Bessel potentials and optimal Hardy and Hardy-Rellich inequalities

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Abstract

We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball Ω of radius R in R^n, $n \geq 2$, so that the following inequalities hold for all $u \in C_0^\infty(\Omega)$:

$$\int_\Omega V(x)|\nabla u|^2 dx \geq \int_\Omega W(x) u^2 dx \tag{1}$$

and

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B \left(\frac{V(x)}{|x|^2} - \frac{v'(|x|)}{|x|}\right)|\nabla u|^2 dx. \tag{2}$$

We then identify a large number of such couples (V, W) – that we call Bessel pairs – and the best constants in the corresponding inequalities. This will allow us to complete, improve, extend, and unify most related results –old and new– about Hardy and Hardy-Rellich type inequalities which were obtained by Caffarelli-Kohn-Nirenberg, Brezis-Vázquez, Adimurthi-Chaudhuri-Ramaswamy, Filippas-Tertikas, Adimurthi-Grossi-Santra, as well as some very recent work by Tertikas-Zographopoulos, Liskevich-Lyachova-Moroz, and Blanchet-Bonforte-Dolbeault-Grillo-Vasquez, among others.

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