Mathematical Architecture of Approximate Deconvolution Models of Turbulence

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Abstract

Direct numerical simulation of many turbulent flows is not feasible for the foreseeable future within time and resource constraints of many applications. There are thus many approaches to finding reduced models of turbulent flows whose solutions have a smaller number of persistent scales (and thus can be solved more quickly and economically). However, the associated closure problem cannot be solved exactly. Thus, it is possible (and in fact not uncommon) that a given turbulence model's solution has little physical fidelity, quantitative agreement and qualitative agreement with the flow averages sought. This talk presents some recent models of the large structures of turbulent flows that have a fairly complete mathematical theory, that are provably high accurate on the large scales, truncate scales, conserve appropriate integral invariants and are unconditionally stable. An outline of their mathematical structure, which is the reason for their effectiveness, will be given.

Broadly, if δ is the (user-selected) filter length scale and *overbar* denotes the associated local, spacial averaging, the true averages, $\overline{\mathbf{u}}$, $\overline{p}$, of an incompressible viscous fluid satisfy the well known Space Filtered Navier-Stokes equations given by

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u}\,\mathbf{u}}) - \nu \Delta \overline{\mathbf{u}} + \nabla \overline{p} = \overline{\mathbf{f}} \qquad \textit{and} \qquad \nabla \cdot \overline{\mathbf{u}} = 0.$$

The closure problem (which occurs since $\overline{\mathbf{u}\,\mathbf{u}} \neq \overline{\mathbf{u}}\,\overline{\mathbf{u}}$) thus leads to the deconvolution problem:

given $\overline{\mathbf{u}}$ (*+noise*), find $\mathbf{u}$ approximately to high accuracy.

This is of course the ill-posed deconvolution problem, common in image processing. Our work begins with an approximate deconvolution operator of van Cittert in 1931. Calling the van Cittert approximate deconvolution of $\mathbf{u}$, $D(\overline{\mathbf{u}})$:

$$D(\overline{\mathbf{u}}) = \text{ approximation of } \mathbf{u}.$$

An approximate solution to the closure problem is then $\overline{\mathbf{u}\,\mathbf{u}} \approx \overline{D(\overline{\mathbf{u}})\,D(\overline{\mathbf{u}})}$. This deconvolution operator satisfies the consistency condition:

$$\mathbf{u} = D_N(\overline{\mathbf{u}}) + O(\delta^{2N+2}) \text{ for smooth } \mathbf{u},$$

and thus $\overline{\mathbf{u}\,\mathbf{u}} = \overline{D_N(\overline{\mathbf{u}})\,D_N(\overline{\mathbf{u}})} + O(\delta^{2N+2})$. This model has remarkable mathematical properties and its accuracy has been established in computational tests of Stolz and Adams and in our theoretical studies.The LES model induced is: find $(\mathbf{w}, q)$ (sought to approximate $(\overline{\mathbf{u}}, \overline{p})$) satisfying

$$\mathbf{w}_t + \nabla \cdot (\overline{D(\mathbf{w})\,D(\mathbf{w})}) - \nu\Delta\mathbf{w} + \nabla q = \overline{\mathbf{f}}, \text{ and } \nabla \cdot \mathbf{w} = 0,$$

with initial condition $\mathbf{w}(x, 0) = \mathbf{w}_0(x)$.

This talk will present the mathematical structure of the ADM, beginning with basic results of a Leray-type theory that unique, smooth strong solutions exist and converge, modulo a subsequence, to a weak solution of the NSE as the averaging radius $\delta \to 0$. Next we examine the consequences of the model energy and helicity balance and apply turbulence phenomenology to the model's predictions of turbulent statistics. We show that the model predicts a helicity and energy cascade correctly up to the cutoff frequency: , over $0 < k < \frac{1}{\delta}$,

$$\widehat{Energy}(k) \sim \epsilon_{model}^{2/3} k^{-5/3} \text{ and } \widehat{Helicity}(k) \sim \gamma_{model} \epsilon_{model}^{-1/3} k^{-5/3}.$$

Finally, we shall outline some applications of deconvolution ideas to some regularizations of the NSE such as the recently popular NS-alpha model and the new and promising NS-omega model.

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