedge \omega^f{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$$

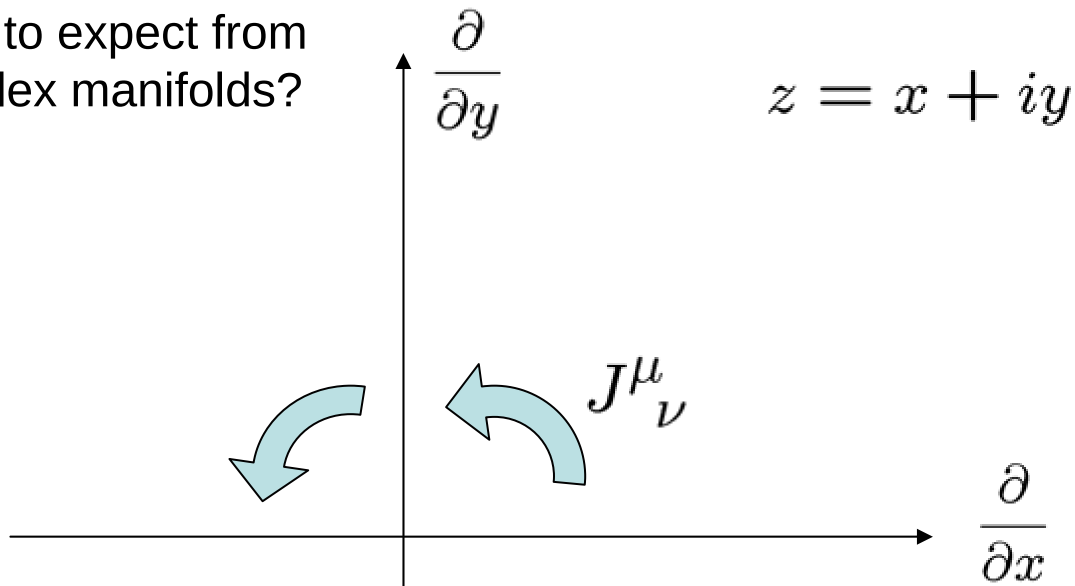
Example of Reduce Holonomy: Kaehler Geometry



smaller set of $O(\gamma(1))$'s?

What to expect from complex manifolds?

$$z = x + iy$$



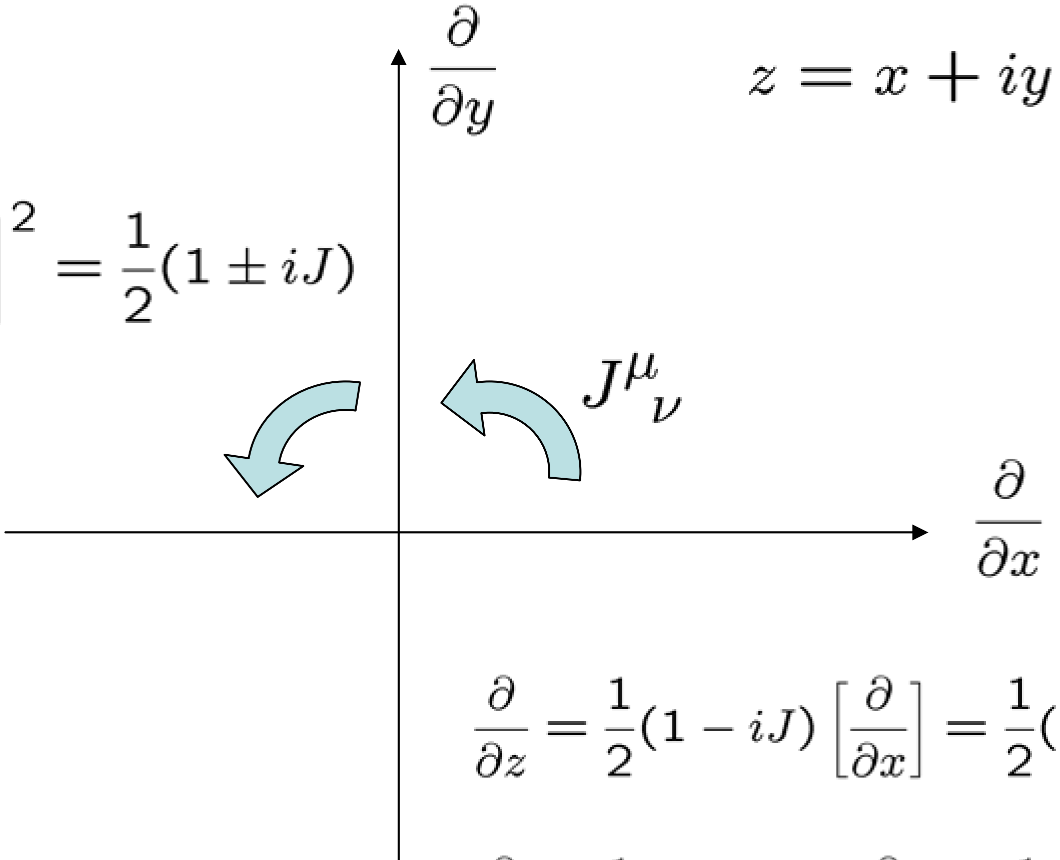
$$J: \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \Rightarrow -i\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \Rightarrow +i\frac{\partial}{\partial \bar{z}}$$

$$z = x + iy$$

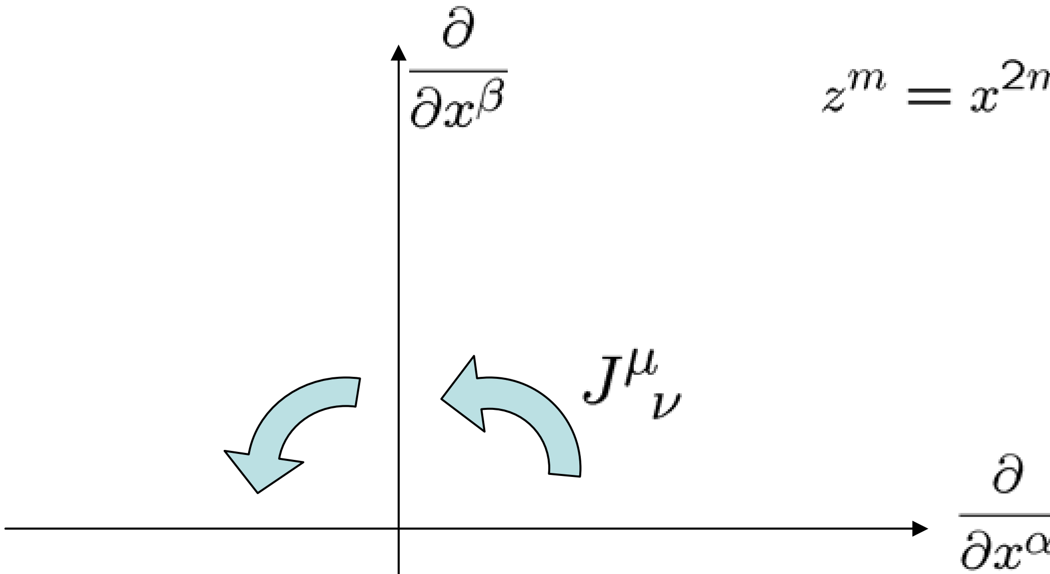
$$\left[\frac{1}{2}(1 \pm iJ)\right]^2 = \frac{1}{2}(1 \pm iJ)$$

projections



$$\frac{\partial}{\partial z} = \frac{1}{2}(1 - iJ)\left[\frac{\partial}{\partial x}\right] = \frac{1}{2}(1 - iJ)\left[-i\frac{\partial}{\partial y}\right]$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(1 + iJ)\left[\frac{\partial}{\partial x}\right] = \frac{1}{2}(1 + iJ)\left[i\frac{\partial}{\partial y}\right]$$



$$z^m = x^{2m-1} + i x^{2m}$$

$$\left\{ \frac{\partial}{\partial z^m} : m = 1, \ldots, d/2 \right\} = \frac{1}{2}(1 - iJ) \left\{ \frac{\partial}{\partial x^\alpha} : \alpha = 1, \ldots, d \right\}$$ holomorphic

$$\left\{ \frac{\partial}{\partial \bar{z}^m} : m = 1, \ldots, d/2 \right\} = \frac{1}{2}(1 + iJ) \left\{ \frac{\partial}{\partial x^\alpha} : \alpha = 1, \ldots, d \right\}$$ anti-holomorphic

Emulate complex vector space $C^{d/2}$:

90 degree rotation: $J^{\mu}{}_{\alpha} J^{\alpha}{}_{\nu} = -\delta^{\mu}{}_{\nu}$

= almost complex structure

lengths preserved: $J^{\mu}{}_{\alpha} J^{\nu}{}_{\beta} g_{\mu\nu} = g_{\alpha\beta}$

= Hermiticity

robust under parallel transport: $\nabla J = 0$

= integrability + Kaehler

metric-independent:
the existence of z^m

metric-dependent

“multiplication by $\pm i$ ” should be constant : $\nabla J = 0$

= integrability + Kaehler

metric-independent

$$[A^m \partial_m, B^n \partial_n] = C^k \partial_k$$

$$[A^{\bar{m}} \partial_{\bar{m}}, B^{\bar{n}} \partial_{\bar{n}}] = C^{\bar{k}} \partial_{\bar{k}}$$

metric-dependent

$$0 = d\left(J_{\mu\nu} dx^\mu \wedge dx^\nu\right)$$

The integrability condition can be formulated as vanishing of

$$N_J(V, W) \equiv Re\,((1 - iJ)[(1 + iJ)V, (1 + iJ)W])$$

for arbitrary pair of vectors V and W.
This defines the Nijenhuis tensor as

$$(N_J)^{\alpha}_{\mu\nu} \equiv \langle dx^{\alpha}, N_J(\partial_\mu, \partial_\nu)\rangle = J_\mu{}^\beta \partial_{[\beta} J_{\nu]}{}^\alpha - J_\nu{}^\beta \partial_{[\beta} J_{\mu]}{}^\alpha$$

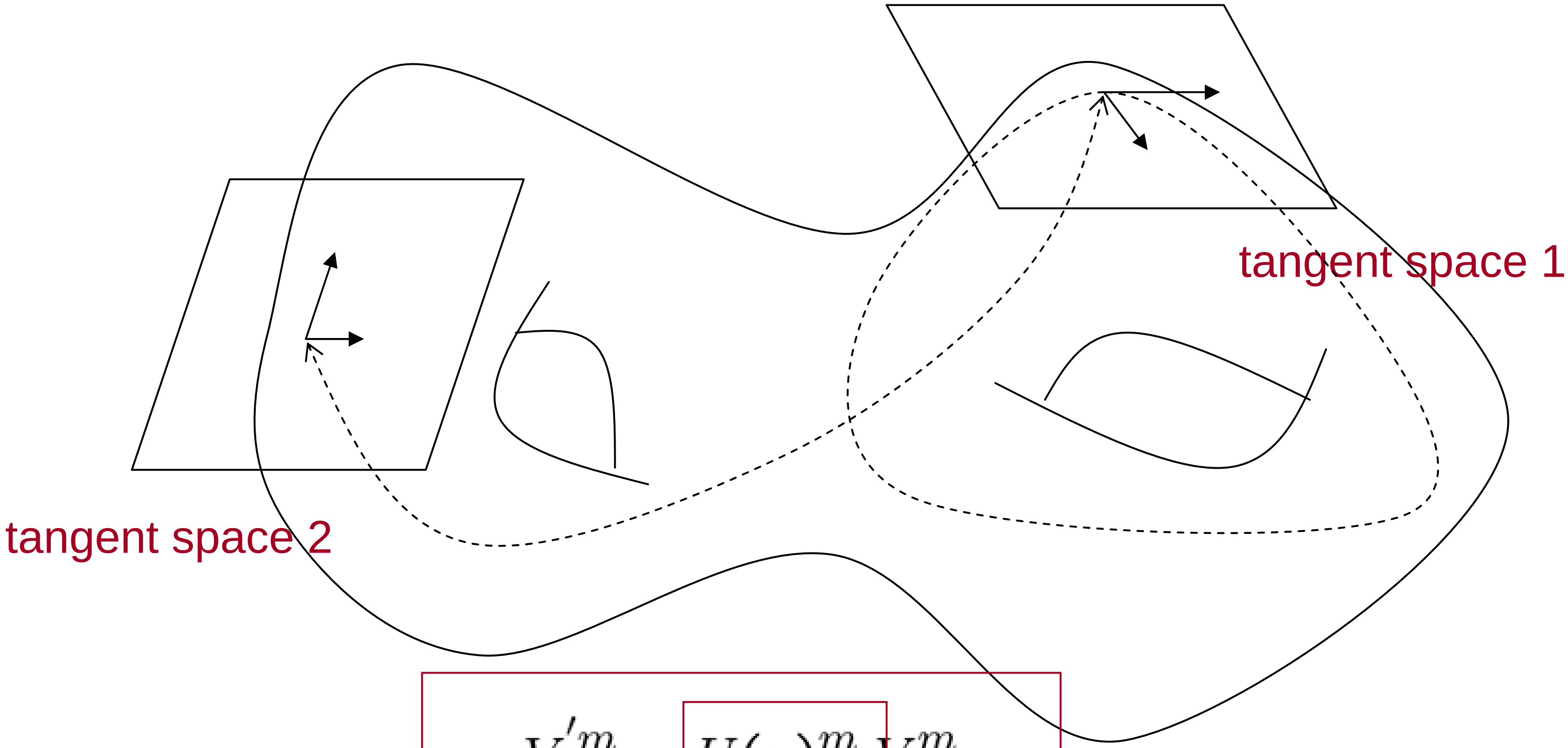
With the metric connection Γ, the partial derivatives may be covariantized, so that $\nabla J = 0$ implies integrability.

$$(N_J)^{\alpha}_{\mu\nu} = J_\mu{}^\beta \nabla_{[\beta} J_{\nu]}{}^\alpha - J_\nu{}^\beta \nabla_{[\beta} J_{\mu]}{}^\alpha \boxed{= 0}$$

Kaehler Condition:

Compatibility of (Anti)Holomorphic Decomposition with Metric

$$V^m \frac{\partial}{\partial z^m} \qquad W^{\bar{m}} \frac{\partial}{\partial \bar{z}^{\bar{m}}}$$



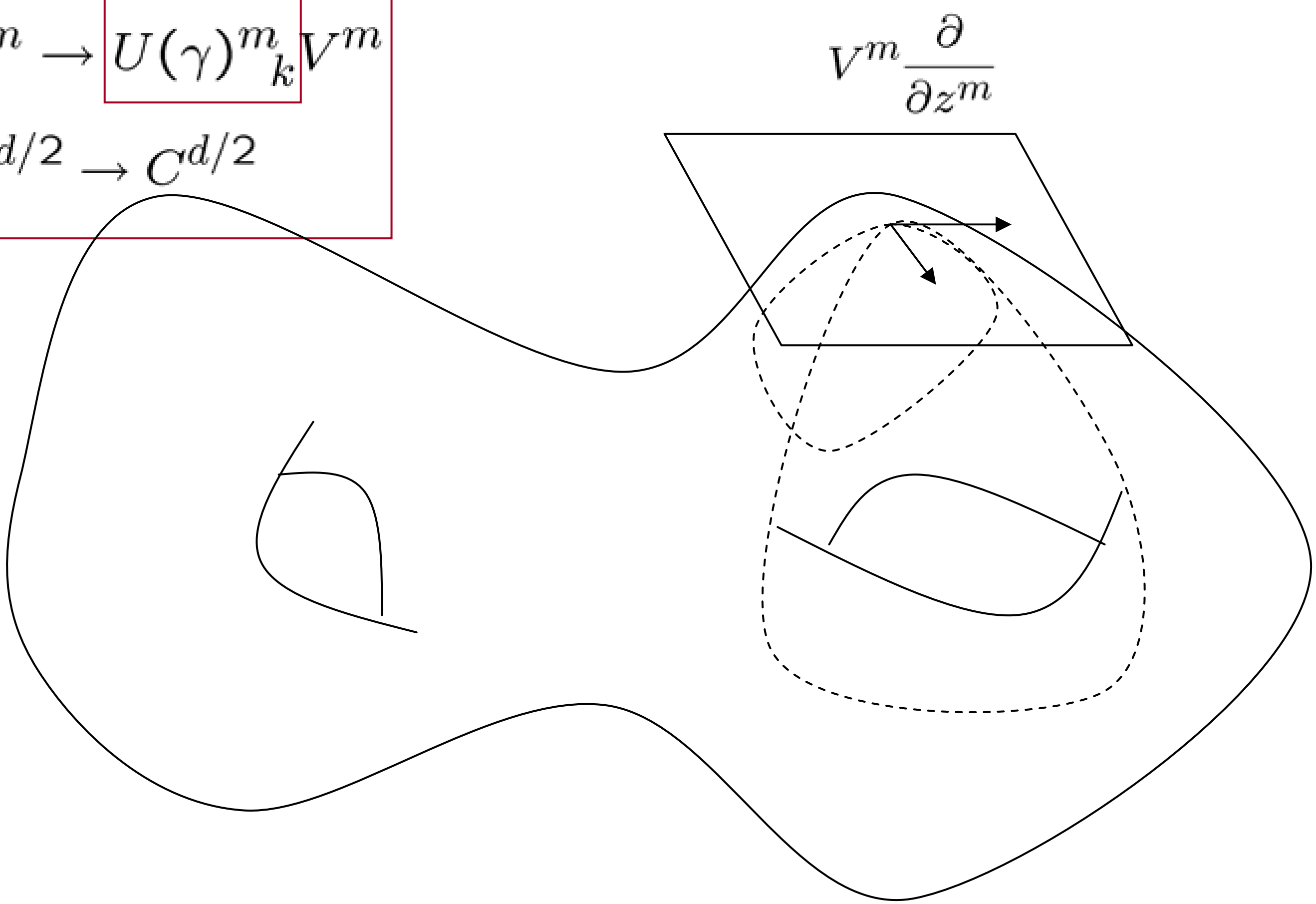
$$V'^m = U(\gamma)^m{}_k V^m$$

$$W'^{\bar{m}} = \bar{U}(\gamma)^{\bar{m}}{}_{\bar{k}} V^{\bar{m}}$$

Kaehler Condition: U(d/2) Holonomy Group

$$V^m \rightarrow U(\gamma)^m{}_k V^m$$

$$C^{d/2} \rightarrow C^{d/2}$$



Kaehler Condition: U(d/2) Holonomy Group

$$V^m \rightarrow U(\gamma)^m{}_k V^m$$

$$C^{d/2} \rightarrow C^{d/2}$$

→

$$\Gamma^m_{kl} \neq 0$$

$$\Gamma^m_{\bar{k}l} = 0$$

$$\Gamma^m_{\bar{k}\bar{l}} = 0$$

↓

$$R_{m\bar{k}p\bar{q}} \neq 0$$

$$R_{mkp\bar{q}} = 0$$

$$R_{mkpq} = 0$$

Reduced Holonomy Group = Existence of Parallel Tensor(s)

