Term Structure Models of Commodity Prices: *A Review*

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This article reviews the literature on term structure of commodity prices with particular reference to recent developments on term structure models and their applications. The term structure is defined as the relationship between the spot price and the futures prices for any delivery date. It provides useful information for hedging or investment decisions because it synthesizes the information available in the market and the operators' expectations concerning the future. This information is very useful for management purposes: it can be used to hedge exposures on the physical market and to adjust the stock level or the production rate. It can also be used to undertake arbitrage transactions, to evaluate derivatives instruments based on futures contracts, and so on.

In many commodity markets, the concept of term structure becomes important because the contracts' maturity increases as the markets come to fruition. In the American crude oil market, this evolution has come a long way because since 1999, there have been futures contracts for maturities as far as seven years. Thus, this market is today the most developed commodity futures market in terms of the volume and the maturity of the transactions. It provides publicly available prices—namely potentially informative and costless signals—whereas in most commodities markets, the only information for far maturities is private and given by forward prices. The existence of these long-term futures contracts authorizes empirical studies on the crude oil prices' curves that were once possible only with forward prices. However, the informational content of the latter is not necessarily reliable or workable (forward contracts are not standardized, and the prices reporting mechanism does not force the operators to disclose their transactions prices).

This specificity of the crude oil market explains why most of the examples used in this article are linked to this commodity. Moreover, even if there exist miscellaneous derivatives markets and hedging instruments in today's commodities markets, we concentrate on the futures contracts in order to focus on the relationship between the physical and paper markets. However, the analysis presented can be extended to other derivative products and markets, because the pricing of complicated instruments can always be reduced to the determination of the term structure of futures prices.

The reader can find more general reviews of the literature on commodities markets. A recent and extensive review can be found in Carter [1999]. Gray and Rutledge [1971] propose a well-known review on the evolutionary aspects of futures markets, on inter-temporal price relationships, on concepts of hedging, etc. Other reviews were done by Tomek and Robinson [1977], Goss and Yamey [1978], Kamara [1982], Blank [1989], and Malliaris [1997].

This survey proceeds as follows. A first section is devoted to the theoretical analysis of the term structure. The second section is centered on term structure models of commodity prices. The third section analyzes the ability of the models to describe the prices curve empirically observed. The last section reviews the two main applications of term structure models: hedging and valuation. The conclusion proposes new directions for future research.

THEORETICAL ANALYSIS OF THE TERM STRUCTURE

This section primarily confines itself to the traditional theories of commodity prices and to their explanation of the relationship between spot and futures prices. However, the theories of normal backwardation and storage are a bit limited when the whole term structure is taken into account. As a result, there is a need for a long-term extension of the analysis, which constitutes the second point of the section. Finally, a dynamic analysis of the term structure is presented.

Traditional Theories and the Relationship Between Spot and Futures Prices

The normal backwardation and the storage theories are traditionally used to explain the relationship between spot and futures prices in commodity markets. Whereas the theory of normal backwardation is centered on the analysis of hedging positions and on the function of transferring the risk provided by the futures market, the storage theory proposes an explanation based on the storage costs. More precisely, the different determinants of the futures prices, in this context, are the spot price, the convenience yield, and the storage cost. The latter includes the pure storage cost and the financing cost.

Keynes introduced the theory of normal backwardation in 1930. Briefly summarized, its central argument is the following: in normal conditions, the commodity market is characterized by a forward price situated below the spot price:

> ". . . in normal conditions the spot price exceeds the forward price i.e. there is backwardation. In other words, the normal supply price on the spot includes remuneration for the risk of price fluctuation during the period of production, whilst the forward price excludes this (Keynes [1930]).

The relationship linking these two prices is due to the relative importance of short and long hedging positions in the futures market. The first assumption of the theory is that short hedging represents a lower volume than long hedging. Consequently, there is a need for speculators to compensate for this market unbalance. In order to motivate the speculators' intervention, there must be a difference between the futures price and the spot price expected at the contract's delivery date. This is the second assumption of the theory. The presence of a positive risk premium associated with the expected spot price explains the difference between the spot and the futures prices. This premium remunerates the speculators for the risks they undertake in their activity.

Until now, the theory of normal backwardation was never truly validated nor rejected. Dusak [1973], Bodie and Rosansky [1980], Richard and Sudaresan [1981], and Bessembinder [1993] use either the static or inter-temporal capital asset pricing model to examine the futures' risk premium and all obtained contradictory results. More generally, the critiques raised against the theory of normal backwardation sustain that whenever the premium exists, chances are slim that it would be positive and constant. Indeed, the net position of the hedgers in the commodity futures markets is not always a short one. Moreover, the risk aversion of the participants can change with time. As a result, the empirical tests carried out in order to validate the theory are contradictory: for the same futures markets and different periods, they can conclude either that there is normal backwardation or conversely that there is "normal contango."

The storage theory relies on the reasons explaining the holding of physical stocks to understand the relationships between spot and futures prices in commodity markets. The analysis of the arbitrage operations between the physical and the futures markets makes it possible to understand the mechanisms causing contango and backwardation. It also shows that the basis evolves differently when it is positive or negative. Indeed, contango is limited to the storage costs between the current date and the contract's expiration. Such a limit does not exist for backwardation.

When physical stocks are invoked to explain the relationship between spot and futures prices, interpreting backwardation becomes tricky. If the futures price corresponds to the spot price increased by positive storage costs, how can we explain that sometimes, the futures price is below the spot price? The concept of convenience yield, introduced by Kaldor in 1939, answers this question. The convenience yield can be briefly defined as the implicit revenue associated with stock holding. The possession of inventories indeed avoids the costs of frequent supply orders and spares the waiting time associated with deliveries:

> In normal circumstances, stocks of all goods possess a yield, measured in terms of themselves, and this yield which is a compensation to the holder of stocks, must be deducted from carrying costs proper in calculating net carrying cost. The latter can, therefore, be negative or positive (Kaldor [1939]).

Moreover, as Brennan [1958] stated, inventories give the possibility to take advantage of sudden and unexpected rises in the demand:

> The convenience yield is attributed to the advantage (in terms of less delay and lower costs) of being able to keep regular customers satisfied or of being able to take advantage of a rise in demand and price without resorting to a revision of the production schedule.

These definitions show that the convenience yield is high when inventories are rare, because stock holding is then more valuable. Conversely, the convenience yield is low when stocks are abundant. Moreover, the convenience yield is positively correlated with the spot price, which is also high when there is a shortage of stocks, and vice versa.

An important part of the research on storage theory was devoted to the definition of the convenience yield. This concept is central to the analysis of the term structure of commodity prices. In the context of other financial markets, the convenience yield corresponds to the coupon linked with the bonds or the dividends given by a stock portfolio.

The storage theory constitutes the main basis for the elaboration of term structure models of commodity prices. Indeed, it brings useful conclusions to construct such a model. First, the relationship between spot and futures prices allows the identification of at least three variables influencing the futures price: the spot price, the convenience yield, and the interest rate, which is implicitly included in the financing costs. Second, convenience yield and spot price are positively correlated: both of them are an inverse function of the stock level. Third, the examination of arbitrage relationships between physical and paper markets shows that the basis has an asymmetrical behavior: in contango, its level is limited to storage costs. This is not the case for backwardation. Furthermore, the basis is stable in contango, and volatile in backwardation, since in this situation stocks cannot absorb price fluctuations. This asymmetry has implications on the dynamic of convenience yield. These implications were exploited in the context of term structure models.

A Long-Term Extension of the Analysis

The most important developments concerning the traditional theories of commodity prices were introduced between 1930 and 1960. At that time, a typical transaction on futures markets was rarely longer than one year. Therefore, the analysis of the prices relationship was originally conceived for the short term, and it must be adapted to enable a long-term analysis.

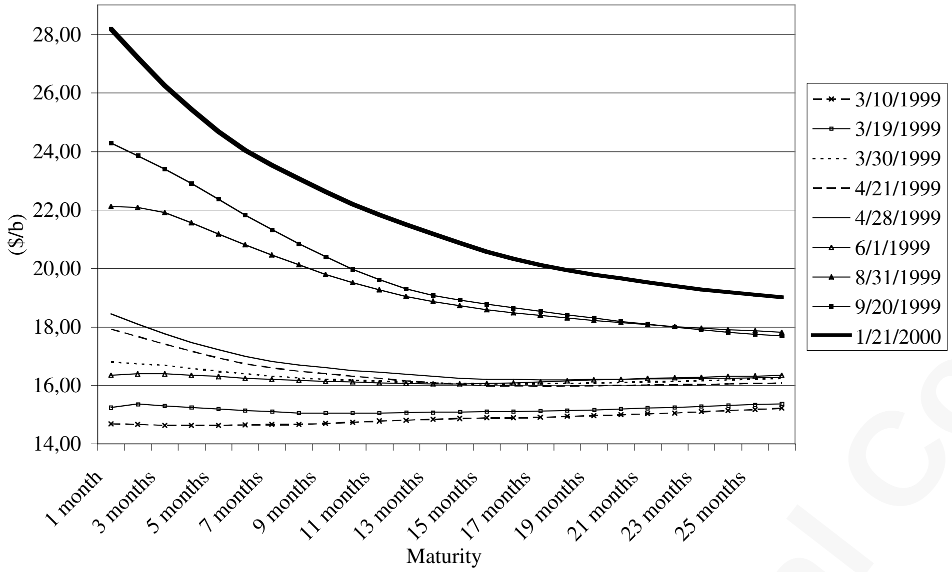
The Keynesian analysis can be extended rather simply. When the whole price curve is taken into account, the eventual simultaneous presence of contangos and backwardations along the curve can be explained by a surplus in the supply or demand of futures contracts for specific maturities. In order to palliate these unbalances, a risk premium is offered to the speculators, provided they accept to take a position in the futures market that compensates for the net position of the other operators. In the way of the preferred habitat theory developed for interest rates (Modigliani and Sutch [1966]), the term structure of commodity prices is then regarded as a succession of segments having different maturities. Market participants select their segment in accordance with their economic needs.

