On a conjectured estimate for solutions of the three-dimensional Stokes equations, with a constant that is optimal and independent of the domain

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Abstract

Let Ω be an arbitrary open subset of R^3, and let

$\mathbf{D}(\Omega) = \{\phi \in \mathbf{C}_0^\infty(\Omega) : \nabla \cdot \phi = 0\}$,
$\mathbf{J}(\Omega) =$ Completion of $\mathbf{D}(\Omega)$ in the $\mathbf{L}^2$-norm $\|\phi\|$, and
$\mathbf{J}_0(\Omega) =$ Completion of $\mathbf{D}(\Omega)$ in Dirichlet-norm $\|\nabla\phi\|$.

Then, for any $\mathbf{f} \in \mathbf{J}(\Omega)$, there is at most one $\mathbf{u} \in \mathbf{J}_0(\Omega)$ such that

$$\int_\Omega \nabla\mathbf{u} \cdot \nabla\phi \, dx = -\int_\Omega \mathbf{f} \cdot \phi \, dx \,, \quad \text{for all } \phi \in \mathbf{D}(\Omega) \,. \tag{1}$$

For such $\mathbf{u}$ and $\mathbf{f}$ we write $\tilde{\triangle}\mathbf{u} = \mathbf{f}$, thereby defining the Stokes operator $\tilde{\triangle}$ on a certain subspace of $\mathbf{J}_0(\Omega)$. We believe that the inequality

$$\sup_\Omega |\mathbf{u}|^2 \le \frac{1}{3\pi} \|\nabla\mathbf{u}\| \left\|\tilde{\triangle}\mathbf{u}\right\| \tag{2}$$

should be valid for all $\mathbf{u} \in \mathbf{J}_0(\Omega)$ and $\mathbf{f} \in \mathbf{J}(\Omega)$ satisfying (1).

Wenzheng Xie has proven this result for $\Omega = R^3$, and shown that the constant $1/3\pi$ is the best possible for any domain. He also proved

an analogous inequality for the Laplacian, valid for arbitrary open sets, with the constant $1/2\pi$. Moreover, he showed that his proof for the Laplacian also applies to the Stokes operator, except at one point where the maximum principle is used.

This lecture concerns our efforts to complete Xie's proof for the case of the Stokes operator by a variational argument. Our reasoning comes to rest on several new conjectures concerning the asymptotic properties of the eigenvalues and eigenfunctions of the Stokes operator. We provide supporting evidence for these new conjectures. Perhaps happily, their analogues for the Laplacian are also not yet proven, lending hope that proofs for the Laplacian might be generalized to the Stokes operator. Expecting this, much of this talk will be presented in the context of the Laplacian. We think our conjectures for the Laplacian may be interesting to the mathematical community at large.

We remark that the inequality (2) would have very important applications to the theory of the Navier-Stokes equations, especially in nonsmooth domains. In particular, the inequality (2) would make possible the estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\nabla\mathbf{u}\right\|^2 + \nu\left\|\tilde{\triangle}\mathbf{u}\right\|^2 &= \left(\mathbf{u}\cdot\nabla\mathbf{u}, \tilde{\triangle}\mathbf{u}\right) \\ &\leq c\left\|\nabla\mathbf{u}\right\|^{\frac{3}{2}}\left\|\tilde{\triangle}\mathbf{u}\right\|^{\frac{3}{2}} \leq \cdots, \end{aligned}$$

in arbitrary domains, and even with a domain independent constant $c = 1/\sqrt{3\pi}$.