GL(d,R)

$$\nabla g = 0$$

Riemann SO(d,R)

Reduced Holonomy Group = Existence of Parallel Tensor(s)

GL(d,R)

$$\nabla g = 0$$

Riemann SO(d,R)

$$\nabla J = 0$$

Kaehler U(d/2)

Reduced Holonomy Group = Existence of Parallel Tensor(s)

GL(d,R) → (symplectic 2-form, $\nabla w^{(2)} = 0$) → Sp(d/2) Symplectic

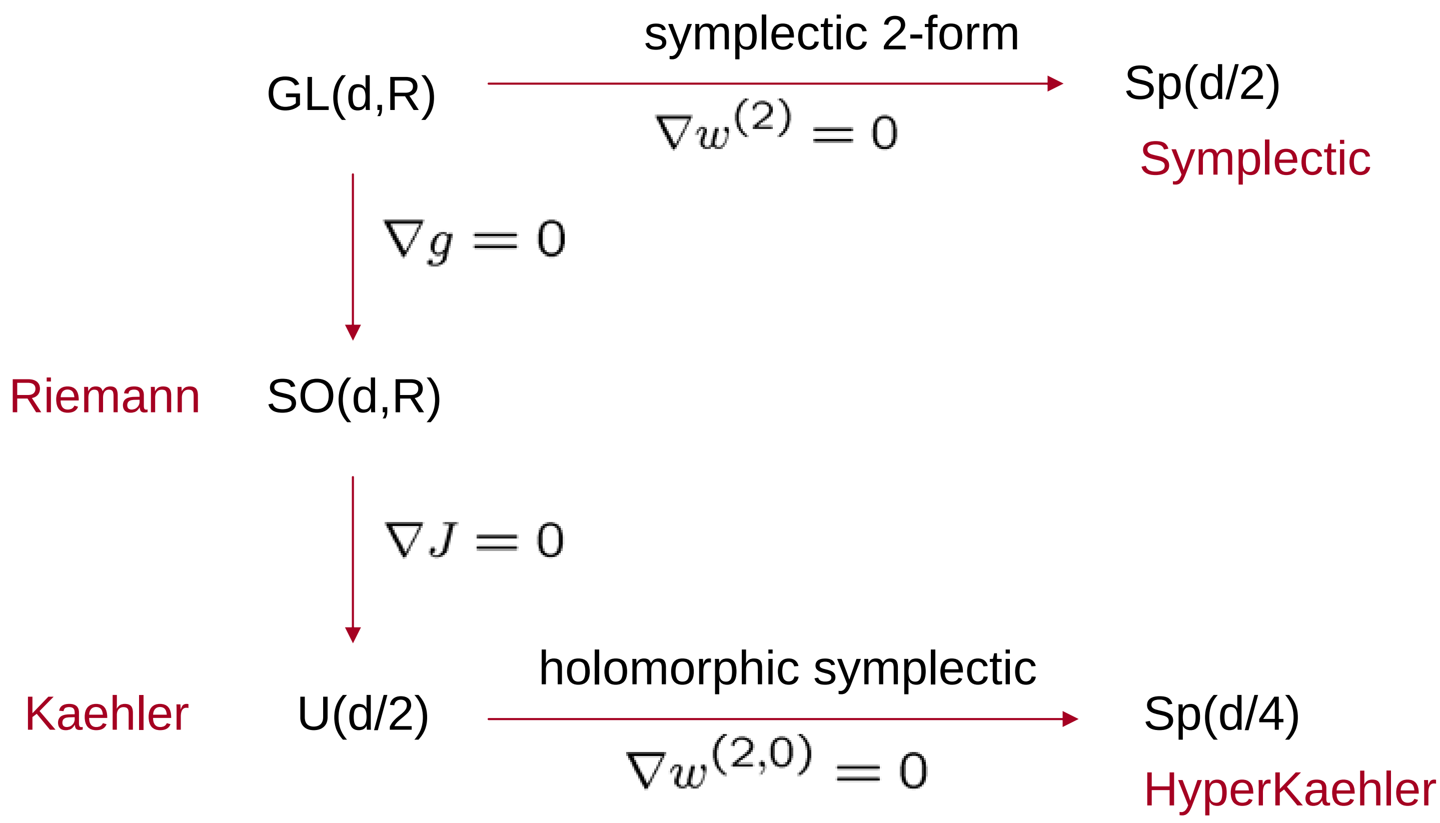
GL(d,R) ↓ $\nabla g = 0$

Riemann SO(d,R)

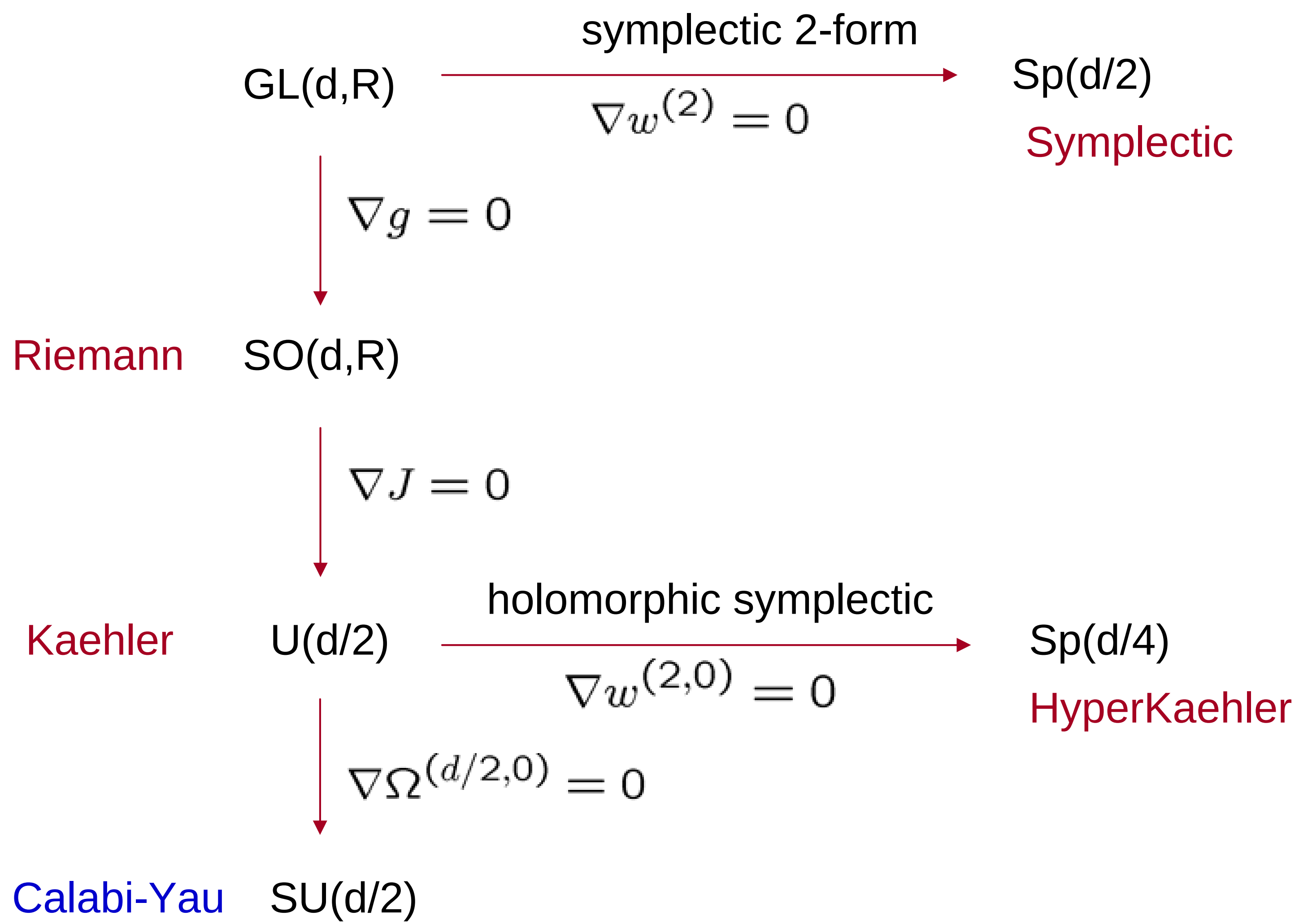
SO(d,R) ↓ $\nabla J = 0$

Kaehler U(d/2)

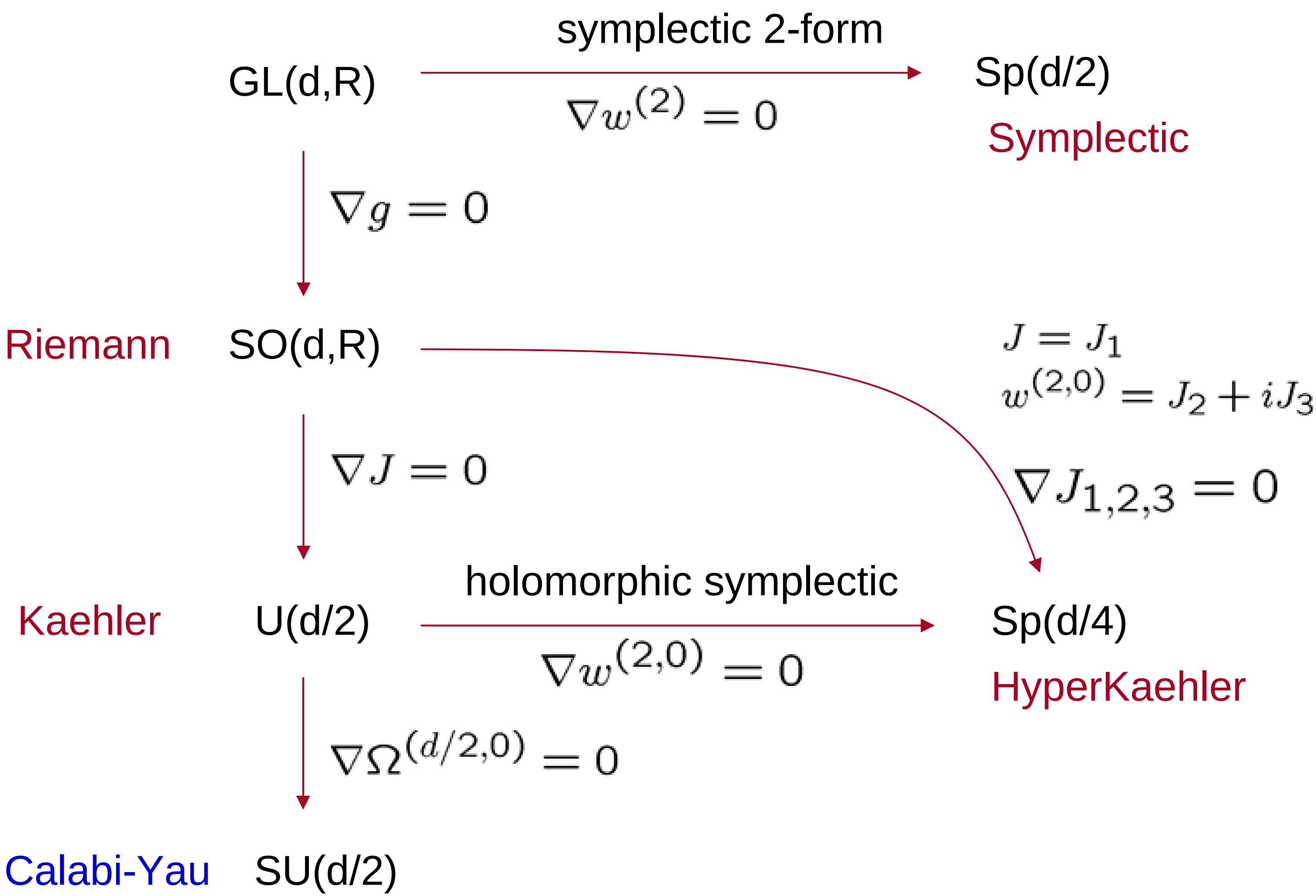
Reduced Holonomy Group = Existence of Parallel Tensor(s)



Reduced Holonomy Group = Existence of Parallel Tensor(s)



Reduced Holonomy Group = Existence of Parallel Tensor(s)



Supersymmetry, Spinors, and Calabi-Yau

Simple Compactification with unbroken SUSY:

$$g_{9+1}^{string} = \eta_{ij}^{3+1} dX^i dX^j + \boxed{g_{\mu\nu}^{(6)}} dy^\mu dy^\nu$$

$$0 = \delta_\epsilon^{SUSY} \Psi_i = D_i \epsilon_{9+1} = \partial_i \epsilon_{9+1}$$
$$0 = \delta_\epsilon^{SUSY} \Psi_\mu = D_\mu \epsilon_{9+1}$$

$$\epsilon_{9+1} = \epsilon_{3+1} \otimes \eta + c.c.$$

$$D_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^{ab}$$

$$0 = \delta_\epsilon^{SUSY} \Psi_\mu \Rightarrow 0 = D_\mu \eta$$

$$\epsilon_{9+1} = \epsilon_{3+1} \otimes \eta + c.c.$$

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}$$

$$\partial_\mu \gamma_a = 0$$

$D_\mu \ vs. \ \nabla_\mu$ is a matter of notational convention. Both denote the covariant derivative with respect to the same connection. The former acts on spinors and the latter acts on tensors.

$$0 = \nabla_\mu \left[\eta^\dagger \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_n} \eta\right]$$

$$
\begin{aligned}
\nabla_\mu \left(\eta^\dagger \gamma ... \eta\right) &= \partial_\mu \left(\eta^\dagger \gamma ... \eta\right) + \eta^\dagger \left[\gamma ..., \frac{1}{4}\omega_{\mu cd}\gamma^{cd}\right]\eta \\
&= (D_\mu \eta)^\dagger \gamma ... \eta + \eta^\dagger \gamma ... D_\mu \eta \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
& \left[\gamma_a, \frac{1}{4}\omega_{\mu cd}\gamma^{cd}\right] \\
&= \frac{1}{4}\omega_{\mu cd}\left(\gamma_a \gamma^c \gamma^d - \gamma^c \gamma^d \gamma_a\right) \\
&= \frac{1}{4}\omega_{\mu cd}\left(2\delta_a^c \gamma^d - 2\delta_a^d \gamma_c\right) \\
&= \omega_{\mu ad}\gamma^d
\end{aligned}
$$

Parallel Spinors

$$D\eta = 0$$

$$J_{\mu\nu} = i\eta^\dagger \gamma_{\mu\nu} \eta$$
$$(\eta^\dagger \eta = 1)$$

Parallel Tensors

$$\nabla J = 0$$

Reduced Holonomy U(d/2):
Kaehler Manifold

Parallel Spinors

$$D\eta = 0$$

$$J_{\mu\nu} = i\eta^\dagger \gamma_{\mu\nu}\eta$$

$$(\eta^\dagger \eta = 1)$$

Parallel Tensors

$$\nabla J = 0$$

Reduced Holonomy U(d/2):
Kaehler Manifold

$$J_{\mu\alpha} g^{\alpha\beta} J_{\nu\beta} = g_{\mu\nu} \quad \equiv \quad J^{\mu}{}_{\alpha} J^{\alpha}{}_{\nu} = -\delta^{\mu}_{\nu}$$

from Fierz Indentity and $\eta^\dagger \eta = 1$

Parallel Spinors

$$D\eta = 0$$

$$J_{\mu\nu} = i\eta^\dagger \gamma_{\mu\nu}\eta$$

$$(\eta^\dagger \eta = 1)$$

Parallel Tensors

$$\nabla J = 0$$

Reduced Holonomy U(d/2):
Kaehler Manifold

$$0 = \eta^\dagger \left[D_\mu, D_\nu\right] \eta = -i\frac{1}{4} R_{ab\mu\nu} J^{ab}$$

Parallel Spinors

$$D\eta = 0$$

$$J_{\mu\nu} = i\eta^{\dagger}\gamma_{\mu\nu}\eta$$

$$(\eta^{\dagger}\eta = 1)$$

Parallel Tensors

$$\nabla J = 0$$

Reduced Holonomy U(d/2):
Kaehler Manifold

$$0 = \eta^{\dagger}\left[D_{\mu}, D_{\nu}\right]\eta = -i\frac{1}{4}R_{ab\mu\nu}J^{ab}$$

SU(d/2) Holonomy:
Calabi-Yau Manifold

$$R_{abcd}J^{ab} = 3R_{a[bcd]}J^{ab} + R_{acbd}J^{ab} + R_{adcb}J^{ab} = 2R_{acbd}J^{cd}$$

$$\equiv 0 \qquad\qquad \propto J_a{}^f (Ricci)_{fb}$$

Calabi-Yau = SU(d/2) Holonomy = Ricci Flat Kaehler

Alternatively

$$\Omega^{(d/2,0)} = \eta^T \gamma_{k_1} \gamma_{k_2} \cdots \gamma_{k_{d/2}} \eta$$

$$\nabla \Omega^{(d/2,0)} = 0$$

This reduces the holonomy group from U(d/2) to SU(d/3) because

$$\Omega^{(d/2,0)} \rightarrow (DetU)\, \Omega^{(d/2,0)}$$

under any holonomy U in U(d/2). The determinant of any element of the U(d/2) holonomy group has to be unit, which means that we actually have SU(d/2) holonomy.