Such an extension of the Keynesian analysis amounts to removing the assumptions concerning the sign and the level of the risk premium. In this context, there are several imbalances between the contract's supply and demand and each segment of the prices curve is presumed to satisfy a specific economic need. Therefore, all the categories of operators do not necessarily intervene on all the maturities. The level of the premium they are willing to pay and the sense of market's imbalance can be different for each segment. Thus, the risk premium is a function of the maturity. Lastly, in order to take into account the eventual distortions of the price curves, this premium must be able to vary with the period, as the operators' expectations and risk aversion change.

As far as the storage theory is concerned, it at first takes into account the existence of a term structure of contangos and backwardations. Such a phenomenon is due to the seasonality of the supply or demand for the commodity. For example, the coexistence of a short-term

EXHIBIT 1

Fluctuation of WTI Futures Prices Curves



backwardation and a long-term contango is interpreted in the following way in the case of agricultural commodities: at the end of the crop year, the inventories reach their lower level. Then, the prices for delivery before and after the harvest reflect two different situations: there is a shortage for the deliveries before harvest, and the prices for these maturities are in backwardation. Simultaneously, prices for post-harvest delivery are in contango.

Thus, the storage theory easily lends itself a priori to an inter-temporal analysis of the prices relationship. However, when the contracts' expiration date exceeds one or two production cycles, one may ask whether the explanatory factors of this theory are still of use. Indeed, the long-term extension of the analysis raises new questions. When the horizon of analysis increases, can the shortages on goods be something other than accidental and unpredictable? Is it possible to invoke, in the long run, other factors than storage costs and convenience yield to explain the shape and the behavior of the prices' term structure?

Gabillon [1995] is the first to respond yes to this last question. Its theoretical analysis reconciles the theory of normal backwardation and the storage theory. It proposes to separate the term structure of crude oil prices into two distinct segments. Each part of the curve reflects a specific economic behavior of the operators. The first segment, corresponding to the shorter maturities (from the 1st to the 18th month), is mostly used for hedging purposes. As a result, production, consumption, stock level, and the fear of inventory disruptions are the most important explanatory factors of the prices relationship. However, for longer maturities the explanatory factors change: interest rates, anticipated inflation, and the prices for competing energies determine the futures prices. In that case, the information provided by the prices is used for investment purposes.

In this analysis, agents have preferred habitats: they are specialized in the holding of certain subsets of maturities and they are reluctant to alter their portfolio to take advantage of arbitrage opportunities. The latter are, therefore, left unexploited. Later, Lautier [2003] showed that there is a segmentation of the crude oil price curve, explained by liquidity factors, and that the segmentation evolves as the futures market matures. The segmentation in 2003 is situated around the 28th month, and not the 18th.

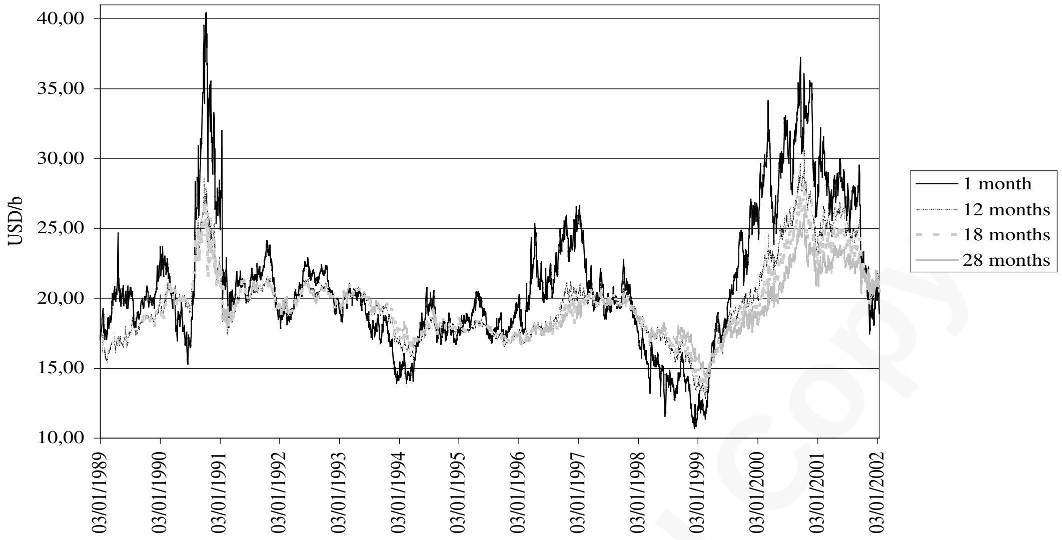
Thus, the extension of the analysis to long-term horizon is relatively natural in the case of normal backwardation theory. It becomes possible provided that one quits the Keynesian framework, which is too rigid. In regards to the storage theory, this enlargement is not simple; it is not sufficient enough to raise simplifying assumptions. Thus, new explicative factors of the prices relationship must be introduced.

Dynamic Analysis of the Term Structure

The most important feature of the commodity price curve's dynamic is probably the difference between the price behavior of first nearby contracts and deferred contracts. The movements in the prices of the prompt contracts are large and erratic, while the prices of long-term contracts are relatively still. This results in a decreasing pattern of volatilities along the price curve. Indeed, the variance of futures prices and the correlation between the nearest and subsequent futures prices decline with maturity. This phenomenon is usually called "the Samuelson effect." Intuitively, it happens because a shock affecting the

EXHIBIT 2

Daily Futures Prices, 1989-2002



nearby contract price has an impact on succeeding prices that decreases as maturity increases (Samuelson [1965]). Indeed, as futures contracts reach their expiration date, they react much more strongly to information shocks, due to the ultimate convergence of futures prices to spot prices upon maturity. These price disturbances influencing mostly the short-term part of the curve are due to the physical market, and to demand and supply shocks. Anderson [1985], Milonas [1986], and Fama and French [1987] have provided empirical support for this hypothesis for a large number of commodities and financial assets. Deaton and Laroque ([1992, 1996]) and Chambers and Bailey [1996] showed that the Samuelson effect is a function of the storage cost. More precisely, a high cost of storage leads to relatively little transmission of shocks via inventory across periods. As a result, futures price's volatility declines rapidly with the maturity. Lastly, in 1988, Fama and French showed that violations of the Samuelson effect might occur at a shorter horizon when inventory is high. In particular, price volatilities can initially increase with the maturity of the contract, because with enough inventories, stocks-outs may not be possible for the nearest delivery months.

Exhibit 1 provides an illustration of the Samuelson effect. It represents the deformations of the crude oil price term structure at different observation dates situated between March 1999 and January 2000. The futures prices correspond to the West Texas Intermediate (WTI) contract traded on the New York Mercantile Exchange (NYMEX). The nearest futures prices appear as more volatile than deferred prices. This phenomenon is especially clear between March and April 1999, when contango disappears and the curve enters into backwardation. Clearly, this transformation affects mostly the short-term prices.

Compared with other commodity markets, the crude oil market displays another characteristic: it is most of the time in backwardation. This phenomenon is well known and has been widely reported (see, e.g., Edwards and Canters [1995] and Litzenberg and Rabinowitz [1995]). This characteristic implies that the crude oil market has been extensively exploited to test the theory of normal backwardation. However, as previously stated, the literature has failed to explain this prevalence of inverted markets in petroleum products.

Exhibit 2 illustrates this characteristic of the crude oil futures market. It represents daily futures prices during 1989-2002 for selected maturities ranging from the first to the 28th months. The graphic shows that there is an alternation of backwardation and contango, and that backwardation is the most frequent situation, with some particularly strong peaks in 1990, in 1997, and in 2000.

Principal component analysis presents another strategy for dealing with the dynamic of futures prices (see, e.g., Cortazar and Schwartz [1994], Tolmasky and Hindanov [2002], and Lautier [2005]). This statistical method reduces the dimensionality of a data set by collapsing the information it contains. In data sets including many variables, groups of variables often move together because they are influenced by the same driving forces. In many systems, there are only a few such driving forces.

Applying a principal component analysis to crude oil futures price curves gives rise to three conclusions. Firstly, it leads to the identification of the type of price curve movements, which are quite simple to describe. Three different kinds of movements can indeed be distinguished: a parallel shift in the curve (level factor), a relative shift of the curve (steepness factor), and the curvature factor. Secondly, the principal component analysis makes it possible to calculate the contribution of each component to volatility. This calculation shows that, in the case of crude oil and of copper, the first two factors account for 99% of the total variance of the futures prices. Therefore, one can consider that most of the risk associated with futures price moves is accounted for two factors, instead of all futures prices.

TERM STRUCTURE MODELS OF COMMODITY PRICES

Term structure models of commodity prices aim to reproduce the futures prices observed in the market as accurately as possible. They also provide a mean for the discovery of futures prices for horizons exceeding exchange-traded maturities. In this section, we review the main term structure models of commodity prices, from the simplest (one-factor models) to the more sophisticated versions (three-factor models). Four different factors are generally used: the spot price, the convenience yield, the interest rate, and the long-term price. Because the models borrow from the contingent claim analysis developed for options and interest rates models, a reminder of the basic principles of contingent claim analysis will be introduced prior this presentation.

