Inverse Problem for the Schrödinger Operator in an Unbounded Strip

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The question of the identification of a diffusion coefficient c, is studied for the Schrödinger problem in an unbounded strip Ω in $\mathbb{R}^2$. Let $\Omega = \mathbb{R} \times (-\frac{d}{2}, \frac{d}{2})$ be an an unbounded strip of $\mathbb{R}^2$ with a fixed width d. We will consider the Schrödinger equation

$$(1) \qquad \begin{cases} Hq := i\partial_t q + \nabla \cdot (c(x,y)\nabla q) = 0 \ \text{ in } \ Q = \Omega \times (0,T), \\ q(x,y,t) = b(x,y,t) \ \text{ on } \ \Sigma = \partial\Omega \times (0,T), \\ q(x,y,0) = q_0(x,y) \ \text{ on } \ \Omega, \end{cases}$$

where $c(x,y) \in \mathcal{C}^3(\overline{\Omega})$ and $c(x,y) \geq c_{min} > 0$. Moreover, we assume that c and all its derivatives up to order three are bounded. The aim of this paper is to give a stability and uniqueness result for the coefficient $c(x,y)$. We prove a global Carleman estimate and an energy estimate for the operator H with a boundary term on Γ^+. Then using these estimates and following the method developed by Imanuvilov, Isakov and Yamamoto for the Lamé system, we give a stability and uniqueness result for the diffusion coefficient $c(x,y)$.

We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{(x,y) \in \Gamma;\ y = \frac{d}{2}\}$ and $\Gamma^- = \{(x,y) \in \Gamma;\ y = -\frac{d}{2}\}$.

Our problem can be stated as follows:
Is it possible to determine the coefficient $c(x,y)$ from the measurement of $\partial_\nu(\partial_t q)$ on Γ^+?

Let q (resp. $\widetilde{q}$) be a solution of (1) associated with (c, b, q_0) (resp. $\widetilde{c}$, b, q_0)) satisfying some regularity properties:

- $\partial_t\widetilde{q}$, $\nabla(\partial_t\widetilde{q})$ and $\Delta(\partial_t\widetilde{q})$ are bounded.
- q_0 is a real valued function in $\mathcal{C}^3(\Omega)$.
- q_0 and all its derivatives up to order three are bounded.

Our main result is

$$|c - \widetilde{c}|^2_{H^1(\Omega)} \leq C|\partial_\nu(\partial_t q) - \partial_\nu(\partial_t\widetilde{q})|^2_{L^2((-T,T)\times\Gamma^+)},$$

where C is a positive constant which depends on (Ω, Γ, T).

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