Cohomologies (More about this from deWolfe later in this school)

d=4

$$[dim H^{p,q}(K3 = CY2)] = \begin{bmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{bmatrix}$$

$$\Omega^{(d/2,0)} \qquad\qquad J$$

d=6

$$[dim H^{p,q}(CY3)] = \begin{bmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{bmatrix}$$

(d=6)

The number of metric deformation preserving the same SUSY

$= h^{1,1} + 2h^{2,1}$ geometric moduli fields

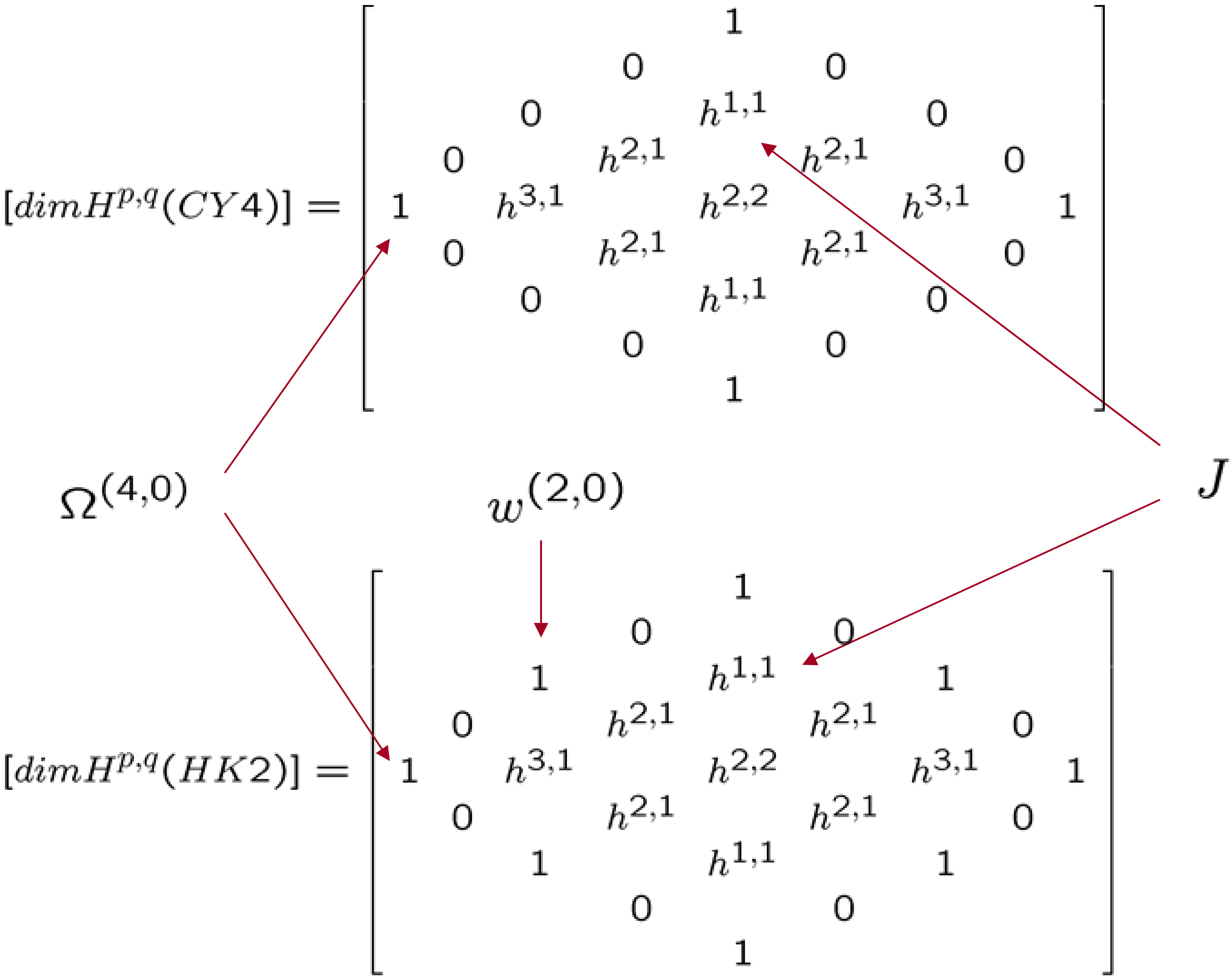
↓ add moduli from other gravity multiplet fields

$2(h^{1,1} + h^{2,1} + 1)$

↓ add moduli from R-R sector tensor fields

$4(h^{1,1} + h^{2,1} + 1)$

d=8



Flux and Backreaction

$$g_{9+1}^{string} = \eta_{ij}^{3+1} dX^i dX^j + \boxed{g_{\mu\nu}^{(6)} dy^\mu dy^\nu}$$

Ricci Flat = Solution to
empty Einstein Equation

Can we ignore other bosonic field strengths
which can contribute to the energy-momentum tensor ?

$$\mathcal{L}_{type\ II} = \sqrt{-g}e^{-2\Phi}\left[R - \frac{1}{2}|H_3|^2 + 4(\nabla\Phi)^2\right] - \sqrt{-g}\sum_p \frac{1}{2}|F^{RR}_{p-form}|^2$$

$$g^{string}_{9+1} = \eta^{3+1}_{ij}dX^i dX^j + g^{(6)}_{\mu\nu}dy^\mu dy^\nu$$

$$\nabla\Phi = 0$$
$$H_3 = 0$$
$$F^{RR}_p = 0$$

$$\mathcal{L}_{heterotic} = \sqrt{-g}e^{-2\Phi}\left[R - \frac{1}{2}|H|^2 + 4(\nabla\Phi)^2 - \frac{\alpha'}{4}\left(trF^2 - trR^2\right)\right]$$

$$g^{string}_{9+1} = \eta^{3+1}_{ij}dX^i dX^j + g^{(6)}_{\mu\nu}dy^\mu dy^\nu$$

$$\nabla\Phi = 0$$
$$H = 0$$
$$F \simeq R$$

(1) Can we turn on various tensor field strengths while maintaining vacuum condition (= minimum energy) ?

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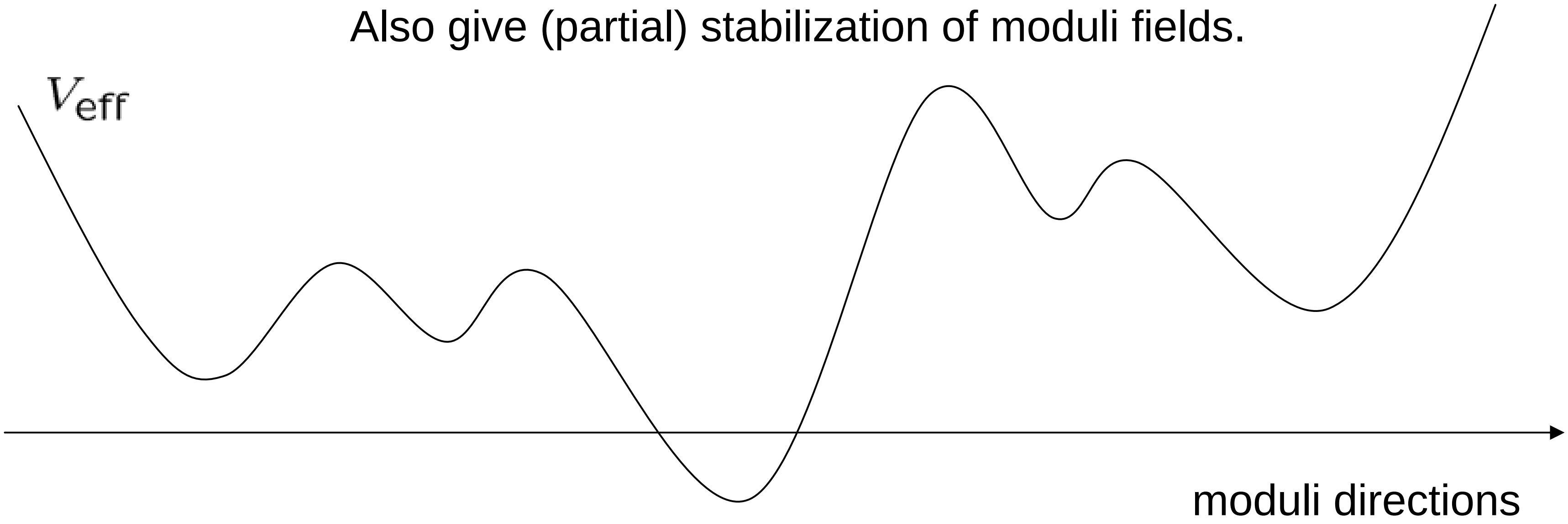
Yes.
Guaranteed if the field strengths involve topological flux.
Also give (partial) stabilization of moduli fields.



(1) Can we turn on various tensor field strengths while maintaining vacuum condition (= minimum energy) ?

Yes.
Guaranteed if the field strengths involve topological flux

(2) Can we turn on various tensor field strengths while maintaining unbroken supersymmetries ?

(1) Can we turn on various tensor field strengths while maintaining vacuum condition (= minimum energy) ?

Yes.
Guaranteed if the field strengths involve topological flux

(2) Can we turn on various tensor field strengths while maintaining unbroken supersymmetries ?

Yes.
Provided that the geometry can backreact and reshape the local form of the flux to be of a definite “chiral” type.

Example 1: IIB on Calabi-Yau with F_3 and H_3 flux

$$\mathcal{L}_{IIB}^{Einstein} = \sqrt{-G}\left[R_G - \frac{\nabla\tau\nabla\bar{\tau}}{2(Im\tau)^2} - \frac{|G_3|^2}{2(Im\tau)} - \frac{|F_5|^2}{2}\right] + \cdots$$

$$G_3 = F_3 - \tau H_3$$

$$F_5 = *F_5$$

$$\tau = C_0 + ie^{-\Phi}$$

Topological Flux in CY3 Background:

$$\oint_I F_3 \ = \ (2\pi)^2 \alpha' N_I$$
$$\oint_I H_3 \ = \ (2\pi)^2 \alpha' K_I$$

Ignoring backreactions, how can we minimize energy incurred by such topological (thus quantized) fluxes?

$$\mathcal{E} \ \sim \ \frac{e^{-\Phi_0}}{2} \int dy^6 \sqrt{g_6(CY3)}\, |G_3|^2 = \int_{CY3} G_3 \wedge *\bar{G}_3$$

$$\int_{CY3} G_3 \wedge *\bar{G}_3 = -i \int_{CY3} G_3 \wedge (\bar{G}_3^{(+)} - \bar{G}_3^{(-)})$$

$$G_3^{(\pm)} = (G_3 \mp i * G_3)/2$$

$$*G^{(\pm)} = \pm i G_3^{(\pm)}$$

$$= -i \int_{CY3} G_3 \wedge (\bar{G}_3 - 2\bar{G}_3^{(-)})$$

$$= -i \int_{CY3} G_3 \wedge \bar{G}_3 + 2i \int_{CY3} G_3^{(-)} \wedge \bar{G}_3^{(-)}$$

metric independent

metric dependent
and nonnegative

This quick and dirty computation produces
an effective potential for Calabi-Yau metrics

$$0 \leq V_{\text{eff}} = 2i \int_{CY3} G_3^{(-)} \wedge \bar{G}_3^{(-)}$$

An approximate vacuum can be found if the metric can be chosen such that the topological F_3 and H_3 potentials combines (with axi-dilaton) to make imaginary anti-self-dual part of G_3 vanishes.

But what if we have chosen to decompose it differently and obtained a potential in terms of imaginary self-dual part of G_3?

Backreaction of IIB Compactification to F_3 and H_3 flux

$$\mathcal{L}_{IIB}^{Einstein} = \sqrt{-G}\left[R_G - \frac{\nabla\tau\nabla\bar{\tau}}{2(Im\tau)^2} - \frac{|G_3|^2}{2(Im\tau)} - \frac{|F_5|^2}{2}\right] - \frac{1}{4iIm\tau} * (C_4 \wedge G_3 \wedge \bar{G}_3)$$

$$F_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}F_3 \wedge B_2$$

$$F_5 = *F_5$$

$$dF_5 = -F_3 \wedge H_3 + \cdots$$

$$G_3 = F_3 - \tau H_3$$

$$F_3 = dC_2$$

$$H_3 = dB_2$$

$$\tau = C_0 + ie^{-\Phi}$$

When H_3 and F_3 has no explicit electric or magnetic sources, the string metric, after backreaction, is still simple:

$$g_{9+1}^{string} = e^{2A(y)} \eta_{ij}^{3+1} dX^i dX^j + e^{-2A(y)} g_{\mu\nu}^{(6)} dy^\mu dy^\nu$$

some Calabi-Yau

whose warp factor is related to F_5 as,

$$F_5 = (1 + *_{9+1}) \left[de^{4A(y)} \wedge dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 \right]$$

$$\nabla^2 e^{-4A(y)} = e^{-6A(y)} \frac{|G_3|^2}{2Im\tau}$$

Net effect of these backreactions to G_3 is to modifiy the energy functional of the compactification from (with constant axi-dilaton)

$$\mathcal{E} \sim \int_{CY3} G_3 \wedge *\bar{G}_3$$

$$= -i \int_{CY3} G_3 \wedge \bar{G}_3 + \boxed{2i \int_{CY3} G_3^{(-)} \wedge \bar{G}_3^{(-)}}$$

$$= V_{\text{eff}}$$

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$$\mathcal{E} \sim \int_{CY3} G_3 \wedge *\bar{G}_3$$

$$= -i \int_{CY3} G_3 \wedge \bar{G}_3 + 2i \int_{CY3} G_3^{(-)} \wedge \bar{G}_3^{(-)}$$

to

$$\mathcal{E} \sim \int_{CY3} \frac{e^{4A}}{Im\tau} [G_3 \wedge *\bar{G}_3 + iG_3 \wedge \bar{G}_3]$$

$$= \int_{CY3} \frac{ie^{4A}}{Im\tau} \left[G_3 \wedge \bar{G}_3^{(-)}\right]$$

$$= \int_{CY3} \frac{ie^{4A}}{Im\tau} \left[G_3^{(-)} \wedge \bar{G}_3^{(-)}\right]$$

$$= V_{\text{eff}}$$

Minimization of the energy functional can be done in 2 steps:

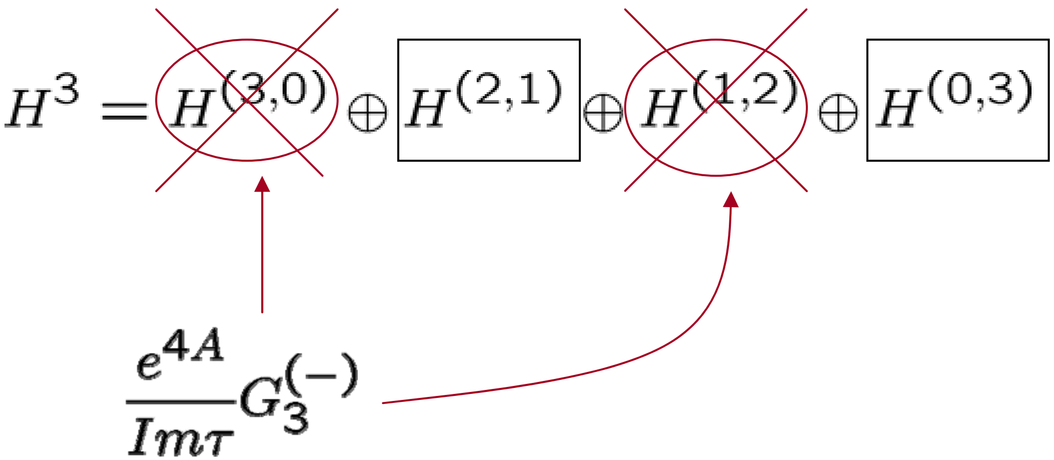
$$0 = \delta \int_{CY3} \frac{ie^{4A}}{Im\tau} \left[G_3 \wedge G_3^{(-)} \right]$$

$$0 = d\left[\frac{e^{4A}}{Im\tau} G_3^{(-)} \right] = d\left[\frac{e^{4A}}{Im\tau} * G_3^{(-)} \right]$$

Therefore,

$\frac{e^{4A}}{Im\tau} G_3^{(-)}$ is a harmonic representative of the cohomology $H^3(CY3)$

And the failure of minimizing the energy functional can be encoded in the finite number of Calabi-Yau moduli.



$[G_3]$ must belong to $H^{(2,1)} \oplus H^{(0,3)}$

$$[dim H^{p,q}(CY3)] = \begin{bmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{bmatrix}$$

IIB Flux compactifications are most widely used for application to real world so far, including

- Explicit solutions (Klebanov-Strassler)
- General form of low energy 4D theory with Gukov-Vafa-Witten superpotential + Nonperturbative corrections
- Complete fixing of moduli (…………)
- Construction of de Sitter vacua (KKLT)
- SUSY breaking phenomenology (Anomaly-Gravity Mediation Mixed)
- Inflationary cosmology (KKLMMT, DBI, D3-D7,….)
- Hierarchy generating geometry (Stringy realization of Randall-Sundrum I)
- Existence of Landscape (Stable and Semistable Vacua)

Some of which will be covered by Oliver’s lecture next week.