Basic Principles of Contingent Claim Analysis

The standard modeling procedure for pricing commodity derivatives typically follows the contingent claim analysis developed for options and interest rates. The term structure models of commodity prices share three assumptions: the market for assets is free of frictions, taxes, or transaction costs; trading takes place continuously; lending and borrowing rates are equal and there are no short sale constraints.

Then, the same method as the one developed in the context of interest rates is used to construct the model. First, the state variables (i.e., the uncertainty sources affecting the futures price) are selected and their dynamic is specified. Then, knowing that the price of a futures contract is a function of the state variables, the time, and the contract's expiration date, using Itô's lemma makes it possible to obtain the dynamic behavior of the futures price. Afterwards, arbitrage reasoning and the elaboration of a hedge portfolio leads to the term premium and to the fundamental valuation equation characterizing the model. Finally, whenever it is possible, the solution of the model is obtained.

However, the transposition of the theoretical framework developed for interest rates in the case of commodities is not straightforward. The reasoning is indeed based on the assumption that the market is complete: in such a market, a derivative asset can be duplicated by a combination of other existing assets. If the latter are sufficiently traded to be arbitrage free evaluated, they can constitute a hedge portfolio whose behavior replicates that of the derivative. Their proportions are fixed so that there are no arbitrage opportunities and the strategy is risk-free. Thus, the return of the portfolio must be the risk-free rate in equilibrium. Valuation is made in a risk-neutral world: it does not depend on the operators' risk aversion.

The transposition problem arises from the fact that commodity markets are not complete. Markets of real assets are far from being free of arbitrage opportunities, as is the case for most financial markets. Thus, valuation will probably not be realized in a risk-neutral world for commodities markets, and several risk-neutral probabilities may coexist.

One-Factor Models

A futures price is often defined as the expectation, conditionally to the available information at a date *t*, of the future spot price. Indeed, the spot price is the main determinant of the futures price. Thus, most one-factor models rely on the spot price.[1]

There have been several one-factor models in the literature on commodity prices. These models can be separated in step with the dynamic behavior that is retained for the spot price: a geometric Brownian motion, or a mean-reverting process. Moreover, the models can be distinguished in step with the assumption they retain concerning the convenience yield.

Spot Price with a Geometric Brownian Motion. Brennan and Schwartz [1985], Gibson and Schwartz [1989, 1990], Brennan [1991], and Gabillon [1992, 1995] use a geometric Brownian motion in their one-factor models.

Among these models, Brennan and Schwartz's model [1985] is the most well known. It has been extensively used in subsequent research on commodity prices (see, e.g., Schwartz [1998], Schwartz and Smith [2000], Nowman and Wang [2001], Cortazar, Schwartz, and Casassus [2001], and Veld-Merkoulova and de Roon [2003]).

The geometric Brownian motion is a dynamic commonly used to represent the behavior of stock prices. When applied to commodities, the spot price's dynamic is the following:

$$dS(t) = \mu S(t)dt + \sigma_S S(t)dz \tag{1}$$

where: S is the spot price;
μ is the drift of the spot price;
σ_S is the spot price volatility; and
dz is an increment to a standard Brownian motion associated with S.

The use of this representation implies that the variation of the spot price at t is supposed to be independent of the previous variations, and that the drift μ conducts the price's evolution. The uncertainty affecting this evolution is proportional to the level of the spot price: when stocks are rare, S is high; in this situation, any change in the demand has an important impact on the spot price, because inventories are not sufficiently abundant to absorb the prices' fluctuations.

Brennan and Schwartz [1985] also fix the definition of the convenience yield. Afterwards, authors always refer to this definition:

> The convenience yield is the flow of services that accrues to an owner of the physical commodity but not to the owner of a contract for future delivery of the commodity. . . . Recognizing the time lost and the costs incurred in transporting a commodity from one location to another, the convenience yield may be thought of as the value of being able to profit from temporary local shortages of the commodity through ownership of the physical commodity. The profit may arise either from local price variations or from the ability to maintain a production process as a result of ownership of an inventory of raw material.

The Brennan-Schwartz model is probably the most simple term structure model of commodity prices and consequently it is very popular. However, the geometric Brownian motion is probably not the best way to represent the price dynamic. Indeed, the storage theory and the Samuelson effect show that the mean-reverting process is probably more relevant.

Mean-Reverting Process. Schwartz [1997], Cortazar and Schwartz [1997], and Routledge, Seppi, and Spatt [2000] retain a mean- reverting process for their one-factor models. Among these models, that of Schwartz [1997], inspired by Ross [1995], is probably the most well known.[2] In that case, the dynamic of the spot price is the following:

$$dS = S\kappa(\mu - \ln S)dt + \sigma_S S dz_S \tag{2}$$

where: S is the spot price;
κ is the speed of adjustment of the spot price;
μ is the long-run mean log price;
σ_S is the spot price volatility; and
dz_S is an increment to a standard Brownian motion associated with the spot price.

In this situation, the spot price fluctuates around its long-run mean. The presence of a speed of adjustment insures that the state variable will always return to its long-term mean μ. Therefore, two characteristics describe the spot price behavior. It has a propensity to return to its long-term mean, but simultaneously, random shocks can move it away from μ.

The use of a mean reversion process for the spot price makes it possible to take into account the behavior of the operators in the physical market. When the spot price is lower than its long-run mean, the industrials, expecting a rise in the spot price, reconstitute their stocks, whereas the producers reduce their production rate. The increasing demand and the simultaneous reduction of supply have a rising influence on the spot price. Conversely, when the spot price is higher than its long-run mean, industrials try to reduce their surplus stocks and producers increase their production rate, thus pushing the spot price to lower levels.

This formulation of the spot price behavior is preferable to the geometric Brownian motion, but it is not without flaws. For example, the storage theory shows that in commodity markets, the basis does not behave similarly in backwardation and in contango. Initially, the mean-reverting process does not allow taking into account that characteristic.

The mean-reverting process was also used by Cortazar and Schwartz in 1997 in a more sophisticated model. The authors introduce a variable convenience yield that depends on the deviation of the spot price to a long-term average price.

Other One-Factor Models. Among the other one-factor models, the models studied by Brennan [1991] are quite interesting because they all rely on a specific hypothesis concerning the convenience yield, which is an endogenous variable. The first model was developed with Schwartz in 1985. In that case, the convenience yield is a simple linear function of the spot price. The second model expresses the convenience yield as a non-linear function of the price:

$$C(S) = a + bS + cS^2 \tag{3}$$

This formula is chosen because it is more flexible than the one used in 1985. However, it was afterwards forgotten, as the third model, which is also non-linear, was preferred.

The third model underlines that when the convenience yield is low, it cannot be lower than the opposite of the storage cost. The latter is supposed to be constant for a large spread of stock levels as long as the storage capacities are not saturated. The non-linearity comes from the fact that either the convenience yield is equal to the opposite of the storage cost, or it is proportional to the spot price:

$$C(S) = \max\,(a,\ b + cS) \tag{4}$$

Brennan's 1991 study, and particularly his pioneering use of a non-linear representation of the behavior of the convenience yield, has stimulated a great deal of further research. Brennan's findings, as well as those of Gibson and Schwartz [1990], lead however to the conclusion that there are limits to one-factor models.

Two-Factor Models

The homogeneity in the choice of the state variables disappears when a second stochastic variable is introduced in term structure models of commodity prices. Most of the time, the second state variable is the convenience yield. However, models based on long-term price or on volatility of the spot price have also been developed. In all such models, the introduction of a second state variable allows for richer shapes of curves than one-factor models (especially for long-term maturities) and richer volatility structures. Yet this improvement is rather costly, because two-factor models are more complex.

The Convenience Yield as the Second State Variable. Schwartz's model [1997] is probably the most famous term structure model of commodity prices. It was used as a reference to develop several models that are more sophisticated (Hilliard and Reis [1998]; Schwartz [1998]; Neuberger [1999]; Schwartz and Smith [2000]; Lautier and Galli [2001]; Yan [2002]; Richter and Sorensen [2002]; Veld-Merkoulova and de Roon [2003]).

Inspired by the one proposed by Gibson and Schwartz in 1990, the latest model is more tractable than its former version: it has an analytical solution. The two-factor model supposes that the spot price *S* and the convenience yield *C* can explain the behavior of the futures price *F*. The dynamic of these state variables is:

$$\begin{cases} dS = (\mu - C)Sdt + \sigma_S S dz_S \\ dC = \left[k(\alpha - C)\right]dt + \sigma_C dz_C \end{cases} \tag{5}$$

with: $\kappa, \sigma_S, \sigma_C > 0$
where: μ is the drift of the spot price;
σ_S is the spot price volatility;
dz_S is an increment to a standard Brownian motion associated with *S*;
α is the long-run mean of the convenience yield;
κ is the speed of adjustment of the convenience yield;
σ_C is the volatility of the convenience yield; and
dz_C is an increment to a standard Brownian motion associated with *C*.

The idea of a mean-reverting process is retained in this model. However, it is applied to the convenience yield. This choice is not incoherent with what was done in the case of one-factor models, because the convenience yield intervenes in the spot price dynamic and gives it—though marginally—a mean-reverting tendency. The convenience yield becomes then a stochastic dividend affecting the spot price dynamic. Such a formulation is a good illustration of the fact that the convenience yield is an implicit revenue associated with physical stocks. Moreover, it authorizes some comparisons with other financial assets such as bonds and securities.

