Characterizations of the Polynomial Numerical Hull

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Abstract

The polynomial numerical hull of degree k for a square matrix or bounded linear operator A is defined as

$$\mathcal{H}_k(A) = \{z \in \mathbf{C} : \ \|p(A)\| \geq |p(z)| \ \text{ for all polynomials } p \text{ of degree } k \text{ or less}\}.$$

Here $\|\cdot\|$ is the operator norm corresponding to the 2-norm for vectors. These sets are designed to give more information than the spectrum alone can provide about the about quantities such as $\|A^j\|$, $j = 1, 2, \ldots$ or $\|e^{tA}\|$, $t > 0$. While we still do not have good ways to compute these sets, we have found a number of equivalent characterizations that have proved useful in various circumstances.

For example, it is easy to show that $z \in \mathcal{H}_k(A)$ if and only if

$$\min_{\substack{p \in \mathcal{P}_k \\ p(0)=1}} \|p(A - zI)\| = 1,$$

where $\mathcal{P}_k$ is the set of polynomials of degree k or less. This enables one to relate the polynomial numerical hull of degree k to the convergence or lack of convergence after k steps of the GMRES algorithm for solving linear systems. Conversely, new results about the convergence of GMRES give us more information about polynomial numerical hulls.

A more recent characterization is:

$$\mathcal{H}_k(A) = \{z \in \mathbf{C} : \ p(z) \in \mathcal{F}(p(A)) \ \ \forall p \in \mathcal{P}_k\},$$

where $\mathcal{F}(\cdot)$ denotes the field of values, or, numerical range. This close relationship with the field of values has enabled us to prove new results about polynomial numerical hulls of Toeplitz matrices and operators, for example.

These and other relationships between polynomial numerical hulls and problems in complex approximation theory will be discussed in this talk.

This is joint work with James Burke.