Example 2: Flux Compactification of M Theory to 2+1 Dimensions

K.Becker+M.Becker 1996

$$G_{10+1} = H^{-2/3}(y)\eta_{2+1} + H^{1/3}(y)g_8(y)$$

Calabi-Yau 4-fold

$$G_4 = dC_3 = -d\left(\frac{1}{H}\right) vol_{2+1} + G_4^{int}(y)$$

Primitive (2,2) Flux

$$d * G_4 = G_4 \wedge G_4 - l_p^6 X_8(R) + M2 \text{ sources}$$

$$X_8 = \frac{1}{192(2\pi)^4}\left(trR^4 - \frac{1}{4}(trR^2)^2\right)$$

$$vol_8 \wedge \nabla_g^2 H = G^{int} \wedge G^{int} - l_p^6 X_8(R_8) + \delta_{M2}$$

$$\downarrow$$

$$\int_{M_8} G^{int} \wedge G^{int} + N_{M2} = l_p^6 \int_{M_8} X_8$$

Tadpole Condition

The energy functional of the compactification is similar to that of IIB on CY3

$$\mathcal{E} \;\sim\; \int dy^8 \sqrt{g_4(CY4)}\,|G_g|^2 = \int_{CY4} G_4 \wedge *\bar{G}_4$$

$$\int_{CY4} G_4 \wedge *\bar{G}_4 = -i \int_{CY4} G_4 \wedge (\bar{G}_4^{(+)} - \bar{G}_4^{(-)})$$

$$= -i \int_{CY4} G_4 \wedge \bar{G}_4 + 2i \int_{CY4} G_4^{(-)} \wedge \bar{G}_4^{(-)}$$

Disallowed part of G_4 is then $H^{(3,1)} \oplus H^{(1,3)} \oplus [J \wedge H^{(1,1)}]$

$$
[dim H^{p,q}(CY4)] = \left[\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & \xcancel{h^{1,1}} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
\boxed{1} & & \xcancel{h^{3,1}} & & \boxed{h^{2,2}} & & \xcancel{h^{3,1}} & & \boxed{1} \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}\right]
$$

Supersymmetry is more restrictive and allows only (2,2) “primitive” part.

Supersymmetry is actually more restrictive:

(2+1 Minkowski Assumed)

(2,2) : $G^{int} = F_{AB\bar{C}\bar{D}} dy^A dy^B dy^{\bar{C}} dy^{\bar{D}}$ $\longleftarrow$ $D_\alpha W_{complex} = 0$

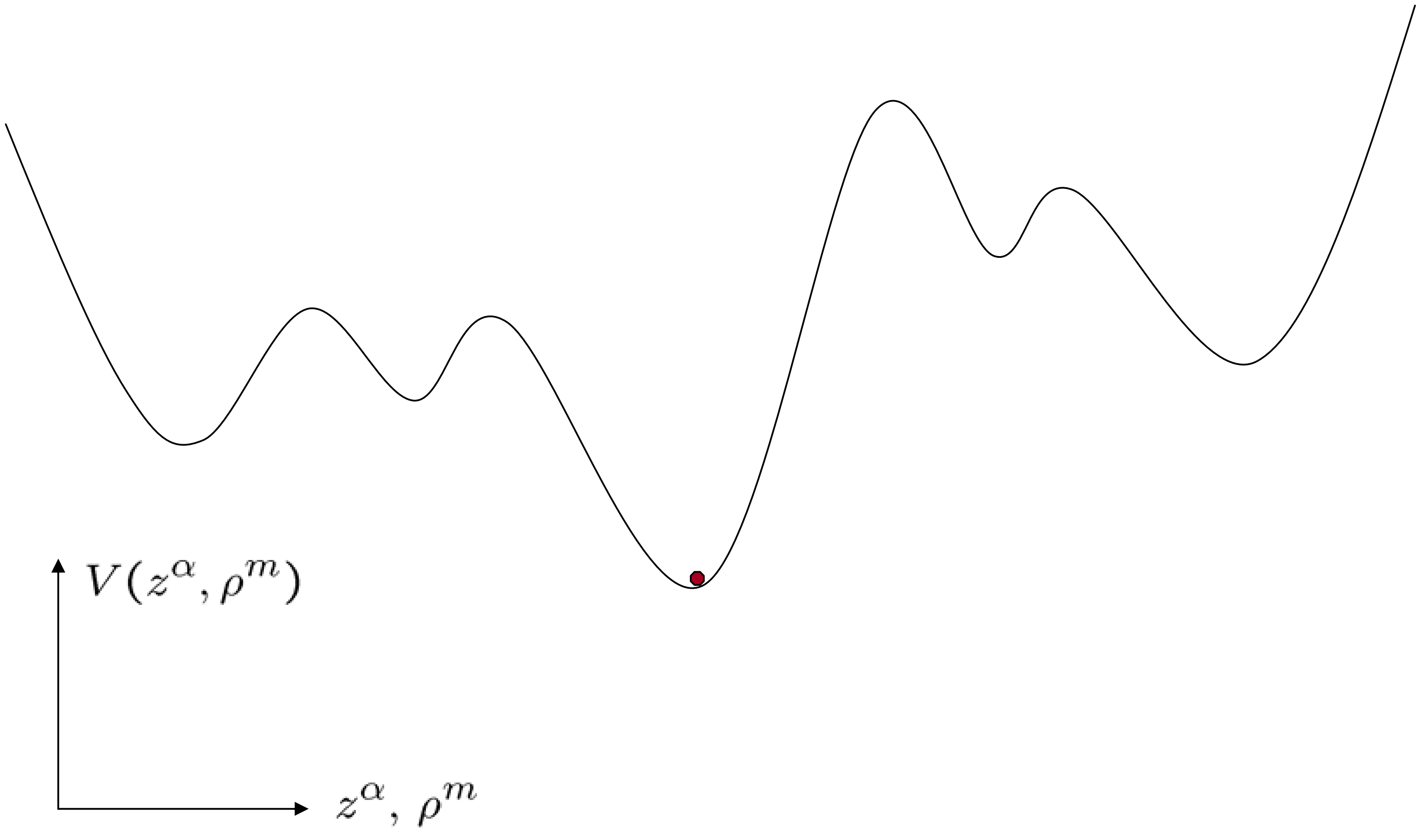
Primitivity : $0 = F_{AB\bar{C}\bar{D}} J^{A\bar{C}}$ $\longleftarrow$ $D_m W_{Kaehler} = 0$

$$W_{complex}(z^\alpha) = \int_{M_8} G^{int} \wedge \Omega^{(4,0)}(z^\alpha)$$

(Haack+Louis)

$$W_{Kaehler}(\rho^m) = \int_{M_8} G^{int} \wedge J(\rho^m) \wedge J(\rho^m)$$

Flux chooses a Calabi-Yau, and gravity backreacts and dresses the Calabi-Yau



Addendum 1:

Complete classification of the holonomy group is known. Apart from those shown, i.e., generic SO(d), Kaehler U(d/2)=U(1)xSU(d/2), Calabi-Yau SU(d/2), symplectic Sp(d/2), and hyperKaehler Sp(d/4), we also have quarternionic Sp(1)x Sp(d/4).

There are two more special cases. A 7-manifold can have G_2 holonomy, and a 8-manifold can have Spin(7) holonomy. However, the other exceptional groups of F_4, E_6, E_7, E_8 do not appear as a holonomy group associated with Levi-Civita connection (=the connection associated with Christoffel symbols).

Addendum 2:

$D_\mu \; vs. \; \nabla_\mu$ is a matter of notational convention. Both denote the covariant derivative with respect to the same connection. The former notation is used on spinors exclusively. That is, we write $D_\mu \eta$ and $\nabla_\mu J_{ab}$ but the same connection $\omega_{\mu ab}$ is used.

Excursion: Primitivity of Differential Form in Complex Geometry

(from an Appendix of Gukov, Vafa, and Witten)

Outline: Part II

- Geometry and Holomony
- Supersymmetry, Spinors, and Calabi-Yau
- Flux and Backreaction
- Energetics of Heterotic Flux Compactification
- Strominger System and Heterotic Flux as a Torsion
- A Supersymmetric Solution to Heterotic Flux Compactification
- Global Issues: Index Counting, Smoothness, etc

Energetics of Heterotic Flux Compactification

$$\mathcal{L}_{heterotic} = \sqrt{-g}e^{-2\Phi}\left[R - \frac{1}{2}|H|^2 + 4(\nabla\Phi)^2 - \frac{\alpha'}{4}\left(trF^2 - trR^2\right)\right]$$

with $dH = \frac{\alpha'}{4}(trF \wedge F - trR \wedge R)$

we will ignore these
for a while

$$g_{9+1}^{string} = \eta_{ij}^{3+1}dX^i dX^j + g_{\mu\nu}^{(6)}dy^\mu dy^\nu$$

Assume existence of a complex structure J ,
and try to see heuristically what would be the analog of
the compactification energy functional in the Heterotic case.

Cardoso, Curio, Dall'Agatha, Luest, 2003

$$\mathcal{E} \sim \int dy^6 \sqrt{-g} e^{-2\Phi} \left[\cdots - 4(\nabla\Phi)^2 + \frac{1}{2}H^2 + \frac{\alpha'}{4} trF^2 + \cdots \right]$$

Generally one can decompose the square of the gauge field strength

$$trF^2 \sim - * (J \wedge trF \wedge F) + \boxed{|F^{(2,0)}|^2 + c|J \cdot F|^2}$$

So that the last two terms in the energy functional are organized into

$$\int dy^6 e^{-2\Phi} \left[\frac{1}{2} H \wedge *H - \frac{\alpha'}{4} J \wedge trF \wedge F + \text{nonnegative} \right]$$

$$\frac{1}{2}\int e^{-2\Phi}\left[H\wedge *H-\frac{\alpha'}{2}J\wedge trF\wedge F+\text{nonnegative}\right]$$

$$=\frac{1}{2}\int e^{-2\Phi}H\wedge *H-2e^{-2\Phi}J\wedge dH+\text{nonnegative}$$

$$=\frac{1}{2}\int e^{-2\Phi}H\wedge *H-2d(e^{-2\Phi}J)\wedge H+\text{total derivative}+\text{nonnegtive}$$

$$=\frac{1}{2}\int e^{-2\Phi}\left(H+e^{2\Phi}*d(e^{-2\Phi}J)\right)\wedge *\left(H+e^{2\Phi}*d(e^{-2\Phi}J)\right)+\cdots$$

Assuming that the rest of the terms also combine into total derivatives and complete squares, we find that minimization of energy functional requires,

$$H=-e^{2\Phi}*d\left(e^{-2\Phi}J\right)$$

$$F=F^{(1,1)},\qquad J\cdot F=0$$

Terms neglected above are those involving dilaton and the metric only,
And these can also be made into complete squares. Vanishing of these sqaured terms demand that

$$0 = N(J)^{\alpha}_{\mu\nu} \equiv J_{\mu}{}^{\beta}\partial_{[\beta}J_{\nu]}{}^{\alpha} - J_{\nu}{}^{\beta}\partial_{[\beta}J_{\mu]}{}^{\alpha}$$

= Nijenhuis tensor

justifying the computation above after assuming a complex structure, and also

$$0 = d\left(e^{-2\Phi} \wedge J \wedge J\right) \longleftrightarrow 0 = \nabla^{\alpha}\left(e^{-2\Phi} J_{\alpha}{}^{\beta}\right)$$

which is also known as the conformal balancing condition.

Inclusion of R^2 term would have given constraints

$$R = R^{(1,1)}, \qquad J \cdot R = 0$$

which would have implies again a Ricci flat metric, except that these equations are not trustworthy because it came from a higher order terms. We will see later how this is precisely fixed by supersymmetry.

$$0 = \nabla^{\alpha}\left(e^{-2\Phi} J_{\alpha\beta}\right)$$

Gradient of Dilaton ~ Derivative of J

$$H = -e^{2\Phi} * d\left(e^{-2\Phi} J\right)$$

Flux ~ Gradient of Dilaton + Derivative of J

$$H \neq 0 \quad \longleftrightarrow \quad dJ \neq 0$$

Heterotic flux compactification gives complex 6-manifolds which are NEVER Kaehler !!!!!!

Strominger System and Heterotic Flux as a Torsion

We will start with the nontrivial observation by Strominger that in string frame, the warp factor is absent when SUSY is required.

$$g_{9+1}^{string} = \eta_{ij}^{3+1} dX^i dX^j + \boxed{g_{\mu\nu}^{(6)}} dy^\mu dy^\nu$$

The backreaction of the geometry to the flux will be all encoded in the geometry of the six-dimensional internal manifold which is no longer Calabi-Yau

Supersymmetric Compactification of Heterotic Superstring without H

$$0 = \delta_\epsilon^{SUSY} \Psi_i = \partial_i \epsilon_{9+1} = \partial_i \epsilon_{9+1}$$

$$0 = \delta_\epsilon^{SUSY} \Psi_\mu = D_\mu \epsilon_{9+1}$$

$$D_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu ab} \gamma^{ab}$$

$$0 = \delta_\epsilon^{SUSY} \lambda = 0 \cdot \epsilon_{9+1}$$

Supersymmetric Compactification of Heterotic Superstring with H

$$0 = \delta_\epsilon^{SUSY}\Psi_i = \partial_i \epsilon_{9+1} = \partial_i \epsilon_{9+1}$$

$$0 = \delta_\epsilon^{SUSY}\Psi_\mu = \left(D_\mu + \boxed{AH_{\mu\alpha\beta}\gamma^{\alpha\beta} + B\gamma_\mu H_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma} + E\partial_\mu\Phi + F\partial^\alpha\Phi\gamma_{\alpha\mu}}\right)\epsilon_{9+1}$$

$$0 = \delta_\epsilon^{SUSY}\lambda = \boxed{\left(G\gamma^\alpha\partial_\alpha\Phi + CH_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\right)}\epsilon_{9+1}$$

A, B, C, E, F, G definite constants

simplificiation is possible by making use of the gravitino variation and dilatino variations together.

Supersymmetric Compactification with H

$$0 = \delta_\epsilon^{SUSY}\Psi_\mu = \left(D_\mu + AH_{\mu\alpha\beta}\gamma^{\alpha\beta} + B\gamma_\mu H_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma} + E\partial_\mu\Phi + F\partial^\alpha\Phi\gamma_{\alpha\mu}\right)\epsilon_{9+1}$$

$$0 = \delta_\epsilon^{SUSY}\lambda = \left(G\gamma^\alpha\partial_\alpha\Phi + CH_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\right)\epsilon_{9+1}$$

$$0 = \delta_\epsilon^{SUSY}\left(C\Psi_\mu - B\gamma_\mu\lambda\right)$$

$$0 = \left(D_\mu + AH_{\mu\alpha\beta}\gamma^{\alpha\beta} + E\partial_\mu\Phi + F'\partial^\alpha\Phi\gamma_{\alpha\mu}\right)\epsilon_{9+1}$$

Rescale ϵ_{9+1} by $e^{-E\Phi}$

$$0 = \left(D_\mu + AH_{\mu\alpha\beta}\gamma^{\alpha\beta} + F'\partial^\alpha\Phi\gamma_{\alpha\mu}\right)\tilde{\epsilon}_{9+1}$$

Supersymmetric Compactification with H

$$0 = \left(D_\mu + A H_{\mu\alpha\beta}\gamma^{\alpha\beta} + \boxed{F' \partial^\alpha \Phi \gamma_{\alpha\mu}} \right) \tilde{\epsilon}_{9+1}$$

This piece can be removed by a conformal transformation of the metric by an exponentiated dilaton factor.