When applied to the convenience yield, the Ornstein-Uhlenbeck process relies on the hypothesis that there is a regeneration property of inventories, namely that there is a level of stocks which satisfies the needs of industry under normal conditions. The behavior of the operators in the physical market guarantees the bearing of this normal level. When the convenience yield is low, the stocks are abundant and the operators sustain a high storage cost compared with the benefits related to holding the raw materials. Therefore, if they are rational, they try to reduce these surplus stocks. Conversely, when the stocks are rare the operators tend to reconstitute them.

Moreover, as the storage theory shows, the two state variables are correlated. Both the spot price and the convenience yield are indeed an inverse function of the inventory level. Nevertheless, as Gibson and Schwartz [1990] have demonstrated, the correlation between these two variables is not perfect. Therefore, the increments to standard Brownian motions are correlated, with:

$$E\left[dz_S \times dz_C\right] = \rho dt \qquad (6)$$

where ρ is the correlation coefficient.

This model, which is quite tractable, presents a limit. Indeed, it ignores that in commodity markets, the price's volatility is positively correlated with the degree of backwardation. This phenomenon has been widely reported and commented (see, e.g., Williams and Wright [1991], Ng and Pirrong [1994], and Litzenberg and Rabinowitz [1995]) and can be explained by the examination of arbitrage relationships between the physical and the futures markets. Such a study shows that the basis has an asymmetrical behavior: in contango, its level is limited to storage costs. This is not the case in backwardation:

> Arbitrage can always be relied upon to prevent the forward price from exceeding the spot price by more than net carrying cost . . . [but] can not be equally effective in preventing the forward price from exceeding the spot price by less than net carrying cost (Blau [1944]).

Furthermore, the basis is stable in contango, and volatile in backwardation, since in the latter situation stocks cannot absorb price fluctuations. This phenomenon can lead to the assumption that the convenience yield is an option (Heinkel, Howe, and Hughes [1990]; Milonas and Thomadakis [1997]; Milonas and Henker [2001]) or that it exhibits asymmetrical behavior. This assumption has been introduced in term structure models by Brennan [1991], Routledge, Seppi, and Spatt [2000], and Lautier and Galli [2001].

Brennan [1991] introduces an asymmetric convenience yield in his model because he takes into account a non-negativity constraint on inventory. However, he supposes that the convenience yield is deterministic. In the model presented by Routledge et al., the asymmetry in the behavior of the convenience yield is introduced in the correlation between the spot price and the convenience yield. This correlation is higher in backwardation than in contango. The convenience yield is endogenous, and it is determined by the storage process. However, it is stochastic. The two factors are the spot price and exogenous transitory shocks affecting supply and demand. Lautier and Galli [2001] propose a two-factor model inspired by the Schwartz model [1997], in which the convenience yield is also mean reverting and acts as a continuous dividend. However, an asymmetry is introduced in the convenience yield dynamic: it is high and volatile in backwardation, when inventories are rare. It is conversely low and stable when inventories are abundant. The asymmetry is measured by the parameter β. When the latter is set to zero, the asymmetrical model is reduced to Schwartz's model.

The Long-Term Price as a Second State Variable. Another approach of the term structure of commodity prices consists in considering the decreasing pattern of volatilities along the price curve. In that situation, it is possible to infer that the two state variables correspond to the two extremities of the price curve, the spot and the long-term price. This kind of approach was followed by Gabillon [1992] and Schwartz and Smith [2000].

Gabillon [1992] uses the spot and long-term prices as state variables. In this model, the convenience yield is an endogenous variable, dependent on the two factors. The use of the long-term price as a second state variable is justified by the fact that this price can be influenced by elements that are exogenous to the physical market, such as expected inflation, interest rates, or prices for renewable energies. Thus, the spot and long-term prices reassemble all the factors allowing the description of the term structure movements. The author retains a geometric Brownian motion to represent the behavior of the long-term price. Moreover, the two state variables are assumed to be positively correlated.

Schwartz and Smith [2000] propose a two-factor model where the state variables come from the decomposition of the spot price. This decomposition authorizes the distinction between short-term variations and long-term equilibrium:

$$\ln(S_t) = \chi_t + \xi_t \qquad (7)$$

where: S_t is the spot price at t;
χ_t is the short-term deviation in prices; and
ξ_t is the equilibrium price level.

These two factors are not directly observable, but they are estimated from spot and futures prices. More precisely, the movements in prices for long-maturity futures contracts provide information about the equilibrium price level, whereas the differences between the short- and long-term futures prices provide information about short-term variations in prices. This model does not explicitly consider changes in convenience yields over time, but it is equivalent to the two-factor model proposed by Gibson and Schwartz [1990], in that the state variables in one of the models can be expressed as linear combinations of the state variables in the other model.

The short-term deviation is assumed to revert to zero, following an Ornstein-Uhlenbeck process. The equilibrium level is assumed to follow a Brownian motion process. Thus, the dynamic of these two state variables is the following:

$$\begin{cases} d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \\ d\xi_t = \mu dt + \sigma_\xi dz_\xi \end{cases} \tag{8}$$

where: κ is the speed of adjustment of the short-term deviation;
σ_x is the volatility of the short-term prices;
dz_x is an increment to a standard Brownian motion associated with χ_t;
μ is the drift of the equilibrium price level;
σ_ξ is the volatility of the equilibrium price level; and
dz_ξ is an increment to a standard Brownian motion associated with ξ_t.

Changes in the short-term deviations represent temporary changes in prices (caused by abrupt weather alteration, supply disruption, etc.) and are not expected to persist. They are tempered by the ability of market participants to adjust to inventory levels in response to changing market conditions. Changes in the long-term level represent fundamental modifications which are expected to persist. The latter are due, for example, to shifts in the number of producers in the industry. The long-term equilibrium is also determined by expectations of exhausting supply, improving technology for the production and discovery of the commodity, inflation, as well as political and regulatory effects.

The most important advantage of this model is that it avoids the questions concerning the convenience yield, its estimation, and its economic significance (on this particular point, see, e.g., Williams and Wright [1989], Brennan, Williams, and Wright [1997], and Frechette and Fackler [1999]). The idea of a long-term equilibrium is also in line with recent works on long memory[3] in the commodity futures markets. Long memory in convenience yields has been studied by Mazaheri [1999], and Barkoulas, Labys, and Onochie [1999] showed that there is long memory in futures prices. However, using the long-term price as a state variable introduces a new problem: is it interesting to represent a stable equilibrium with a stochastic variable?

Seasonality. Apart from the nature of state variables, research has been conducted on the seasonality of commodity prices. In this field, Gabillon once again forged new territory in 1992 with a model including a seasonal function as a composite of sine and cosine functions. The same type of formalization was retained by Richter and Sorensen [2002]. However, the latter model takes the spot price and the convenience yield as state variables, whereas Gabillon retains the spot and the long-term prices.

Three-Factor Models

Until 1997, every term structure model of commodity prices assumed the interest rate constant. Such a hypothesis amounts to saying that the term structure of interest rates is flat, which is all the more reductive as the horizon of analysis is remote. Schwartz [1997] proposes a model including three state variables: the spot price, the convenience yield, and the interest rate. The latter has a mean-reverting behavior.

The introduction of a stochastic interest rate in the analysis of price relationships is important from a theoretical point of view: the assumption of a constant interest rate amounts to saying that futures and forward prices are equivalent, which is not the case (Cox, Ingersoll, and Ross [1981]). With a stochastic interest rate, it is possible to take into account the margin call mechanism of the futures market. Thus, two distinct payoff structures can be taken into account for futures and forward contracts. Finally, the presence of the interest rate as a third explicative factor of the futures price is consistent with the storage theory.

Since 1997, several three-factor models have been proposed. In 1998, Hilliard and Reis modified the three models proposed by Schwartz in 1997. They introduced jumps in the spot price process in order to take into account the large and abrupt changes due to supply and demand shocks affecting certain commodity markets, especially the energy commodities used for heating. In

2000, Schwartz and Smith proposed an extension of their short-term/long-term model in which the growth rate for the equilibrium price level is stochastic. Such an extension improves the model's ability to fit long-term futures prices. Another improvement of three-factor models was proposed by Yan in 2002. His model incorporates stochastic convenience yields, interest rates, and volatility. He also introduces simultaneous jumps in the spot price and volatility. The convenience yield follows an Ornstein-Uhlenbeck process, whereas the interest rate follows a square-root process and the volatility follows a square-root jump-diffusion process. However, Yan finds that stochastic volatility and jumps do not alter the futures price at a given point in time. Nevertheless, they play important roles in pricing options on futures.

Lastly, Cortazar and Schwartz [2003] proposed a three-factor model related to Schwartz [1997], in which all three factors are calibrated using only commodity prices.[4] In this model, the authors consider as a third risk factor the long-term spot price return, allowing it to be stochastic and to return to a long-term average. The two other stochastic variables are the spot price and the convenience yield. The convenience yield models temporary variations in prices due to changes in inventories, whereas the long-term return is due to changes in technologies, inflation, or demand pattern. The dynamic of the state variable is the following:

$$\begin{cases} dS = (v - y)Sdt + \sigma_1 Sdz_1 \\ dy = -\kappa ydt + \sigma_2 dz_2 \\ dv = a(\bar{v} - v)dt + \sigma_3 dz_3 \end{cases} \tag{9}$$

where: S is the spot price;
y is the demeaned convenience yield, with $y = C - \alpha$, where α is the long run mean of the convenience yield C;
v is the expected long-term spot price return, with $v = \mu - \alpha$, where μ is the drift of S;
κ is the speed of adjustment of the demeaned convenience yield;
a is the speed of adjustment of v;
$\bar{v}$ is the long-run mean of the expected long-term spot price return;
σ_i is the volatility of the variable i; and
dz_i is the increment of a standard Brownian motion associated with the variable i.