However, the nontrivial fact here is that this piece disappears already in the string frame.

$$A = 1/8$$

$$0 = \left(D_\mu + \frac{1}{8} H_{\mu\alpha\beta}\gamma^{\alpha\beta} \right) \tilde{\epsilon}_{9+1}$$

Supersymmetric Compactification with H

$$0 = \left(D_\mu + \frac{1}{8} H_{\mu\alpha\beta} \gamma^{\alpha\beta} \right) \epsilon_{9+1}$$

$$\epsilon_{9+1} = \epsilon_{3+1} \otimes \eta + c.c.$$

$$D^{(+)}_\mu \equiv D_\mu + \frac{1}{8} H_{\mu ab} \gamma^{ab} = \partial_\mu + \frac{1}{4} \boxed{\left(\omega_{\mu ab} + \frac{1}{2} H_{\mu ab} \right)} \gamma^{ab}$$

the connection is twisted by a torsion

$$0 = D^{(+)}_\mu \eta$$

$$\omega_{\mu ab} + \omega_{\mu ba} = 0$$

$$\omega^{(+)}_{\mu ab} + \omega^{(+)}_{\mu ba} = 0$$

$$de^a + \omega^a{}_b \wedge e^b = 0$$

$$de^a + \omega^a_{(+)b} \wedge e^b = \frac{1}{2} H_\mu{}^a{}_b e^b_\nu dy^\mu \wedge dy^\nu \neq 0$$

$$d\omega^a{}_b + \omega^a{}_f \wedge \omega^f{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu$$

$$d\omega^a_{(+)b} + \omega^a_{(+)f} \wedge \omega^f_{(+)b} = \frac{1}{2} R^a_{(+)b\mu\nu} dx^\mu \wedge dx^\nu$$

Despite this twist, the metric is still covariantly constant:

$$\omega \to \omega + H/2$$

$$\downarrow$$

$$\Gamma^{\alpha}{}_{\mu\nu} \to \Gamma^{\alpha}{}_{\mu\nu} - \frac{1}{2} H^{\alpha}{}_{\mu\nu}$$

$$\downarrow$$

$$\begin{aligned} \nabla_\mu g^{\alpha\beta} = \nabla^{(+)}_\mu g_{\alpha\beta} &= \nabla_\mu g_{\alpha\beta} + \frac{1}{2} H^{\gamma}{}_{\mu\alpha} g_{\gamma\beta} + \frac{1}{2} H^{\gamma}{}_{\mu\beta} g_{\alpha\gamma} \\ &= \nabla_\mu g_{\alpha\beta} + \frac{1}{2}\left(H_{\beta\mu\alpha} + H_{\alpha\mu\beta} \right) = 0 + 0 \end{aligned}$$

=0 by antisymmetric nature of H

In particular, this allows raising and lowering of indices by metric

$$0 = \delta_\epsilon^{SUSY} \Psi_\mu \Rightarrow 0 = D_\mu^{(+)} \eta$$

$$\epsilon_{9+1} = \epsilon_{3+1} \otimes \eta + c.c.$$

$$\begin{aligned} D_\mu^{(+)} &= \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab} + \frac{1}{8}H_{\mu ab}\gamma^{ab} \\ &= \partial_\mu + \frac{1}{4}\omega_{\mu ab}^{(+)}\gamma^{ab} \end{aligned}$$

$$\partial_\mu \gamma_a = 0$$

$D_\mu \; vs. \; \nabla_\mu$ is a matter of notational convention. Both denote the covariant derivative with respect to the same connection. The former acts on spinors and the latter acts on tensors.

$$0 = \nabla_\mu^{(+)} \left[\eta^\dagger \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_n} \eta \right]$$

$$
\begin{aligned}
\nabla_\mu \left(\eta^\dagger \gamma ... \eta\right) &= \partial_\mu^{(+)} \left(\eta^\dagger \gamma ... \eta\right) + \eta^\dagger \left[\gamma ..., \frac{1}{4}\omega_{\mu cd}^{(+)} \gamma^{cd}\right] \eta \\
&= (D_\mu^{(+)} \eta)^\dagger \gamma ... \eta + \eta^\dagger \gamma ... D_\mu^{(+)} \eta \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\left[\gamma_a, \frac{1}{4}\omega_{\mu cd}^{(+)} \gamma^{cd}\right] \\
&= \frac{1}{4}\omega_{\mu cd}^{(+)} \left(\gamma_a \gamma^c \gamma^d - \gamma^c \gamma^d \gamma_a\right) \\
&= \frac{1}{4}\omega_{\mu cd}^{(+)} \left(2\delta_a^c \gamma^d - 2\delta_a^d \gamma_c\right) \\
&= \omega_{\mu ad}^{(+)} \gamma^d
\end{aligned}
$$

Parallel Spinors

$$D^{(+)}\eta = 0$$

$$J_{\mu\nu} = i\eta^\dagger\gamma_{\mu\nu}\eta$$

$$(\eta^\dagger\eta = 1)$$

Parallel Tensors

$$\nabla^{(+)}J = 0$$

Reduced Holonomy U(d/2):

~~Kaehler Manifold~~

Parallel Spinors

$$D^{(+)}\eta = 0$$

$$J_{\mu\nu} = i\eta^{\dagger}\gamma_{\mu\nu}\eta$$

$$(\eta^{\dagger}\eta = 1)$$

Parallel Tensors

$$\nabla^{(+)}J = 0$$

Reduced Holonomy U(d/2):

~~Kaehler Manifold~~

$$J_{\mu\alpha}g^{\alpha\beta}J_{\nu\beta} = g_{\mu\nu} \quad \equiv \quad J^{\mu}{}_{\alpha}J^{\alpha}{}_{\nu} = -\delta^{\mu}_{\nu}$$

from Fierz Indentity and $\eta^{\dagger}\eta = 1$

Is J an integrable complex strujcture ?
No longer an automatic consequence of covariantly constant J !

$$0 \overset{?}{=} (N_J)^{\alpha}_{\mu\nu} \;=\; J_{\mu}{}^{\beta}\nabla_{[\beta}J_{\nu]}{}^{\alpha} - J_{\nu}{}^{\beta}\nabla_{[\beta}J_{\mu]}{}^{\alpha}$$

$$0 = \nabla^{(+)}_{\beta}J_{\nu}{}^{\alpha} = \nabla_{\beta}J_{\nu}{}^{\alpha} + \frac{1}{2}H_{\beta}{}^{\alpha}{}_{\delta}J_{\nu}{}^{\delta} + \frac{1}{2}H_{\beta}{}^{\delta}{}_{\nu}J_{\delta}{}^{\alpha}$$

$$\downarrow$$

$$2(N_J)_{\alpha\beta\gamma} = -H_{\alpha\beta\gamma} + 3J_{[\alpha}{}^{\mu}J_{\beta}{}^{\nu}H_{\gamma]\mu\nu}$$

$N_J = 0$ is demonstrated by making use of dilatino variation, which we will refer to Strominger’s original work.

Parallel Spinors

$$D^{(+)}\eta = 0$$

→

$$J_{\mu\nu} = i\eta^\dagger\gamma_{\mu\nu}\eta$$

$$(\eta^\dagger\eta = 1)$$

Parallel Tensors

$$\nabla^{(+)}J = 0$$

↓

Reduced Holonomy U(d/2):

~~Kaehler Manifold~~

↓

Reduced Holonomy SU(d/2):

~~Calabi-Yau~~

↓

$$0 = \eta^\dagger \left[D_\mu^{(+)}, D_\nu^{(+)}\right] \eta = -i\frac{1}{4}R_{ab\mu\nu}^{(+)}J^{ab}$$

$$
\begin{aligned}
0 = R^{(+)}_{abcd} J^{ab} &= 3R^{(+)}_{a[bcd]} J^{ab} + R^{(+)}_{acbd} J^{ab} + R^{(+)}_{adcb} J^{ab} \\
&= 3\boxed{R^{(+)}_{a[bcd]}} J^{ab} + \boxed{2R^{(+)}_{acbd} J^{cd}}
\end{aligned}
$$

$$
\neq 0 \qquad \propto J_a{}^f (Ricci)^{(+)}_{fb}
$$

Calabi-Yau $\neq$ SU(d/2) Holonomy with Torsion $\neq$ Ricci Flat Kaehler

SU(3) Holonomy $R^{(+)}_{abcd} J^{cd} = 0$

Ricci Scalar with Torsion

$$0 = \frac{1}{2} R^{(+)}_{abcd} J^{ab} J^{cd} = \frac{3}{2} R^{(+)}_{a[bcd]} J^{ab} J^{cd} + R^{(+)}$$

$$R^{ab}_{(+)cd} = R^{ab}{}_{cd} - \frac{1}{2} \nabla_{[c} H^{ab}{}_{d]} + \frac{1}{4} H^{a}_{f[c} H^{fb}{}_{d]}$$

$$H_{abc} J^{bc} = J^{b}_{a} \nabla_{b} \Phi$$

$$0 = R + \frac{1}{2} H^2 + 6 \nabla^2 \Phi - 8 (\nabla \Phi)^2$$

A traced form of a gravity equation, showing how the geometry reacts to energy-momentum tensor due to fluxes.

Likewise

$$\Omega^{(d/2,0)} = \eta^T \gamma_{k_1} \gamma_{k_2} \cdots \gamma_{k_{d/2}} \eta$$

$$\nabla^{(+)} \Omega^{(d/2,0)} = 0$$

A General Solution for H:

A General Solution for H: Need to invert the condition

$$0 = \nabla_\alpha^{(+)} J_{\beta\gamma} = \nabla_\alpha J_{\beta\gamma} + \frac{1}{2} H_{\lambda\alpha\beta} J^\lambda{}_\gamma + \frac{1}{2} H_{\lambda\alpha\gamma} J_\beta{}^\lambda$$

A General Solution for H: Need to invert the condition

$$0 = \nabla^{(+)}_\alpha J_{\beta\gamma} = \nabla_\alpha J_{\beta\gamma} + \frac{1}{2} H_{\lambda\alpha\beta} J^\lambda{}_\gamma + \frac{1}{2} H_{\lambda\alpha\gamma} J_\beta{}^\lambda$$

$$J = \frac{1}{2} J_{m\bar{k}}\, dz^m \wedge d\bar{z}^{\bar{k}}$$

Hermiticity + Integrability

$$0 = \nabla_k J_{m\bar{n}} + \frac{1}{2} H_{\lambda k m} J^\lambda{}_{\bar{n}} + \frac{1}{2} H_{\lambda k \bar{n}} J_m{}^\lambda$$

A General Solution for H: Need to invert the condition

$$0 = \nabla^{(+)}_{\alpha} J_{\beta\gamma} = \nabla_{\alpha} J_{\beta\gamma} + \frac{1}{2} H_{\lambda\alpha\beta} J^{\lambda}{}_{\gamma} + \frac{1}{2} H_{\lambda\alpha\gamma} J_{\beta}{}^{\lambda}$$

$$J = \frac{1}{2} J_{m\bar{k}}\, dz^m \wedge d\bar{z}^{\bar{k}}$$

$$0 = \nabla_k J_{m\bar{n}} + \frac{1}{2} H_{\lambda k m} J^{\lambda}{}_{\bar{n}} + \frac{1}{2} H_{\lambda k \bar{n}} J_m{}^{\lambda}$$

antisymmetrize indices to decouple Γ

$$0 = \partial_{[k} J_{m]\bar{n}} - i\frac{1}{2} H_{\bar{n}[km]} + i\frac{1}{2} H_{[mk]\bar{n}} = \partial_{[k} J_{m]\bar{n}} - iH_{[km]\bar{n}}$$

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antisymmetrize indices to decouple Γ

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$H = i(\bar{\partial} - \partial) J$ or equivalently $H_{\alpha\beta\gamma} = -3 J_\alpha{}^\mu J_\beta{}^\nu J_\gamma{}^\lambda \nabla_{[\mu} J_{\nu\lambda]}$

How to relate the dilaton to H ?

$$0 = \delta_\epsilon^{SUSY} \lambda = \left(G \gamma^\alpha \partial_\alpha \Phi + C H_{\alpha\beta\gamma} \gamma^{\alpha\beta\gamma} \right) \epsilon_{9+1}$$

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$$\epsilon_{9+1} = \epsilon_{3+1}\otimes\eta + c.c.$$

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constract with $\eta^\dagger\gamma_\mu$ and take the imaginary part

$$0 = \eta^\dagger\left([\gamma_\mu,\gamma^\alpha]\partial_\alpha\Phi + C'H_{\alpha\beta\gamma}\{\gamma_\mu,\gamma^{\alpha\beta\gamma}\}\right)\eta$$

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$$= 2\delta_\mu^{[\alpha}\gamma^{\beta\gamma]} - 2\delta_\mu^{[\beta}\gamma^{\alpha\gamma]} + 2\delta_\mu^{[\gamma}\gamma^{\alpha\beta]}$$

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↓

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$$\nabla_\alpha\Phi = \frac{1}{4}J_{\mu\nu}J_{\lambda\alpha}H^{\mu\nu\lambda}$$

Summary of the Strominger System for Heterotic Flux Compactification:

$$\mathcal{L}_{heterotic} = \sqrt{-G}e^{-2\Phi}\left[R - \frac{1}{2}H^2 + 4(\nabla\Phi)^2 - \frac{\alpha'}{4}\left(trF^2 - trR^2_{(-)}\right)\right]$$

$$dH = \frac{\alpha'}{4}(-trR_{(-)} \wedge R_{(-)} + trF \wedge F) - \delta_{fivebranes}$$

$$G_{9+1} = \eta_{3+1} + g_6(y)$$ no warp factor in string frame !

↑

SU(3) holonomy with torsion

$$\nabla^{(+)}J^{(1,1)} = 0$$

$$\nabla^{(+)}\Omega^{(3,0)} = 0$$

Integrable complex structure

$$0 = N(X,Y) \equiv Re\{(1+iJ)[(1-iJ)X,(1-iJ)Y]\}$$

Hermitian metric

$$g(X,Y) = g(JX,JY)$$

Conformal balancing with Dilaton

$$0 = d\left(e^{-2\Phi} \wedge J \wedge J\right)$$

General form of the solution:

$$\frac{1}{2}H_{\alpha\beta\gamma} = -\frac{3}{2}J_{\alpha}{}^{\mu}J_{\beta}{}^{\nu}J_{\gamma}{}^{\lambda}\nabla_{[\mu}J_{\nu\lambda]}$$ = Bismut torsion

$$\nabla_{\alpha}\Phi = \frac{1}{4}J_{\mu\nu}J_{\lambda\alpha}H^{\mu\nu\lambda} = \frac{3}{4}J^{\mu\nu}\nabla_{[\alpha}J_{\mu\nu]}$$

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Combining the two reproduce the energy minimizing condition:

$H = -e^{2\Phi} * d\left(e^{-2\Phi}J\right)$ solving E.O.M. $d * \left(e^{-2\Phi}H\right) = 0$

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Finally the Bianchi indentity must be solved in favor of J

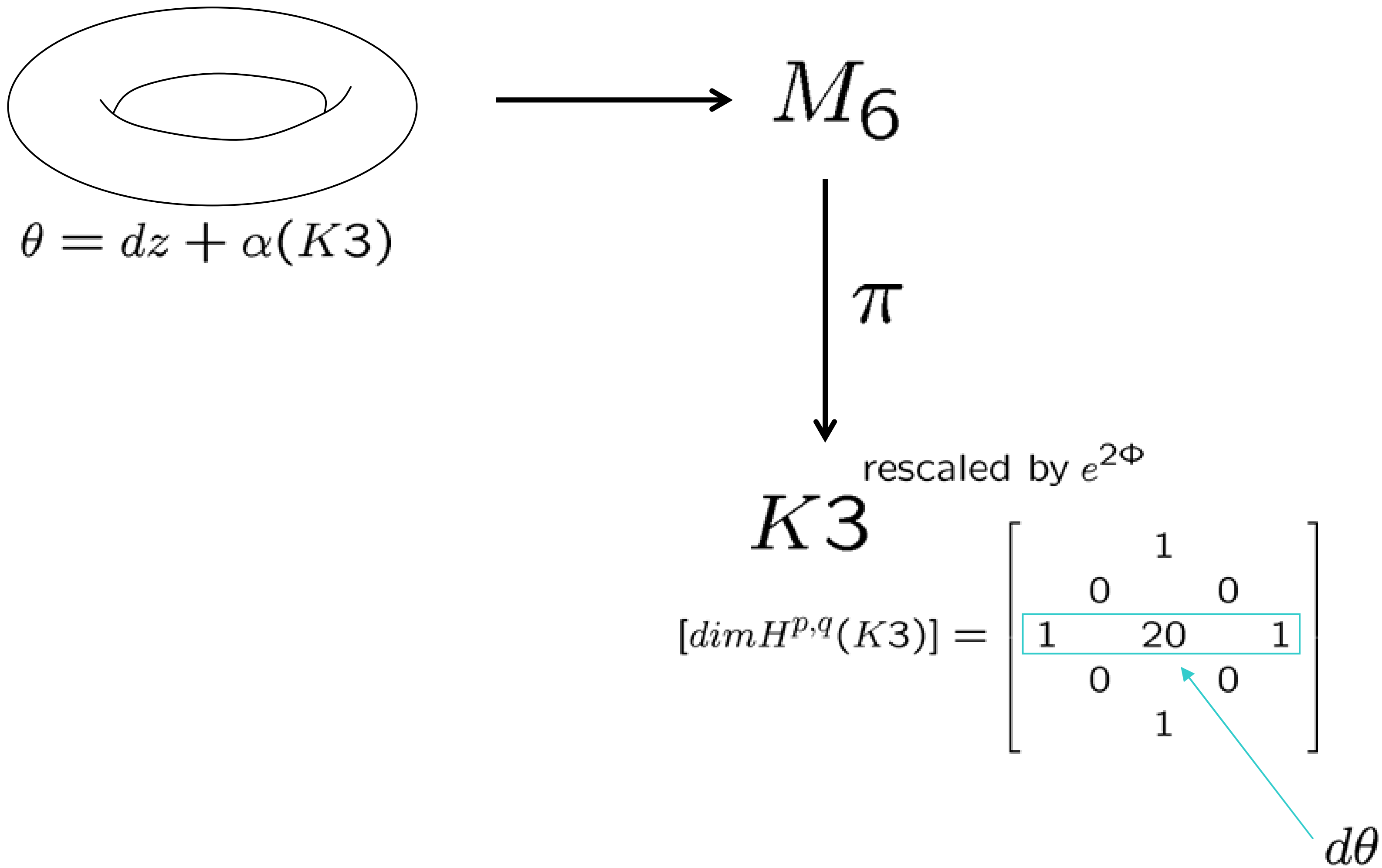
$$dH = \frac{\alpha'}{4}\left(-trR_{(-)}\wedge R_{(-)} + trF\wedge F\right) + \cdots$$

Solutions are difficult to find.

So far, only one family of reasonably explicit and smooth solutions is found, in 2006.

This family of solutions has been expected based on U-duality to F-theory for several years.

A smooth and supersymmetric solution to heterotic flux compactification



$$J = e^{2\Phi} J_{K3} + \frac{i}{2}\theta \wedge \bar{\theta}$$

$$g_6 = e^{2\Phi} g_{K3} + |\theta|^2$$

base fibre

well-defined holomorphic bundle

$$\theta = 2\pi\sqrt{\alpha'}\,(dz + \beta)$$

$$d\beta \in H^2(K3, Z)$$

conformally balanced metric

$$\Phi = \Phi_{K3}$$

$$d\theta \wedge J_{K3} = 0$$

$$H \sim -e^{2\Phi} * d\left(e^{-2\Phi}\theta \wedge \bar{\theta}\right)$$

$$\sim i(\partial - \bar{\partial})J$$

Hermitian YM bundle

$$F \wedge J_{K3} = 0$$

$$F \wedge \Omega_{K3} = 0$$

$$J = e^{2\Phi} J_{K3} + \frac{i}{2}\theta \wedge \bar{\theta}$$

$$g_6 = e^{2\Phi} g_{K3} + |\theta|^2$$

base fibre

$$H \sim -e^{2\Phi} * d\left(e^{-2\Phi}\theta \wedge \bar{\theta}\right)$$

$$\sim i(\bar{\partial} - \partial)J$$

The Bianchi Identity

$$dH = 2i\partial\bar{\partial}\left(e^{2\Phi}\right) J_{K3} + \cdots = \frac{\alpha'}{4}(-trR \wedge R + trF \wedge F)$$

is now a nonlinear partial differential equation for dilaton

$$dH = \frac{\alpha'}{4}(-trR \wedge R + trF \wedge F)$$

$$\int_{M_6} J \wedge dH = \int_{M_6} J \wedge \left[\frac{\alpha'}{4}(-trR \wedge R + trF \wedge F)\right]$$

$$= \int_{M_6} \frac{i}{2}\theta \wedge \bar{\theta} \wedge \left[\frac{\alpha'}{4}(-trR \wedge R + trF \wedge F)\right]$$

$$= vol_{T^2}\frac{\alpha'}{4}\int_{K3}(-trR \wedge R + trF \wedge F)$$

$$\int_{K3} |d\beta|^2 = \left[-\frac{1}{2}p_1(K3)\right] - \left[-\frac{1}{2}p_1(Gauge)\right]$$

=24 nonnegative by SUSY

Fu and Yau states that the Bianchi identity

$$dH = \frac{\alpha'}{4}(-trR \wedge R + trF \wedge F)$$

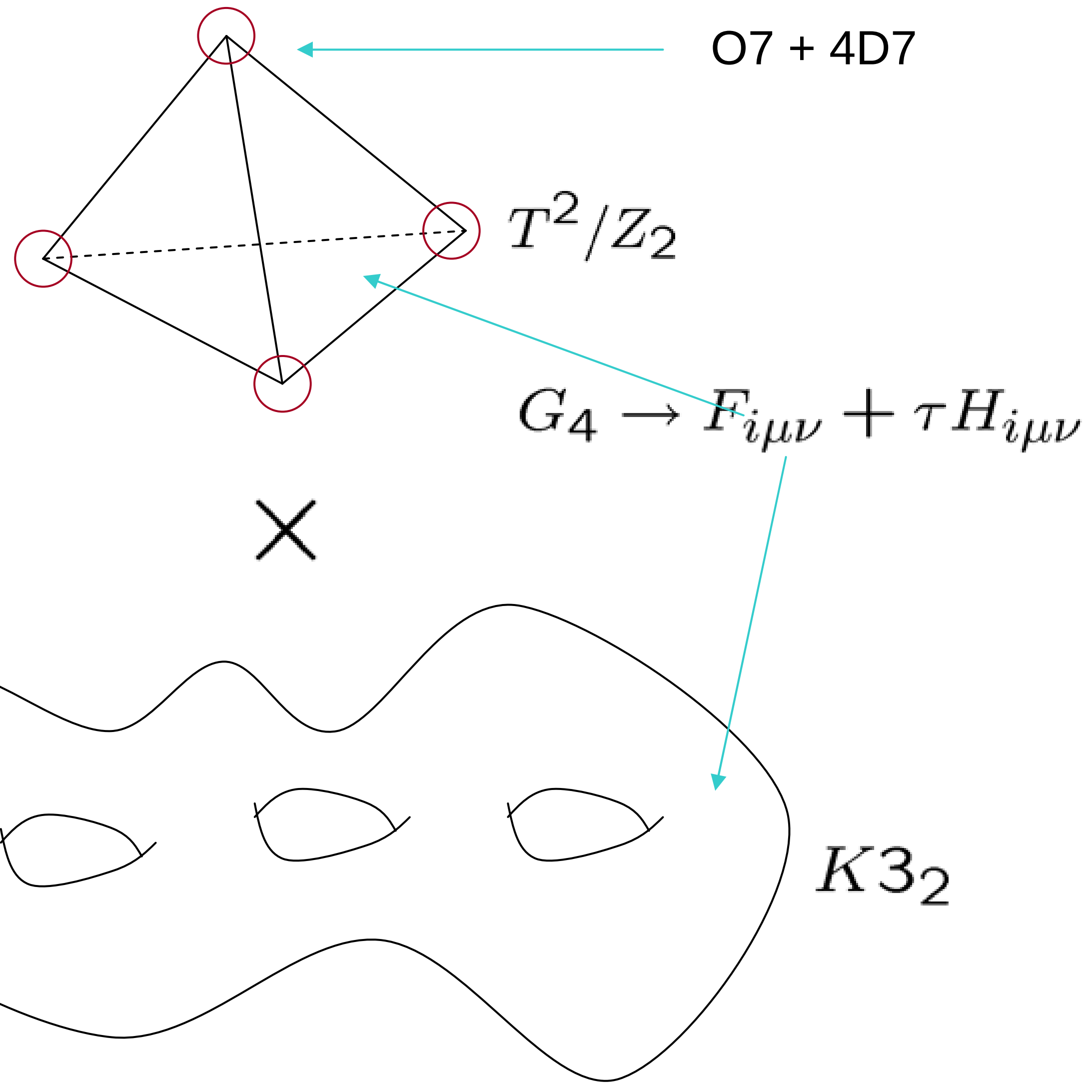
can be solved for a smooth dilaton, when the tadpole condition

$$\int_{K3} |d\beta|^2 = \left[-\frac{1}{2}p_1(K3)\right] - \left[-\frac{1}{2}p_1(Gauge)\right]$$

is satisfied, provided that the base K3 is sufficiently large.

Recall IIB/F Theory on

$$R^{3+1} \times$$



U-Dual Story: Heterotic ← Type I ←F/IIB

IIB on $T^2/Z_2 \times K3$ with H_3 and F_3 Fluxes

= Orientifold Limit of F-Theory with Flux on $K3 \times K3$

T-duality on T^2

Type I on T^2 fibered over $K3$ with F_3 Fluxes

S-duality

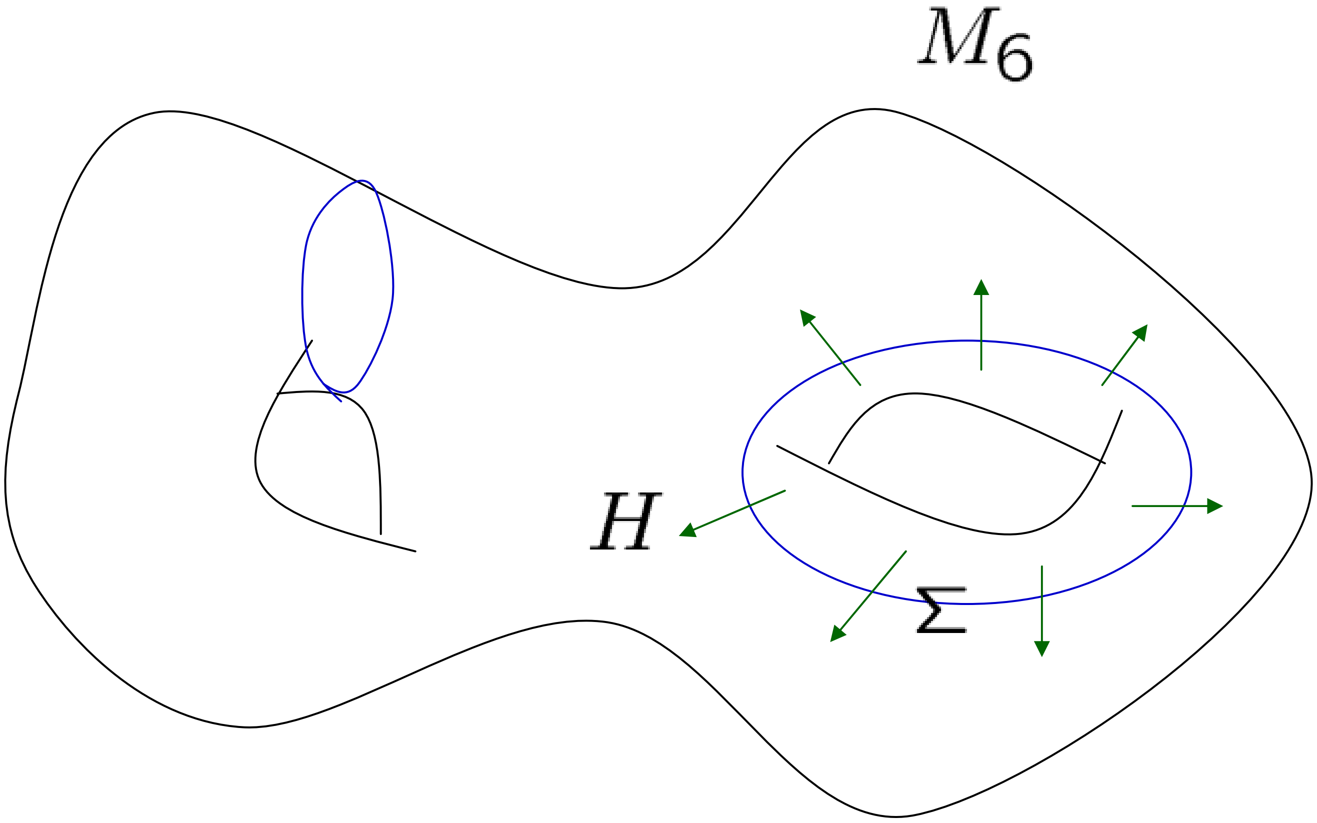
Heterotic on T^2 fibered over $K3$ with H Fluxes

Duality Dictionary for Branes:

IIB	Type I	Heterotic
O7-D7's on K3	O9-D9's	gauge bundle
mobile D3's	D5 on T^2	fivebranes on T^2
the other D7's	the other D5's	the other fivebranes
D3 instantons	D1 instantons D5 instantons	worldsheet instantons fivebrane instantons

Calibrating fivebranes under Flux

& application to BBFTY Solution



Usual calibration in a Calabi-Yau background without Flux
= volume minimization to a specific topological bound

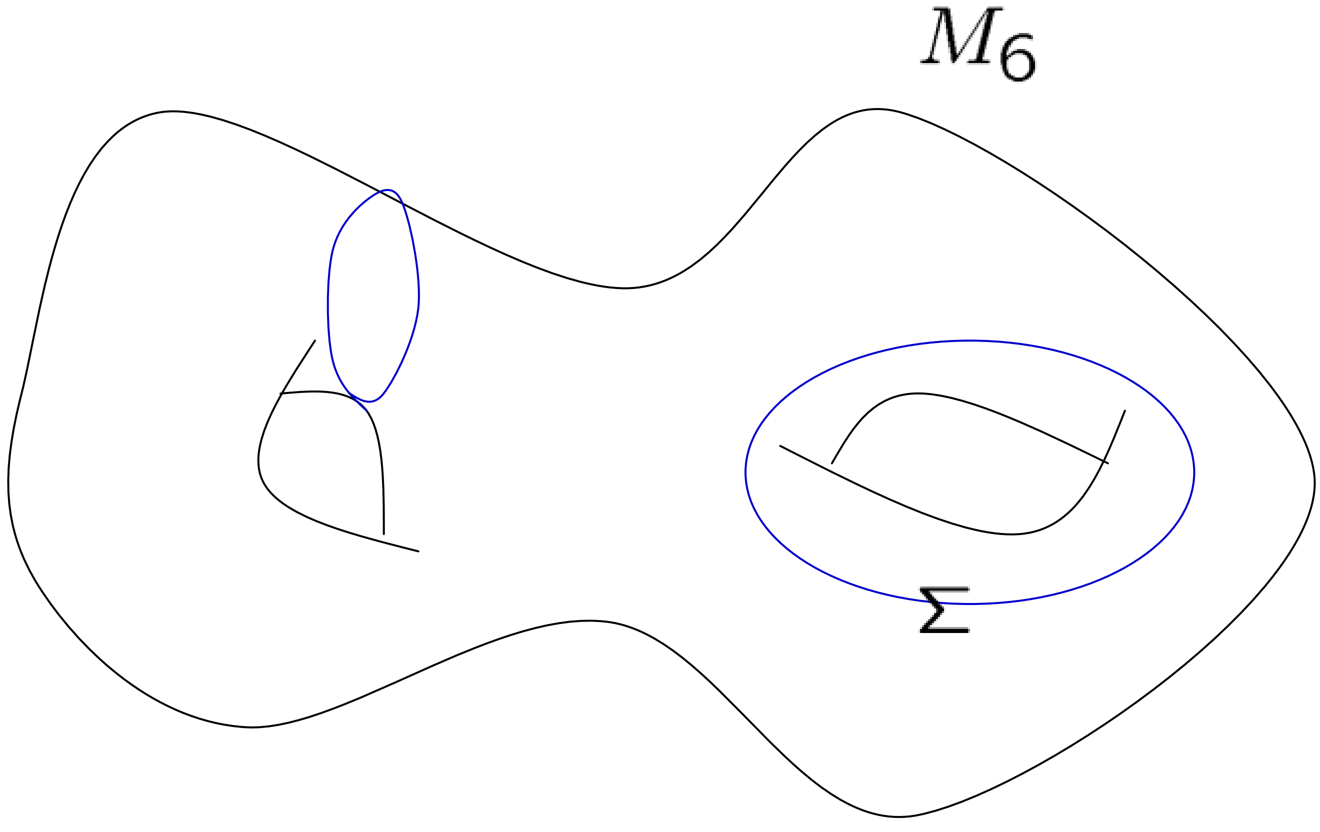
$$0 \leq Q^2_\Sigma = \int_\Sigma vol - \int \Sigma^*(\Omega) \quad \text{with} \quad d\Omega = 0$$

$$\Delta \int \Sigma^*(\Omega) = \int \Sigma_1{}^*(\Omega) - \Sigma_2^*(\Omega) = \int B_{12}{}^*(d\Omega) = 0$$

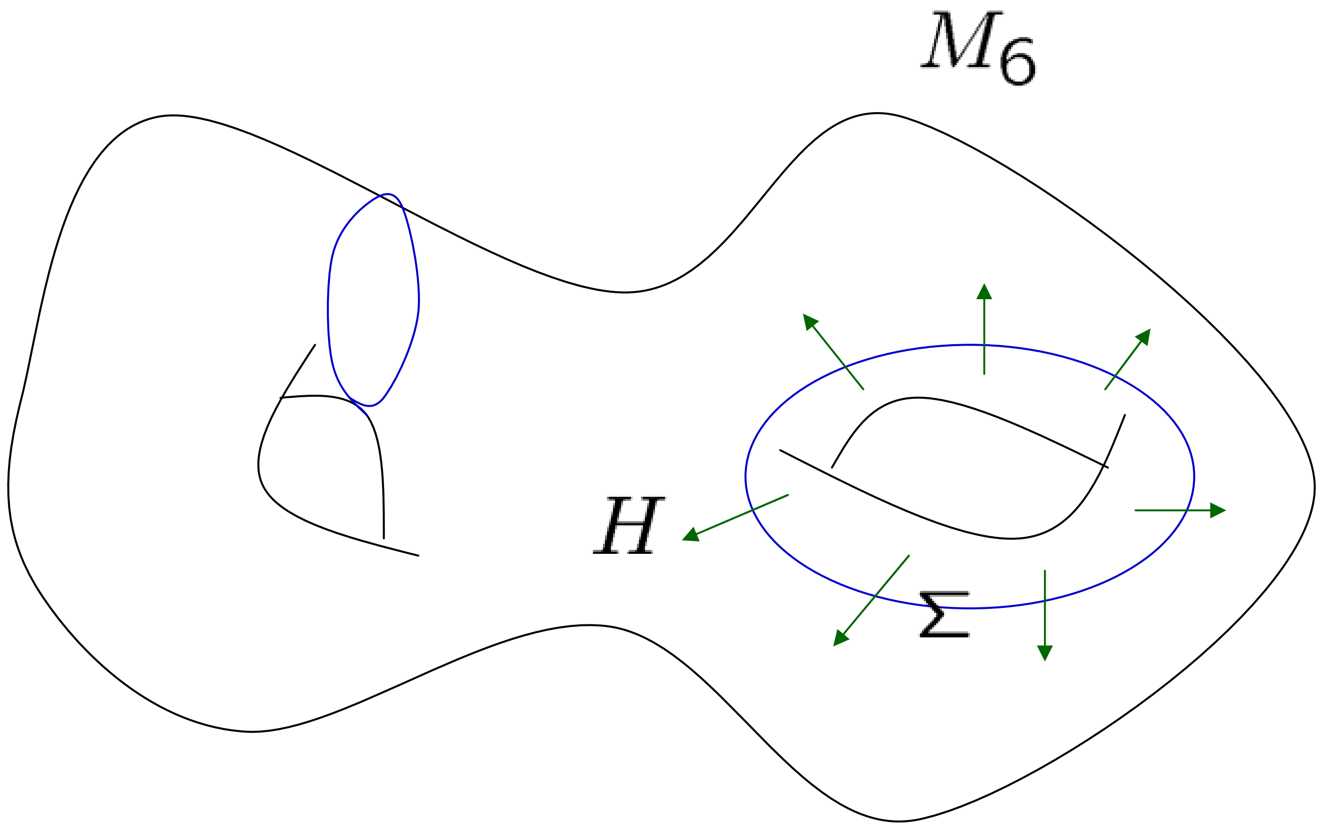
a topological bound & BPS bound

supercharge

energy (per unit Minkowski volume)



M_6
Σ


M_6
H
Σ

$\mathcal{E}_\Sigma =$ energy of wrapped fivebrane $\Sigma \times R^{3+1}$ under Flux ?

$+ \int_\Sigma e^{-2\Phi} vol_2$ the rest mass per unit 3-volume

$- \int \Sigma^* \left(\tilde{B}_2\right)$ the magnetic potential energy per unit 3-volume:

$$0 = d\left(2e^{-2\Phi} *_{9+1} H\right) = dd\tilde{B}_6 = (dd\tilde{B}_2) \wedge vol_{3+1}$$

$$\tilde{B}_2 = e^{-2\Phi} J$$

$$0 \leq \mathcal{E}_\Sigma = \int_\Sigma e^{-2\Phi} vol_2 - \int \Sigma^* \left(e^{-2\Phi} J\right)$$

not closed;
may not be topological

Is $\mathcal{E} = 0$ a supersymmetry condition ? Yes !