In practice, the development of three-factor models posits the question of the arbitrage between reality and simplicity. Although the introduction of a third factor may improve the performance of the models in terms of their ability to describe the evolution of futures prices, there is always a balance to be struck between the fidelity of the models and the need for parsimony, especially when the models are conceived for the evaluation of more complex derivatives products.

TERM STRUCTURE MODELS AND THE DESCRIPTION OF PRICE CURVES

This section reviews the main empirical results obtained with term structure models of commodity prices. First of all, simulations highlight the influence of the assumptions concerning the stochastic process retained for the state variables and the impact of the number of state variables. Then, in order to test the model, parameter values are needed. Thus, the methods generally used for the estimation of the parameters are set forth. Lastly, the models' performances, namely their ability to reproduce the term structure of commodity prices, are presented.

Simulations

Simulations make it possible to compare the curves extracted from the models and to appreciate how realistic the models are. Among the different term structure models presented below, two are retained for simulation purposes: the Brennan-Schwartz model [1985] and the two-factor Schwartz model [1997]. These models are probably now the most well known and they are extensively used. The values retained for the state variables and parameters are inspired by empirical tests carried out on the crude oil market (Schwartz [1997], Lautier and Galli [2001], Lautier [2003]). Their maximal and minimal levels are presented in Exhibit 3.

Brennan-Schwartz Model [1985]. In Brennan and Schwartz's model, the relative values of the two parameters (the interest rate *r* and the convenience yield *c*) determine the whole shape of the term structure of futures prices. When the interest rate is superior to the convenience yield, the curve is in contango. Conversely, the curve is in backwardation. Exhibit 4 illustrates that with this model, price curves can be only monotonically decreasing, monotonically increasing, or flat. The growth rate of the futures price is indeed a constant:

EXHIBIT 3

Values Retained for State Variables and Parameters

	S	C	r	α	κ	σ_S	σ_C	ρ	λ
Min	12	-0.3	0.02	-0.1	0.5	0.1	0.3	0	-1
Max	20	0.2	0.06	0.1	2	0.5	0.7	1	1

$$\partial F / \left(F\delta\tau\right) = r - c \tag{10}$$

When the difference ($r - c$) is positive (as is the case when c is set to –0.2) prices are upward sloping and they can reach a level without real economic significance: almost US\$65 per barrel for a seven-year expiration date. Conversely, when the growth rate is negative, the curve is in backwardation and prices tend toward zero.

Moreover, this model considers that the convenience yield is a constant, and it supposes that the volatility of the returns is the same for all the maturities:

$$\delta F / F = \sigma_S dz \tag{11}$$

Therefore, Brennan and Schwartz's model presents important drawbacks, especially for long-term analysis. However, its simplicity renders it very tractable and it is still in use today.

Schwartz Model [1997]. With Schwartz's model, we can obtain various shapes of price curves, as is shown in Exhibit 5, which presents simulations with different values of convenience yields: the curves can be sunken ($C = 0.1$), humped ($C = –0.3$), or flat ($C = 0$). The level of the futures prices is a decreasing function of the convenience yield: the more the latter increases, the more the price level diminishes.

The simulations also show that the gap between the convenience yield and its long-run mean α influences the shape of the prices curves. When the convenience yield is far from its long-run value ($C = –0.3$ and $\alpha = 0$), it takes 3.4 years until the curve becomes stable. When conversely the convenience yield is equal to its long-run mean ($C = \alpha = 0$), the growth rate of the futures prices becomes stable as early on as 1.8 years.

Another interesting series of simulations can be made regarding the speed of adjustment of the convenience yield. The latter are illustrated by Exhibit 6. They show that the growth rate of the futures prices rises when the speed of adjustment diminishes. The gap between the nearest and the farthest maturities increases when mean reversion decreases.

Thus, the introduction of a second state variable makes it possible to obtain richer price curves than with a one-factor model (this remark is also valuable in the field of interest rates). Furthermore, the Schwartz model is more realistic than that of Brennan and Schwartz, because in the two-factor model the volatility of the futures prices decreases with the maturity τ:

$$\sigma_F^2(\tau) = \sigma_S^2 + \sigma_C^2\left(\frac{1-e^{-\kappa\tau}}{\kappa}\right)^2 - \left[2 \times \frac{1-e^{-\kappa\tau}}{\kappa} \times \rho\sigma_S\sigma_C\right] \tag{12}$$

When the contract reaches its expiration date, the futures price's volatility converges toward the spot price's volatility.

EXHIBIT 4

Brennan and Schwartz Model: Impact of a Variation in the Convenience Yield

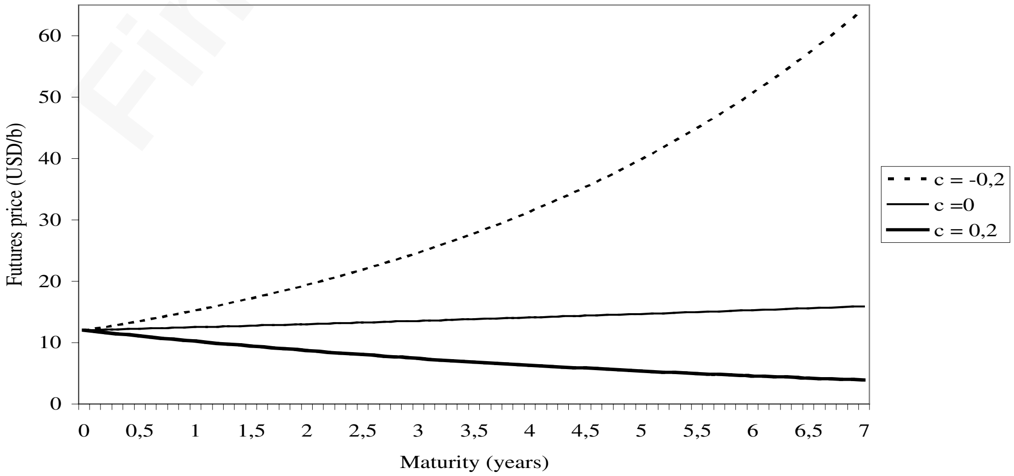


EXHIBIT 5

Schwartz Model Impact of Variation in the Convenience Yield

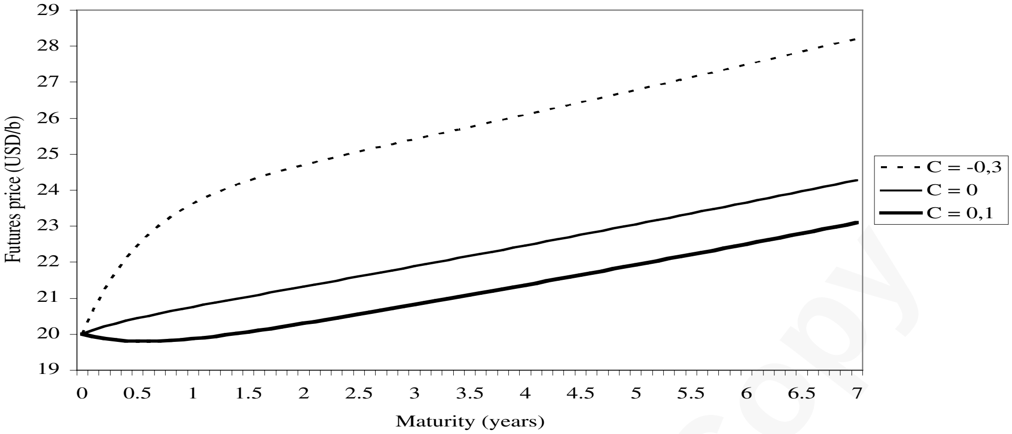
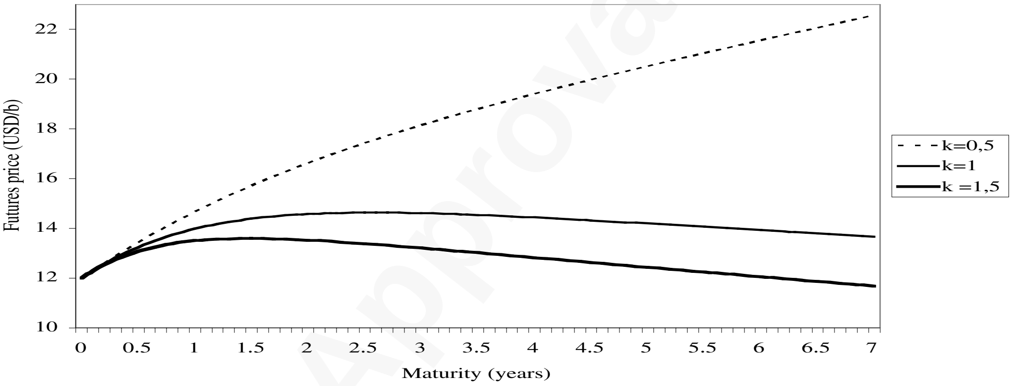


EXHIBIT 6

Schwarz Model: Impact of a Variation in the Speed of Adjustment



Conversely, when the maturity tends toward infinity, the volatility of the futures price tends toward a fixed value:

$$\lim_{\tau \to \infty} \sigma_F^2 = \sigma_S^2 + \frac{\sigma_C^2}{\kappa^2} - \frac{2\rho\sigma_S\sigma_C}{\kappa} \qquad (13)$$

However, the two-factor model is also more complex, because it includes six parameters, as opposed to two for the one-factor model.