It follows from adapting the generalized calibration for M5 branes

Gutowski,Papadopoulos & Townsend, 1999

$$\begin{aligned} G_{10+1} &= e^{4\Phi/3} dx_{11}^2 + e^{-2\Phi/3} G_{9+1} \\ &= e^{4\Phi/3} dx_{11}^2 + e^{-2\Phi/3} g_6 + e^{-2\Phi/3} \eta_{3+1} \end{aligned}$$

$$*_{10+1} dC_3 = d\tilde{C}_6 = d\tilde{B}_6$$

$$E_{M5} = \int_\Sigma vol_2^{G_{10+1}} \wedge vol_{3+1}^{G_{10+1}} - \int \left[\Sigma \times R^{3+1}\right]^* (\tilde{C}_6) = \mathcal{E}_\Sigma^{M5} \wedge vol_{3+1}^\eta$$

$vol_2^{G_{10+1}} \rightarrow e^{-2\Phi/3} vol_\Sigma$

$vol_{3+1}^{G_{10+1}} \rightarrow e^{-4\Phi/3} vol_{3+1}^\eta$

$\left[\Sigma \times R^{3+1}\right]^* (\tilde{C}_6) \rightarrow (\tilde{B}_2 + d\Lambda) \wedge vol_{3+1}$

calibrated by

$$\mathcal{E}_{\Sigma}^{M5} \geq \int \Sigma^*(\Omega \equiv e^{-2\Phi} \boxed{\epsilon^\dagger \gamma \epsilon} - vol_{3+1}^{\eta} \cdot \tilde{C}_6)$$

$d\Omega = 0$ by SUSY

fundamental 2-form J for g_6

$$Q(\epsilon)^2 = \mathcal{E}_{\Sigma}^{M5} - \int \Sigma^*(\Omega) = \mathcal{E}_{\Sigma} \geq 0$$

BBFTY Geometry has

$$g_6 = e^{2\Phi} g_{K3} + |\theta|^2 \qquad e^{-2\Phi} J = J_{K3} + \frac{i}{2} e^{-2\Phi} \theta \wedge \bar{\theta}$$

$$\mathcal{E}_\Sigma \geq 0 \longleftrightarrow \int_\Sigma e^{-2\Phi} vol - \int \Sigma^* \left(\frac{i}{2} e^{-2\Phi} \theta \wedge \bar{\theta} \right) \geq \int \Sigma^* (J_{K3})$$

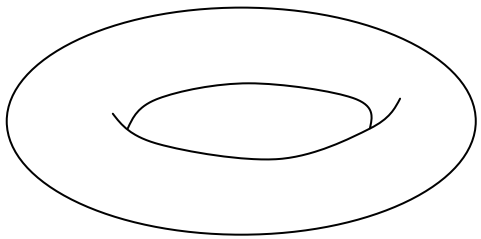
closed/ topological bound

similar to usual calibration, except that the calibrating 2-form is degenerate in the full manifold

$$\Sigma^*(J_{K3}) \neq 0$$

$$\Sigma^*(J_{K3}) = 0$$

$$\int \Sigma^*(J_{K3}) = \int (\pi_*\Sigma)^*(J_{K3})$$



$\theta = dz + \alpha(K3)$

$\longrightarrow M_6$

$\downarrow \pi$

$K3$ rescaled by $e^{2\Phi}$

$$[dim H^{p,q}(K3)] = \begin{bmatrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{bmatrix}$$

$d\theta$

A necessary condition: $\int_{\pi_*\Sigma} vol = \int (\pi_*\Sigma)^*(J_{K3})$

$$vol_\Sigma = \sqrt{|\Sigma^*\Omega^{(2,0)}|^2 + |\Sigma^* J|^2}$$

for any 4D hyperKaehler/Calabi-Yau

$$(\pi_*\Sigma)^*\Omega_{K3}^{(2,0)} = 0$$

holomorphic embedding

$\Sigma^*(J_{K3}) = 0$ $\pi_*\Sigma$ is point-like in K3

$\Sigma^*(J_{K3}) \neq 0$ $\pi_*\Sigma$ is a holomorphic surface in K3

$\Sigma^*(J_{K3}) = 0$ $\pi_*\Sigma$ is point-like in K3

must wraps the fibre completely

$\mathcal{E}_\Sigma = 0$ regardless of its position on K3

moduli space = K3

$\Sigma^*(J_{K3}) \neq 0$ $\pi_*\Sigma$ is a holomorphic surface in K3

can be lifted into the full manifold only if
the restriction of the bundle to the 2-cycle is trivial

$$\int_{\pi_*\Sigma} d\theta = 0$$

the no-winding lift gives T^2 worth of moduli space

Duality Dictionary for Branes:

IIB	Type I	Heterotic
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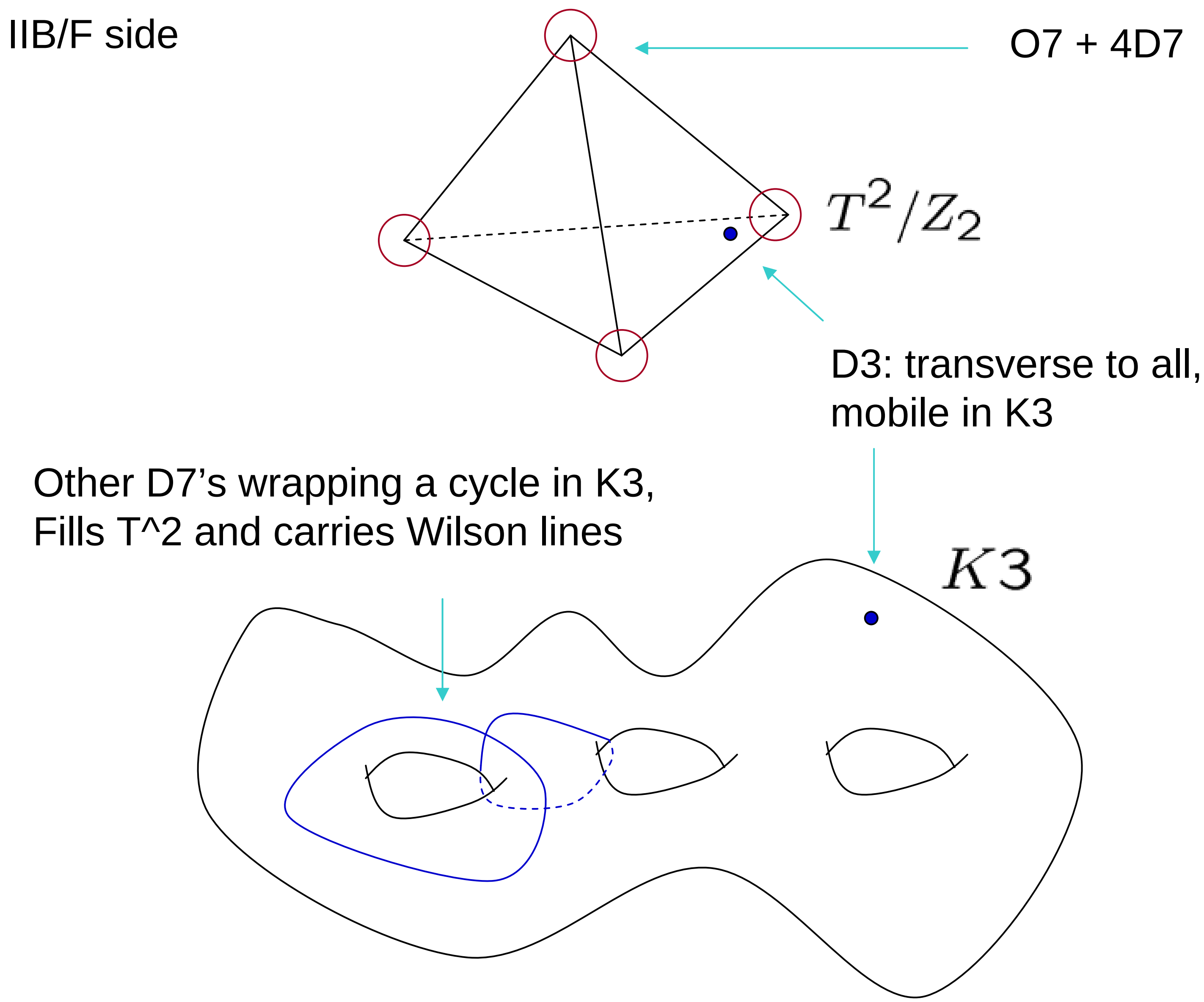
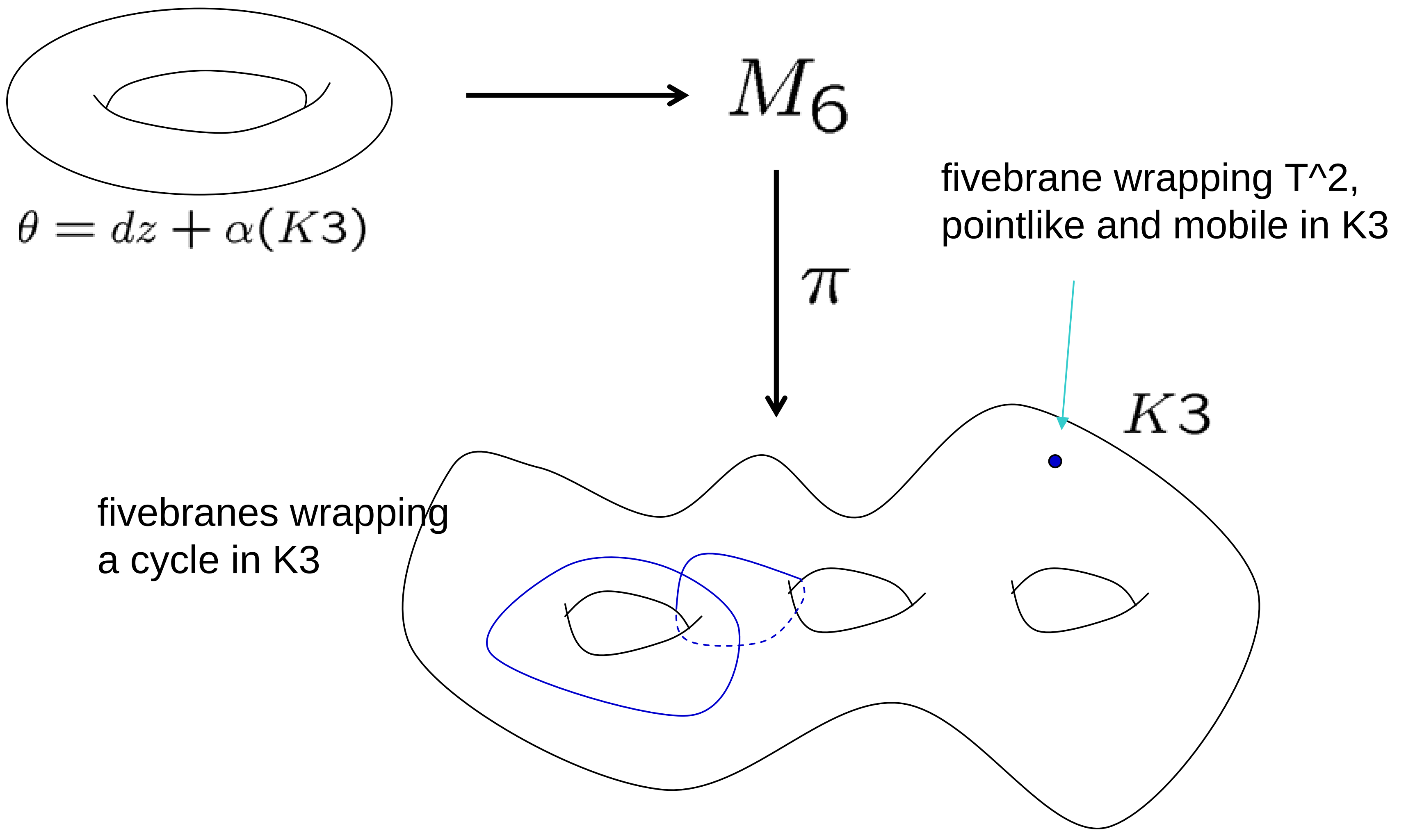

IIB/F side
O7 + 4D7
T^2/Z_2
D3: transverse to all,
mobile in K3
Other D7's wrapping a cycle in K3,
Fills T^2 and carries Wilson lines
$K3$

heterotic side



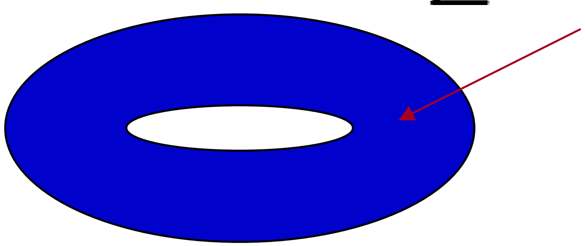
$\Sigma^*(J_{K3}) = 0$ $\pi_* \Sigma$ is point-like in K3

must wraps the fibre completely

$\mathcal{E}_\Sigma = 0$ regardless of its position on K3

moduli space = K3 = moduli space of D3

Σ fivebrane wrapping fibre T^2 completely



Note that the fibre T^2 has a cyclic homotopy:
Each circle is Hopf fibred over a 2-cycle in K3,
as in S^1 fibered over S^2 giving rise to S^3/Z_n.

This is also related to the fact that the base K3 does
not represent a homology cycle either.

$$vol_{K3} \sim d\theta \wedge d\bar{\theta} = d\left(\theta \wedge d\bar{\theta}\right) \to \text{exact 4-form}$$

The n-wrapped fivebrane is stable entirely due to a dynamical
reason: competition of tension energy and magnetic energy.

But, what about the tadpole condition?

$$dH = \alpha' \left(trR \wedge R - trF \wedge F \right) - \delta_{fivebrane}$$

?

$$\int_{K3} dH = \int_{K3} \left[\alpha' \left(trR \wedge R - trF \wedge F \right) - \delta_{fivebrane} \right]$$

$$dH = \alpha' \left(trR \wedge R - trF \wedge F \right) - \delta_{fivebrane}$$

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$$\int_{K3} dH = \int_{K3} \left[\alpha' \left(trR \wedge R - trF \wedge F\right) - \delta_{fivebrane}\right]$$

$$\int_{M_6} J \wedge dH = \int_{M_6} J \wedge \left[\alpha' \left(trR \wedge R - trF \wedge F\right) - \delta_{fivebrane}\right]$$

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$$\int_{M_6} J \wedge dH = \int_{M_6} J \wedge \left[\alpha' \left(trR \wedge R - trF \wedge F\right) - \delta_{fivebrane}\right]$$

when the manifold is written as a T^2 bundle over K3

$$\frac{1}{\alpha'} \int_{K3} |d\theta|^2 + n_{fivebranes\ on\ T^2} = p_1(K3) - p_1(Gauge\ Bundle\ on\ K3)$$

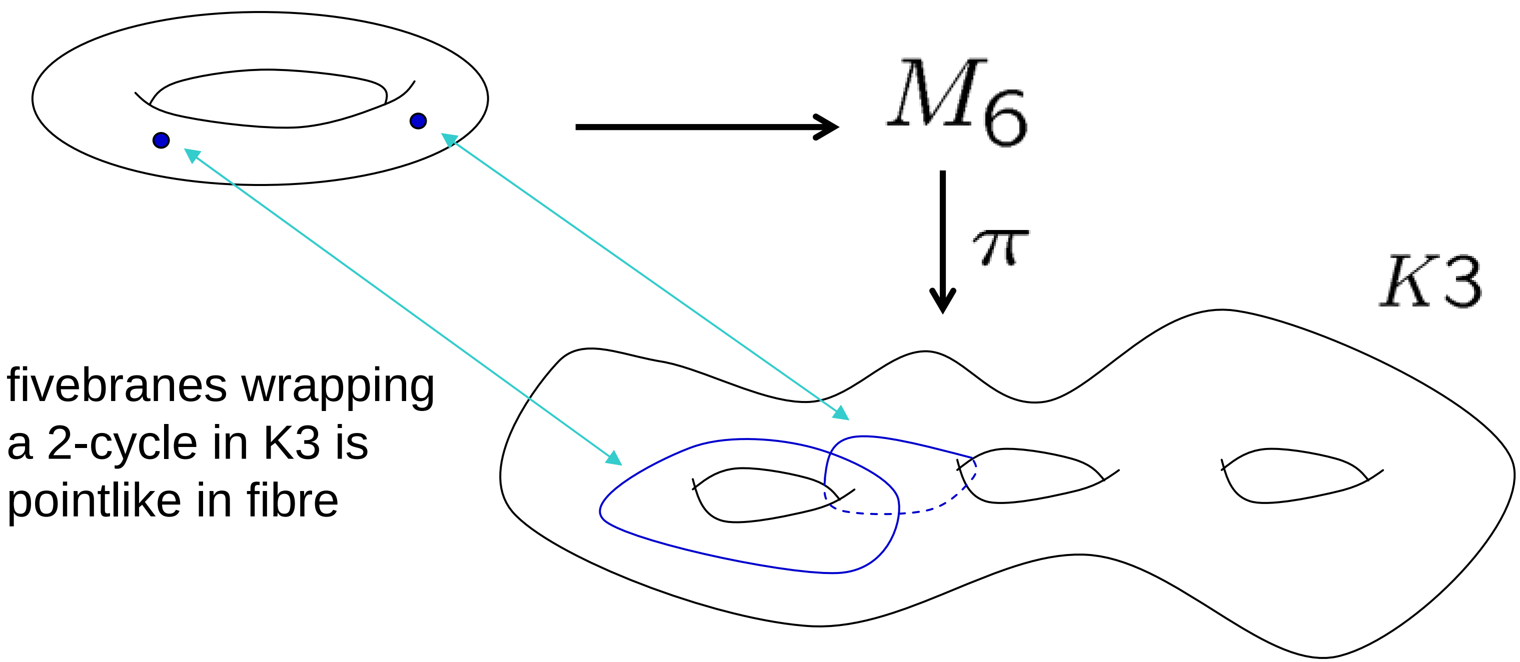
$\Sigma^*(J_{K3}) \neq 0$ $\qquad$ $\pi_*\Sigma$ is a holomorphic surface in K3

can be lifted into the full manifold only if
the restriction of the bundle to the 2-cycle is trivial

$$\int_{\pi_*\Sigma} d\theta = 0$$

the no-winding lift gives T^2 worth of moduli space

= Wilson lines of the other D7



SUSY requirement:

$$\int_{\pi_* \Sigma} \Omega^{2,0}_{K3} = \int_{K3} [\pi_* \Sigma] \wedge \Omega^{(2,0)}_{K3}$$