Parameter Estimation

To assess the performance of a model, parameter values are needed in order to compute the estimated prices and to compare them with empirical data. The parameter estimation proves tricky, however, because many term structure models rely on non-observable variables. In order to overcome this difficulty, a method that was retained by Schwartz in 1997, i.e., a Kalman filter, can be used.

Non-Observable State Variables and Their Approximations. The non-observable state variables are, in the case of models of commodity prices, the spot price, the convenience yield, and the long-term equilibrium price level.

The non-observable nature of the spot price implies that this problem concerns all the models. The usual way to deal with the non-observable nature of the spot price consists in retaining the nearest futures price as an approximation. The spot price is regarded as non-observable because in most commodities markets there is a lack of reliable time series of spot prices: physical markets are geographically dispersed, transaction are not standardized, the price reporting mechanism does not require the operators to disclose their transaction prices, and so on. In the case of the United States crude oil market, spot prices are also affected by other problems specific to this market. The transactions volume for the West Texas Intermediate grade, namely the underlying commodity of the futures contract, is very low and the spot prices provide information only on local supply and demand. Moreover, there are sometimes problems due to an undercapacity of the pipeline system, which can create price jumps. The latter are due to the delivery system, not to general market conditions. This phenomenon, reported by Horsnell and Mabro [1993], is known as the "Cushing Cushion," because usually difficulties arise at Cushing, Oklahoma.

The convenience yield is also a non-observable variable because it does not correspond to a traded asset. The approximation method usually chosen for this variable consists in using the solution of the Brennan-Schwartz model [1985]. The calculation requires the use of two prices: the nearest and the subsequent futures prices. Let us denote the maturities of these prices as T_1 and T_2. The convenience yield c is then:

$$c = r - \frac{\ln\left(F(S,t,T_1)\right) - \ln\left(F(S,t,T_2)\right)}{T_1 - T_2} \tag{14}$$

Lastly, the long-term price presents the same characteristic as the convenience yield: it is not a traded asset. To overcome this difficulty, Schwartz and Smith [2000] use Kalman filtering techniques, which were previously used by Schwartz in 1997.

Kalman Filters. The main principle of Kalman filters is to use temporal series of observable variables in order to reconstitute the values of non-observable variables. In finance, the problem of non-observable variables is not unique to commodity prices. It also arises, for example, with term structure models of interest rates, with market portfolios in the capital asset pricing model, and with credit risk. When associated with an optimization procedure, the Kalman filter provides a way to estimate the model parameters. Finally and most importantly, because it is very fast, the method is also useful for large data sets.

There are different versions of Kalman filters.[5] The simple filter is the most well known, and it is quite frequently used in finance nowadays.[6] Nevertheless, it is not suitable for non-linear models. In that case, an extended filter can be used (Javaheri, Lautier, and Galli [2003]). However, the latter relies on an approximation that influences model performance: the extended filter leads to less precise results than the simple one. Nevertheless, it is still acceptable in the case of term structure models of commodity prices. Apart from the linearization, the two filters rely on the same principles.

The Kalman filter is an iterative process. To use it, the model has to be expressed in a state-space form characterized by a transition equation and a measurement equation.[7] The transition equation describes the dynamics of the state variables $\tilde{\alpha}$, for which there are no empirical data. During the first step of the iteration—the prediction phase—this equation is used to compute the values of the non-observable variables at time t, conditionally on the information available at time $(t-1)$. The predicted values $\tilde{\alpha}_{t/t-1}$ are then introduced into the measurement equation to determine the values of the measures $\tilde{y}_t$. The measurement equation is the relationship linking the observable variables $\tilde{y}$ with the non-observable $\tilde{\alpha}$. In the second iteration step—or innovation phase—the innovation v_t, which are the differences at t between the measures $\tilde{y}_t$ and the empirical data y_t, are calculated. The innovations are used in the third iteration step—or updating phase—to obtain the values of $\tilde{\alpha}_t$ conditionally on the information available at t. Once this calculation has been made, $\tilde{\alpha}_t$ are used to begin a new iteration. Thus, the Kalman filter makes it possible to evaluate the non-observable variables $\tilde{\alpha}$, and it updates their values in each step using the new information.

This brief presentation explains why the Kalman filter is a very fast method. Indeed, to reconstitute the temporal series of the non-observable variables, only two elements are necessary: the transition equation and the innovation v. Because there is an updating phase in the iteration, very little information is needed.

Model Performance

The performance of a model is measured by its ability to reproduce the term structure of commodity prices. To assess performance, criteria are needed. We first present these criteria. Then, we expose the main empirical results obtained with the models.

Performance Criteria. Two criteria are usually retained to measure the performance of a term structure model: the mean pricing error (MPE) and the root mean-squared error (RMSE):

The MPE is defined as follows:

$$MPE = \frac{1}{N}\sum_{n=1}^{N}\left(\tilde{F}(n,\tau) - F(n,\tau)\right)$$

where N is the number of observations, $\widetilde{F}(n, \tau)$ is the estimated futures price for maturity τ at the date n, and $F(n, \tau)$ is the observed futures price. The MPE measures the estimation bias for a given maturity. If the estimation is good, the MPE should be very close to zero.

Using the same notation, the RMSE is, for a given maturity τ:

$$RMSE = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\tilde{F}(n,\tau) - F(n,\tau)\right)^2}$$

The RMSE can be considered as an empirical variance which measures the estimation stability. This second criterion is considered as more representative because price errors can offset themselves and the MPE can be low even if there are strong deviations.

Empirical Results. We first describe some general features of the empirical tests carried out with term structure models. Then, the empirical results obtained with one-factor, two-factor, and three-factor models are presented.

General Features. The term structure models share two general features. Firstly, parameters change with the study period (Schwartz [1997]). Thus, when using the term structure models, parameters should be recomputed regularly. This can become a problem if the model has no analytical solution, because of the computing time. Secondly, parameters vary with the maturity (Schwartz [1997]; Lautier [2003]). When the convenience yield is stochastic and characterized by a mean-reverting process, its speed of adjustment is a decreasing function of the maturity. Indeed, mean reversion concerns the inventories, which are of little importance for long-term maturities. The same kind of explanation can be evoked for the spot price volatility and the convenience yield volatility: their level decreases with the maturity because the shocks on supply and demand have least impact on the long-term futures prices. These parameter changes lead to the conclusion that, ideally, the parameters of the term structure models should depend on the maturity.

One-Factor Models. The empirical tests carried out with one-factor models generally lead to the conclusion that their performance is bad. This result, however, does not apply systematically for all commodities. Comparing the empirical results obtained with three one-factor models for several commodity markets, Brennan [1991] showed that the convenience yield is close to zero for precious metals and is positive for industrial commodities. He interprets that phenomenon as the result of differences in the motivations of the operators holding precious metals and industrial commodities. Precious metals are essentially held for speculative reasons. Inventories are high and they constitute a reserve of value rather than the input of a production process. In that situation, the storage cost is very low compared with the stocks' value. Therefore, the convenience yield plays a marginal role for this specific category of commodities, and one-factor models are suited for them.

Schwartz [1997] confirms this empirical finding. He also validates the assumption of a mean-reverting process for the spot price: indeed, he shows that, in the crude oil and copper markets, the parameter representing the speed of adjustment is statistically significant. Moreover, this dynamic is suited for industrial commodities, but it is not a good way to represent spot price behavior in the case of gold.

Two-Factor Models. Several empirical tests demonstrate the superiority of two-factors models over one-factor models (Brennan [1991]; Schwartz [1997]; Schwartz and Smith [2000]). In all these cases, the performance of the models is greatly improved by the introduction of a second state variable. The tests also show that the mean-reverting process is suited to represent the dynamic behavior of the convenience yield (Brennan [1991]; Schwartz [1997]). The parameter representing the speed of adjustment is significantly different from zero. The same result is obtained for the long-run mean of the convenience yield.

An example of the two-factor model's ability to reproduce the shape of the term structure of commodity prices is given by Exhibit 7, which represents the performance of the Schwartz model on the crude oil market, from 1998 to 2001, for futures prices of one-, three-, six-, and nine-month maturities.

Thus, the performances can be quite good: the average MPE is around 6 cents per barrel on the period. Moreover, they are still excellent when the maturity of the contracts is extended up to seven years (Lautier [2003]). A graphical representation also shows that the model is able to reproduce the price dynamic quite precisely even if, as in 1998-2001, there are very large fluctuations in the futures prices. Exhibit 8 shows the results obtained for the one month's futures prices. During that period, the

EXHIBIT 7
Performance of Schwartz Model on the Crude Oil Market, 1998-2001

Maturity	MPE	RMSE
1 month	- 0.0604	2.3197
3 months	- 0.1078	1.9894
6 months	- 0.0545	1.7152
9 months	- 0.0073	1.5675
Average	***- 0.0575***	***1.8980***

Source: Lautier and Galli [2004].

crude oil price jumps from US$11 per barrel to US$37 per barrel.

Finally, it is possible to underline the model's ability to reproduce the evolution of prices curves through time. Exhibit 9 represents six term structures of crude oil prices, for different maturities, observed on the NYMEX between August 9 and September 14, 1999. During this period, the curves are always in backwardation. Moreover, the intensity of the backwardation increases as the futures prices for all maturities rise.