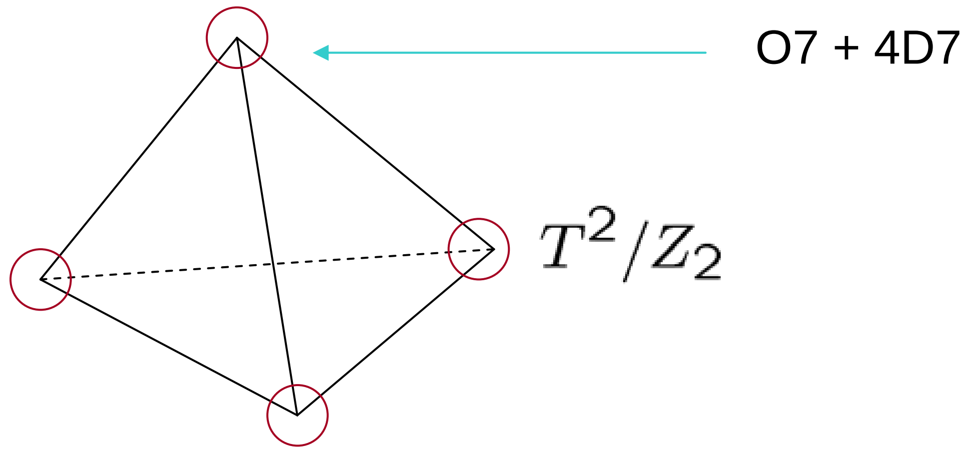
Integral cohomology element

For insertion of one such fivebrane, we fix one “complex” moduli.

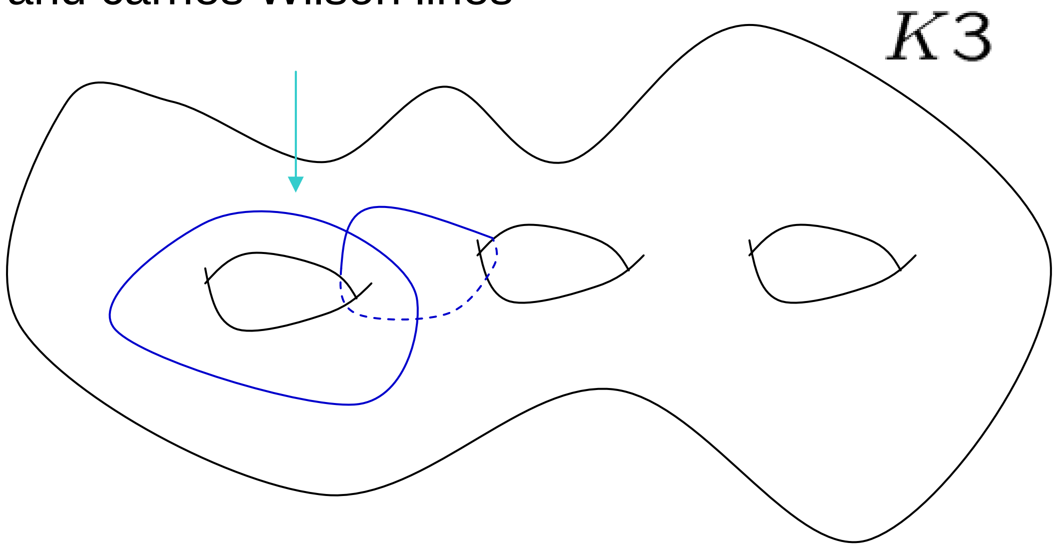
When this procedure is carried out maximally and fixed the holomorphic 2-form completely, the resulting K3 is “attractive,” saturating the bound $Rank\left(H^2(K3,Z) \cap H^{1,1}(K3,R)\right) \leq 20$

#[Complex Moduli from K3] < 20

IIB/F side



Other D7's wrapping a cycle in K3,
Fills T^2 and carries Wilson lines



Global Issues

- 4 Dimensional Gauge Spectra and Index Counting
- Smooth Compactification, and Gauge Bundle
- Gauge moduli and Geometric Moduli
- Near Calabi-Yau Regime ?

Torsion, Torsion, and Gauge Zero Modes

$$\gamma^a D_a(\omega; A)\chi + \frac{1}{24} H_{abc}\gamma^{abc}\chi = F_{ab}\gamma^{ab} \text{ (dilatino/gravitino)}$$

$$\gamma^a D_a(\omega + H/6; A)\chi = F_{ab}\gamma^{ab}(\cdots)$$

exists for $\mathcal{F}$ only

$E_8 \oplus E_8$ or $SO(32)$ ← $\mathcal{G} = \mathcal{F} \oplus \mathcal{Q} \oplus \mathcal{H}$

$\mathcal{F}$: Gauge Bundle

$\mathcal{Q}$: Matter sector

$\mathcal{H}$: Stabilizer of F_{ab} = Unbroken Gauge Group

$$\gamma^a D_a(\omega + H/6)\chi_{\mathcal{H}} = 0 \qquad \gamma^a D_a(\omega + H/6; A_{\mathcal{Q}})\chi_{\mathcal{Q}} = 0$$

There should be a single zero mode for $\chi_{\mathcal{H}}$

$\chi_{\mathcal{Q}}$ zero mode counts the number of chiral matter fermions

Zero modes along $\chi_{\mathcal{F}}$ estimates the gauge bundle moduli

Without Torsion,

$$\gamma^a D_a(\omega)\chi_{\mathcal{H}} = 0 \xrightarrow[\nabla_a \epsilon_{SUSY} = 0]{} \chi_{\mathcal{H}} = \epsilon_{SUSY}$$

With Torsion,

$$\gamma^a D_a(\omega + H/6)\chi_{\mathcal{H}} = 0$$

$$\Big\downarrow \; \nabla_a^{(+)} \epsilon_{SUSY} = 0$$

$$\chi_{\mathcal{H}} \neq \epsilon_{SUSY}$$

$$D_a(\omega + H/6) = e_a^m \left(\partial_m + \frac{1}{4}(\omega_{mab} + H_{mab}/6)\gamma^{ab} \right)$$

Consider this:

$$(\gamma^a D_a(\omega + H/6))^2 = \gamma^a\gamma^b D_a D_b + \gamma^a[D_a,\gamma^b]D_b$$

$$= g^{ab} D_a D_b + \gamma^a[D_a,\gamma^b]D_b + \gamma^{ab}[D_a, D_b]$$

$$= \nabla^{(\omega+H/6)a}\nabla_a^{(\omega+H/6)} + \frac{1}{6}H^a{}_{bc}\gamma^{bc}D_a + \gamma_{ab}[D_a, D_b]$$

$$= \nabla^{(+)a}\nabla_a^{(+)}$$

minus the Laplacian with a torsion

$$+\frac{1}{144}\left(H_{abc}\gamma^{bc}\right)^2 - \frac{1}{12}\left(\nabla^a H_{abc}\right)\gamma^{bc} + \frac{1}{8}R_{abcd}^{(\omega+H/6)}\gamma^{ab}\gamma^{cd}$$

after a little algebra with Dirac matrices

$$-V_{\mathcal{H}} \equiv \frac{1}{4}R - \frac{1}{8}H^2 - \frac{1}{24}dH_{abcd}\gamma^{abcd}$$

Similarly:

$$-(\gamma^a D_a(\omega + H/6, A_{\mathcal{Q}}))^2 = -\nabla(\omega^{(+)}, A_{\mathcal{Q}})^a \nabla(\omega^{(+)}, A_{\mathcal{Q}})_a$$

$$+V_{\mathcal{H}} + \frac{i}{2} F^{\mathcal{Q}}_{ab} \gamma^{ab}$$

$$V_{\mathcal{H}} = \frac{1}{4} R - \frac{1}{8} H^2 - \frac{1}{24} dH_{abcd} \gamma^{abcd}$$

Comments on Index counting:

Generally, torsion can be understood as an innocuous continuous deformation if the Dirac operator defines a Fredholm operator with gap. The Index will be then determined by the metric and the gauge bundle only. However, the metric backreact to torsion strongly, so we cannot rely on old computations.

Explicit computation of index density with torsion have been carried out largely in the context of 4D spacetime, where torsion terms are shown to organize themselves into a total derivative.

Peeters+Waldron 2000

Higher dimensional computation is available for the case of $dH = 0$ where the usual characteristic class form of index densities holds provided that the curvature 2-form replace by the curvature 2-form of $\omega + H$

Mavromatos 1988; Bismut 1988

With $dH \neq 0$, the index densities are unlikely to be given by the familiar characteristic classes, although we could anticipate corrections by at most a total derivative.

“Minimal Embedding”

$$dH = 0$$

$$trR \wedge R = trF \wedge F$$

Do we gain something in this special limit ?
For instance, take a look at

$$V_{\mathcal{H}} = \frac{1}{4}R - \frac{1}{8}H^2$$

with an SU(3) holonomy condition $0 = R + \frac{1}{2}H^2 + 6\nabla^2\Phi - 8(\nabla\Phi)^2$

and $0 = *\,(J \wedge dH)$

$$
\begin{aligned}
0 = J \wedge dH &= e^{2\Phi} d\left(e^{-2\Phi} J \wedge H\right) - e^{2\Phi} d(e^{-2\Phi} J) \wedge H \\
&= e^{2\Phi} d\left(e^{-2\Phi} J \wedge J \wedge K\right) - H^2 \wedge vol_6 \\
&= 2dK \wedge *J - H^2 \wedge vol_6 \\
&= \left(-\nabla^2\Phi + 2(\nabla\Phi)^2 - H^2\right) \wedge vol_6
\end{aligned}
$$

$$
K_a \equiv \frac{1}{8} H_{abc} J^{bc} = \frac{1}{2} J_a^b \nabla_b \Phi
$$

$$
V_{\mathcal{H}} = \frac{1}{8} H^2 - (\nabla\Phi)^2 = \frac{1}{8} |H_{primitive}|^2 \geq 0
$$

$$
H_{primitive} \equiv H - J \wedge K
$$

No massless gaugino in 4 dimensions ?
This cannot be so, unless SUSY is broken !

Formal positivity

$$-\nabla^2 + V \geq V \geq 0$$

must be wrong in heterotic compactification with $H \neq 0$ and $dH = 0$

Consider

$$*(J \wedge dH) = 2\left(-\nabla^2\Phi + 2(\nabla\Phi)^2 - \frac{1}{2}H^2\right) = 0$$

$$\rightarrow \quad e^{-2\Phi}H^2 = \nabla^2 e^{-2\Phi}$$

$$\int_{M_6} e^{-2\Phi} H^2 = \int_{M_6} \nabla^2 e^{-2\Phi} = 0$$

?

$$H = 0$$

$$\int_{M_6} e^{-2\Phi} H^2 = \int_{M_6} \nabla^2 e^{-2\Phi} \neq 0$$

If the manifold is either open or singular somewhere

$$\int_{M_6} e^{-2\Phi} H^2 = \int_{M_6} \nabla^2 e^{-2\Phi} \neq 0$$

If the manifold is either open or singular somewhere

$$\int_{M_6} -\chi^\dagger \nabla^2 \chi = \int_{M_6} |\nabla \chi|^2 - \int_{M_6} \nabla \left(\chi^\dagger \nabla \chi \right) \neq \int_{M_6} |\nabla \chi|^2 \geq 0$$

$$\int_{M_6} e^{-2\Phi} H^2 = \int_{M_6} \nabla^2 e^{-2\Phi} \neq 0$$

If the manifold is either open or singular somewhere

$$\int_{M_6} -\chi^\dagger \nabla^2 \chi = \int_{M_6} |\nabla\chi|^2 - \int_{M_6} \nabla\left(\chi^\dagger \nabla \chi\right) \neq \int_{M_6} |\nabla\chi|^2 \geq 0$$

flux compactification under the approximation

$$H^2 \gg *(J \wedge dH) = \frac{\alpha'}{4} J \wedge (trR \wedge R - trF \wedge F)$$

is necessarily singular

Ivanov+Papadopoulos 2001

Recall that, with minimal embedding without Flux and the breaking of the gauge group

$$E_8 \to SU(3) \oplus E_6$$

the counting of charged matter fermions gives a universal formula

$$\#\text{Family} = Index\left[\chi_{(3,27)}\right] - Index\left[\chi_{(\bar{3},\bar{27})}\right] = \frac{\text{Euler Number}}{2}$$

In extending this to “minimal” Flux compactification with

$$E_8 \to SU(4) \oplus SO(10)$$

all bets are off since we are forced into singular internal manifold; total derivative terms (due to torsion contributions) and the boundary condition must be reconsidered carefully.

Smooth Compactifications and Approximations

L : size of the internal 6-manifold

$\alpha'/L^2 \ll 1$

$$\frac{1}{L^2} \sim H^2 \gg *(J \wedge dH)$$ large torsion, singularities

$$\frac{1}{L^2} \gg H^2 \sim *(J \wedge dH)$$ small torsion, smooth solutions

$$\frac{1}{L^2} \gg H^2 \gg *(J \wedge dH)$$ small torsion, singularities

order of magnitudes favorable for supergravity approach

$$R_{abcd} \sim \frac{1}{L^2} \ll \frac{1}{\alpha'}$$

$$H^2 \sim *(J \wedge dH) \sim \alpha' J \wedge (trR \wedge R - trF \wedge F) \sim \frac{\alpha'}{L^4}$$

$$R \sim H^2 \sim (\nabla\Phi)^2 \sim \frac{\alpha'}{L^4}$$

also

$$R(\omega \pm H)_{abcd} - R_{abcd} \sim \frac{\alpha'}{L^4} \ll R_{abcd}$$

$$G_{9+1}^{Einstein} = e^{-\Phi/2}\eta_{3+1} + e^{-\Phi/2}g_6$$

warp factor

$$e^{-2\Phi} * (J \wedge dH) =$$

$$\nabla^2 e^{-2\Phi} - e^{-2\Phi}H^2 = 0$$

~ also, a warp factor equation of motion simplified by $dH = 0$

For general background without such restriction, this generalizes to

$$0 = \left[\frac{\delta}{\delta\Phi} - \frac{1}{2}G^{AB}\frac{\delta}{\delta G^{AB}}\right]\int \mathcal{L}_{heterotic}$$

with

$$\mathcal{L}_{heterotic} = \sqrt{-G}e^{-2\Phi}\left[R - \frac{1}{2}H^2 + 4(\nabla\Phi)^2 - \frac{\alpha'}{4}\left(trF^2 - trR_+^2\right)\right]$$

$$\nabla^2 e^{-2\Phi} = e^{-2\Phi}\left[H^2 + \frac{1}{4}\alpha'\left(trF^2 - trR_+^2\right)\right] + \alpha' \cdots$$

$$= e^{-2\Phi}\left[H^2 + \frac{1}{4}\alpha'\left(trF^2 - trR^2\right)\right] + \alpha' \cdots$$

$\sim \alpha'^2/L^6$

potential no-go theorem
against smooth compactificatin

$\sim *(J \wedge dH)$
under SUSY

double divergences
of Ricci tensors or

RdH
HHR
$HHdH$
$HHHH$

negative tadpole contribution from $dH \neq 0$
overcome the potential no-go theorem

The fact that

$$dH$$

controls/resolves singularities of the manifold suggest that
the singularities are related BPS objects with negative fivebrane charges,

reminiscent of type IIB compactifications where singularities
due to fluxes are realized as orientifold planes of some RR fields.

Summary

- Supersymmetry demands special geometric structure on the internal manifold. In some special cases, such as some limiting cases of type IIB theory and the Heterotic theory, the holonomy group reduces to SU(3). In type IIB without fivebrane sources, the geometry is a Calabi-Yau up to a conformal factor, while in the Heterotic case the torsion, or H flux, preserves SU(3) holonomy group while destroying Kaehler property of the metric.

- Effect of the flux, in general, is to introduce additional potential to minimize. This in general reduces the number of massless scalar fields, alleviating the moduli problems with conventional string theory compactifications.

- As with type II theories, the heterotic flux compactification is also plagued by a gravitational tadpole condition which can render the geometry singular. The situation is worse than type II in that we do not have orientifold plane description of such singular places, but is better in that we can make the geometry smooth by going beyond the “minimal embedding” of the gauge bundle.

- Global issues in flux compactification remain largely unsolved, and must be investigated further. No general tools for addressing glocal problems are available at the moment.

Prospects

- More solutions and model building in the Heterotic theory
(De Sitte vacua; Hierarchy; Inflation; Axion; Cosmological constant; Standard model vacua;………….)

- Duality maps between type IIA, IIB, and Heterotic Theories
(Mirror; Heterotic dual to non-GKP IIB; Complete fixing of moduli on the Heterotic side)]

- New Physics in the Heterotic-M with Flux?

- Landscape of the Heterotic String Theory