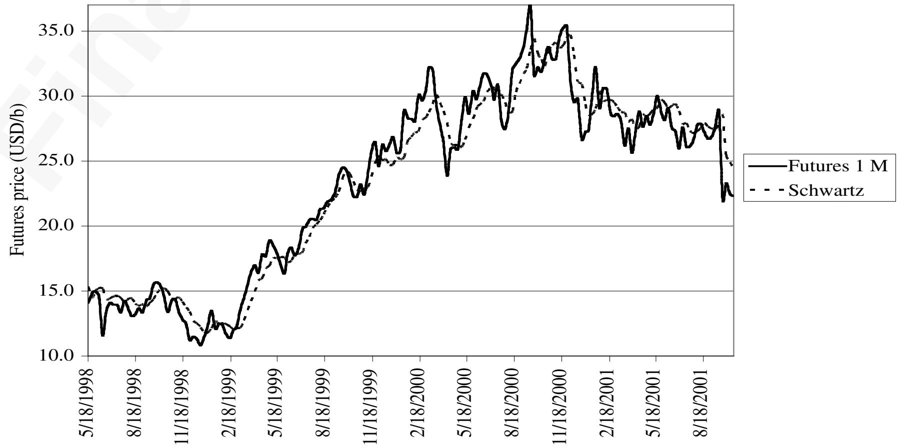
Exhibit 10 shows how the model reproduces this evolution. It represents, for the same observation dates, the estimated term structure of crude oil prices. The model is able to replicate correctly not only the displacement toward the heights, but also the slope's intensification. However, in this example, the estimated price curves decrease more regularly than those observed empirically.

Empirical tests (Lautier and Galli [2001]) also show that the introduction of an asymmetry in the convenience yield behavior improves the performance of the model.

All models including the convenience yield as a second state variable demonstrate a high correlation between the spot price and the convenience yield. Most of the time, the level of the correlation coefficient attains 0.9 for the Schwartz model (Schwartz [1997]; Lautier and Galli [2001]; Lautier [2003]). Therefore, the correlation is so high that one might ask whether the choice of the convenience yield as the second state variable is relevant. In reply, the selection of the two state variables of the short-term/long-term model offers an improvement because the short-term and the long-term deviations are more "orthogonal" in their dynamics (the correlation between these two variables is lower: it was estimated by Schwartz and Smith at 0.189 and 0.3 for two datasets). This orthogonality allows for a clearer distinction of the impact of each factor. However, one might also ask whether the long-term price, since it is fairly stable, is a good candidate for a stochastic variable.

Three-Factor Models. The empirical results obtained by Schwartz [1997] with regard to his three-factor model raise questions as to the relevance of such models. Indeed, using long-term forward prices, Schwartz shows the two- and three-factor models to be empirically very similar. Indeed, these models have very similar structures of volatility. This result is consistent with the principal component analysis of term structure performed on the copper and crude oil markets.

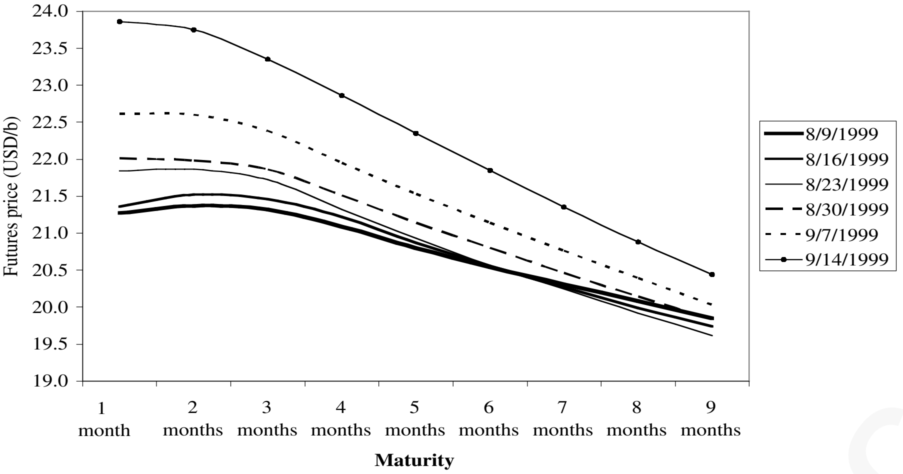
EXHIBIT 8
Estimated and Observed Futures Prices for the One-Month Maturity, 1998-2001



Source: Lautier and Galli [2004].

EXHIBIT 9

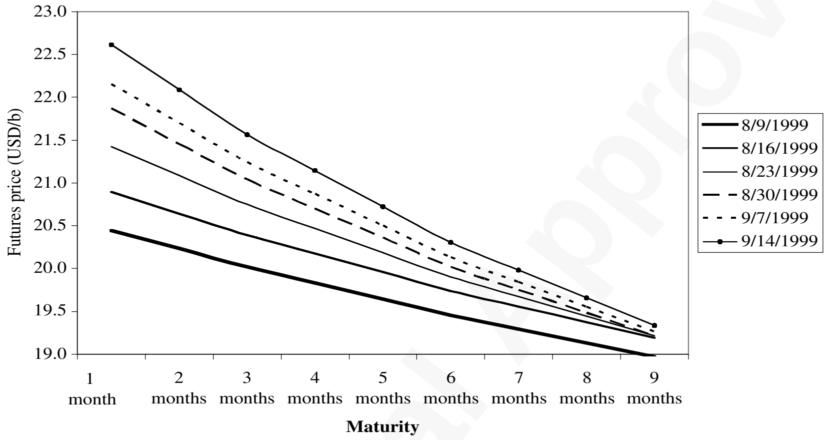
Observed Term Structures of Crude Oil Prices at Different Dates



Source: Lautier and Galli [2002].

EXHIBIT 10

Estimated Term Structure of Crude Oil Prices



Source: Lautier and Galli [2002].

APPLICATIONS OF TERM STRUCTURE MODELS

Two important applications have been considered for term structure models of commodity prices: dynamic hedging strategies and investment decisions. These two applications rely on the relationship between futures prices of different maturities.

Dynamic Hedging Strategies

The first application of term structure models is the hedge of long-term commitments on commodity markets when "naïve" hedging strategies are neglected. Naïve strategies suggest taking a short (long) position in the futures market in order to hedge a long (short) position in the physical market, the two positions having the same size and expiration date. These strategies cannot be initiated when the position to hedge has a horizon superior to the exchange-traded contracts. Whereas this kind of problem has been tackled by Ederington [1979], it acquires a new dimension with term structure models. Indeed, these models rely on arbitrage reasoning and on the construction of a hedge portfolio. Therefore, their elaboration leads naturally to the study of hedging strategies.

Reflection on the use of term structure models for hedging purposes was motivated by Metallgesellchaft's experiment. At the beginning of the 1990s, the firm tried to cover long-term forward commitments on the physical market with short-term futures contracts. This attempt ended in a resounding failure: US$2.4 billion were lost. However, it initiated research, the goal of which was knowing whether such an operation could be safely undertaken. The interest of the Metallgesellschaft case is twofold. Firstly, being able to have long-term positions on the physical market while hedging the product of such sales constitutes an important stake. The success encountered by Metallgesellschaft when it proposed forward sales for a horizon of 5 or 10 years is indicative of that point. Secondly, Metallgesellschaft tried to hedge its long-term commitments in the physical market with short-term positions in the futures market. This kind of strategy is particularly interesting when the maturity of actively traded futures contracts is limited to a few months. The use of the nearest maturities supposes, however, that the hedge portfolio is rebalanced regularly as the futures' expiration date approaches, in order to constantly maintain the position on the paper market. Thus, this strategy entails rollover basis risk, which, associated with a bad hedging ratio, contributed to the ruin of Metallgesellschaft.[8]

The various hedging strategies based on term structure models differ from each other mainly in the assumptions concerning the behavior of the futures prices. Brennan and Crew [1997] compared the hedging strategy initiated by Metallgesellschaft on the crude oil market with other strategies relying on the models of Brennan and Schwartz [1985] and Gibson and Schwartz [1990]. The authors showed that the strategies relying on the term structure models outperform by far that of the German firm, all the more as the term structure model is able to correctly replicate the price curve empirically observed. Schwartz [1997] also calculated the hedge ratios associated with each of the three models he studied. However, he did not test the efficiency of the related hedging strategies. Neuberger [1999] compared the performance of hedging strategies relying on the two-factor model developed by Schwartz in 1997 and on a new model. In this new model, no assumption is made on the number of variables, about the process they follow, or about the way risk is priced. The key assumption is that the expected price at which the long-dated contract starts trading is a linear function of the price of existing contracts (whereas in Schwartz's model, the futures price is a non-linear function of the state variables). While theoretically inconsistent with some models of the term structure of commodity prices, in practice this new model still gives good results. With this model, the author constructed hedge portfolios based on an arbitrary number of futures contracts with different maturities. His strategy was quite successful in eliminating most of the risk exposure (approximately 85% as measured by the standard deviation of the hedge error) in the crude oil market. The author believes that the robustness of his model derives from the paucity of assumptions. The drawback of hedging strategies based on term structure models is that the latter are efficient only if futures prices are fairly priced relative to each other. However, Neuberger underlines that his model requires balancing the portfolio a bit more frequently than Schwartz's. Routledge, Seppi, and Spatt [2000] showed that hedge ratios based on their term structure model are not constant, but rather conditional on the current demand shock and the endogenous inventory level. These ratios take into account the fact that short-term and long-term forward prices differ, in that long-term prices do not include the option to consume the good between the two delivery dates. Lastly, Veld-Merkoula and de Roon [2003] used a one-factor term structure model based on convenience yield to construct hedge strategies that minimize both spot price risk and rollover risk by using futures of two different maturities. They take into account the transaction costs associated with their strategy, which outperforms the naïve hedging strategy. However, the authors do not compare their results with previous works.

These studies of hedging strategies share some general features. Firstly, the hedging problem is always approached modeling the relationship between the futures prices and using combinations of futures contracts having different maturities. The number of futures positions is at least equal to the number of underlying factors (i.e., state variables) included in the term structure model. Secondly, to properly hedge a forward commitment, the sensitivity of the present value of the commitment with respect to each one of the underlying factors must equal the sensitivity of the portfolio of futures contracts used to hedge the commitment with respect to the same factors. Therefore, the hedge ratios are state dependent and they decline with the maturity of the forward position. Thirdly, until now, the maturities of the futures contracts forming the hedge portfolio are always chosen arbitrarily, like the date of the rebalancing of the portfolio. Lastly, little work has been done on transaction costs and on the financing costs associated with the positions on the futures market.

Investment Decision

The second application of term structure models of commodity prices is the investment decision. The use of term structure models in the case of investment decision is rather intuitive: with such a model, it is possible to compute a futures price for any expiration date, even if the latter is very far away. Thus, such a model enables the valuation of the net cash flow associated with an investment project. All the studies using term structure models for the investment decision are conceived in the framework of real options, and the commodities considered are mineral reserves. The real option theory is based on an analogy with financial options.[9] It aims to identify the optional component included in most investment projects, and when possible, to evaluate it. The main advantage of this theory is that, contrary to the methods traditionally used for the selection of investment projects—like net present value—it takes into account the flexibility of a project. This is all the more important since irreversibility is associated with the project, as is the case with most mineral investments. The theory leads to the identification of different families of real options and underlines that most investment projects include several options. Therefore, the studies realized in the field of commodities take into account various real options.

All the studies on investment decision share certain general features. First, the analysis framework is usually quite simple, because the valuation of an option is generally much more complicated than the valuation of a futures contract. Thus, the authors tend to use simple models, inspired by those developed in the case of futures contracts and presented in section two. Second, poor empirical work has been done in this field, simply because parameter estimation becomes tricky when the horizon of analysis exceeds the exchange-traded maturities.

The pioneering article on investment decision was that of Brennan and Schwartz [1985]. The authors considered a mine where the resource to be exploited is homogenous and of a known volume, extraction costs are known, and interest rates are non-stochastic. There is an upper limit to the output rate, and the study takes into account the possibility of closing and reopening the mine in response to current market conditions. The main source of uncertainty, in that case, is the commodity's price, whose dynamic behavior is represented with the one-factor term structure model previously mentioned. In this framework, there are several real options associated with the possession of the mine: the option to shut down the mine temporarily, the abandonment option, and the option to defer investment. The latter is the simplest real option and undoubtedly the most frequently evoked in the literature. It represents the possibility to wait before investing in order to collect useful information. With the parameter values they chose for simulations, the authors find that it is never optimal, under uncertainty, to close or abandon the mine. They also show how the option value changes with the volume of the reserve, with the initial amount to invest, and so on.

Cortazar and Schwartz [1997] use a one-factor model based on mean-reverting spot price, in which the convenience yield is variable and depends on the deviation of the spot price to a long-term average price. Using this model, they calculate the value of the field at different stages. Stage one corresponds to the field before committing to the development, stage two is the field during development, and stage three is the field during production. They show that the flexibility to wait before investment can amount 10% to 40% of the field value, and that the timing option is an increasing function of the spot price and a decreasing function of the available time to develop.

The same year, Schwartz shows how the value of a copper mine varies with the underlying term structure models chosen for the valuation. He considers only the option to delay and he computes the trigger price at which it is optimal to invest. The study shows that the value of a real option and the investment decision depend strongly on the method used for the valuation of the net future cash flows associated with an investment project. Indeed, the simulations indicate that the assumptions about the dynamic behavior of the state variables in the term structure model have a considerable influence.

Schwartz [1998] develops a one-factor model that retains most of the characteristics of the more complex two-factor model proposed in 1997 in terms of its ability to price the term structure of futures prices and volatilities. The new model is based on the pricing and volatility results of the two-factor model, but, when applied to value long-term commodity projects, it only requires the numerical solution corresponding to a typical one-factor model. Schwartz shows that this one-factor model has practically the same implications as the two-factor model. Thus, this one-factor model can be used to value complex real options without sacrificing any of the advantages of the two-factor model. Still, the latter is needed because the value of its parameters is used as an input for the one-factor model.

The same year, Smith and McCardle propose a model of an oil property where both the price of oil and the production rate vary stochastically over time and the decision maker can opt at any time to terminate or accelerate production by drilling additional wells. He can also hedge some, but not all of the risks associated with the project.

Schwartz and Smith [2000] apply their short-term/long-term model to a hypothetical real options problem. They consider two real options: the option to defer investment for a long-term project, and the development option for a short-term project. To simplify the analysis, they assume there are no operating costs, royalties, or taxes, and that once production starts, it continues indefinitely. The authors show that in the short-term project, the values and policies are sensitive to both state variables and the value increases with both the short-term deviations and the equilibrium price. In contrast, the value and policies of the long-term project are quite insensitive to the short-term deviations. Therefore, the authors propose, for long-term analysis, to reduce this two-factor model to a one-factor model that considers uncertainty in the equilibrium price only. However, the two-factor model would always be needed to estimate the equilibrium price.

Cortazar, Schwartz, and Casassus [2001] collapse price and geological-technical uncertainty into a one-factor model. Using this model, they determine the value of several options: flexible investment schedules for all exploration stages, and a timing option for the development investment. Moreover, once the mine is developed, there are closing, opening, and abandonment options. The model is applied to the copper market. The authors find that a significant portion of the field value is due to the operational, development, and exploitation options available to managers.

Thus, in the beginning, research on investment decision was made in a quite idealistic analysis framework: everything except the price of the commodity was supposed to be known. Since then, however, other sources of uncertainty have also been taken into account. The stake and the interest of these works are beyond the scope of the simple interrogations concerning the optimal exploitation of a field or a mine. In the petroleum industry, for example, the exploration phase is particularly risky and the length of the return on investment period is important. Consequently, for the most part only private funds can be used to finance such projects. However, provided it benefits from reliable methods to valuate a field and to hedge the associated cash flows, a producer could use its guaranteed profits on the field as collateral to banking funds, thus reaching funding previously inaccessible.

CONCLUSION

This last section aims to identify some of the broad trends in the literature on commodity pricing during the 1990s and early 2000s.

Firstly, considering the main developments regarding term structure models of commodity prices, it is possible to determine certain specificities of commodities which distinguish them from other assets. Commodities are characterized by mean reversion in spot and futures prices. Moreover, because arbitrage relationships between the futures and physical markets are limited, price volatility is positively correlated with the degree of backwardation. Prices are also sometimes affected by seasonality. Lastly, the behavior of the term structure is explained by the Samuelson effect.

Secondly, independently of the number of state variables included, the term structure models of commodity prices are generally conceived in a partial equilibrium framework.[10] Consequently, the selection of the state variables can somehow be considered as arbitrary. However, the choice of these variables is usually based on the traditional theories (normal backwardation and storage theories). Moreover, a model with autonomous spot price, convenience yield, and/or long-term price may be regarded as the reduced form of a more general model in which these variables would be endogenously determined by production, consumption, and storage decisions. Still, until now, nobody has really proved that the convenience yield is a better choice than the long-term price as a second factor, because the comparison between different models is quite difficult to undertake. Improvements could probably be made in this field.

Future developments in term structure models of commodity prices most likely will also introduce a more precise description of price behavior. Until now, for example, the leptokurtic nature of commodity prices (Dusak [1973]) and the fact that their distribution is negatively skewed were ignored. The introduction of such characteristics in term structure models could lead to an improvement of the performances. However, in that case, the question of the arbitrage between realism and simplicity arises. There will be a balance to find, especially when the models are conceived for the evaluation of more complex derivative products, as real options.

As for the applications of term structure models, two paths can be explored. For hedging purposes, in order to be adapted by practitioners, the literature would need to progress toward practical considerations such as the transaction costs associated with hedging portfolios or the rebalancing of these portfolios. Moreover, there is a need to quantify the risk associated with these portfolios, for example using "value at risk" methods (Cabedo and Moya [2003]). For the valuation of real assets relying on the theory of real options, it may prove interesting to introduce other sources of uncertainty into the valuation process. Generally speaking the analytical framework employed has been simplistic, the main source of uncertainty being the price of the commodity. However, any improvement in this regard would seem to preclude the pricing of several options simultaneously. Once again, choices must be made.

ENDNOTES

[1] An exception is the one-factor model relying on the convenience yield developed by Veld-Merkoulova and de Roon [2003].

[2] This model was also used by Schwartz and Smith [2000].

[3] Long memory or long-term dependence describes the correlation structure of series at long lags. If a series exhibits long memory, there is persistent temporal dependence between distant observations. Such series are characterized by distinct but non-periodic cyclical patterns.

[4] Schwartz [1997] calibrated the third factor of his model

(the interest rate) using bond prices.

[5] For a presentation of Kalman filters, see for example Harvey [1989].

[6] See for example Schwartz [1997] or Babbs and Nowman [1999].

[7] There is more than one state-space form for some models. Because some of them are more stable, the choice of a specific representation is important.

[8] For more information on the Metallgesellschaft case, see for example Culp and Miller ([1994, 1995]) and Edwards and Canter [1995].

[9] A presentation of the real options theory can be found in Trigeorgis [1999], Copeland and Antikarov [2001], and Grinblatt and Titman [2001].

[10] To be exhaustive on term structure models, one must also quote the studies undertaken by Cortazar and Schwartz [1994] and Miltersen and Schwartz [1998]. They were the only ones to propose a general framework of pricing commodity futures options using the Heath, Jarrow, and Morton [1992] methodology (probabilistic approach). Their model takes the entire term structure of futures prices as given.

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