SAFEGUARDING IN OPTIMIZATION UNDER UNCERTAINTY

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a common situation:

- decision must be taken now
- consequences not known until later

uncertainty tied to the future
→ risk!

the challenge for optimization:

- how to understand constraints and objectives?
- how to characterize solutions and compute them?

safeguarding:
approaches for handling risks, related to "safety margins"

- engineering design
- systems management
- financial planning ....

UNCERTAINTY IN OPTIMIZATION

decisions $\longrightarrow$ uncertain consequences

standard problem with certainty:

choose $x \in S \subset \mathbb{R}^n$ to

minimize $f_0(x)$ subject to $f_i(x) \leq 0$, $i = 1, \ldots$

basic uncertainty:

Ω = space of future states ω

$\leftarrow$ probability structure $(\Omega, \mathcal{F}, P)$

$\begin{cases} \text{choose } x \in S \text{ now} \\ \text{observe } \omega \in \Omega \text{ later} \end{cases}$

$\longrightarrow f_i(x, \omega)$ for $i = 0, 1, \ldots, m$

decision x only determines

<u>functions</u>: $\underbrace{\omega \longmapsto f_i(x, \omega)}_{\underset{\sim}{f}_i(x) \text{ "random variable"}}$

How should a problem of optimization be formulated in this context?

How to think about objectives, constraints

TRADITIONAL APPROACHES

1. guessing the future:
   $\bar{\omega}$ = "estimate" of future state in Ω
   minimize $f_0(x,\bar{\omega})$ subject to
   $f_i(x,\bar{\omega}) \le 0,\ i=1,\dots,m$
   common, but dangerous

2. relying on expectations:
   minimize $E_\omega\{f_0(x,\omega)\}$ subject to
   $E_\omega\{f_i(x,\omega)\} \le 0,\ i=1,\dots,m$
   focuses on long-range behavior

3. worst-case analysis:
   minimize $\sup_\omega f_0(x,\omega)$ subject to
   $\sup_\omega f_i(x,\omega) \le 0,\ i=1,\dots,m$
   pessimistic, maybe infeasible

4. probabilistic constraints:
   find lowest α such that
   $\text{prob}\{f_0(x,\omega) \le \alpha\} \ge \pi_0$
   $\text{prob}\{f_i(x,\omega) \le 0\} \ge \pi_i,\ i=1,\dots,m$
   $\pi_0, \pi_1, \dots, \pi_m$ = selected probabilities
   $\longrightarrow$ technical troubles,
   conceptual controversies

A GENERAL FRAMEWORK

random variables $X : \Omega \to \mathbb{R}$

$$X \in \mathcal{L}^1 = \mathcal{L}^1(\Omega, \mathcal{A}, P)$$

orientation:

$X(\omega)$ = a kind of "cost" incurred in state ω

lower = better, higher = worse

risk functionals: $\mathcal{R} : \mathcal{L}^1 \to (-\infty, \infty]$

$\mathcal{R}(X)$ = single number assigned to the random variable X

a "numerical surrogate" for X

axioms for "coherency":

(R1) $\mathcal{R}(C) = C$ for constants C

(R2) $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for $\lambda > 0$

(R3) $\mathcal{R}(X_1 + X_2) \leq \mathcal{R}(X_1) + \mathcal{R}(X_2)$

(R4) $\mathcal{R}(X_1) \leq \mathcal{R}(X_2)$ when $X_1 \leq X_2$ a.s.

(R5) $\{X \mid \mathcal{R}(X) \leq 0\}$ is closed in $\mathcal{L}^1$

$\Longrightarrow$ $\mathcal{R}$ is convex, positively homogeneous, lower semicontinuous

$$\mathcal{R}(0) = 0, \quad \mathcal{R}(X + C) = \mathcal{R}(X) + C$$

coherency: from Artzner et al., 1999 (with small modifications)

OPTIMIZATION MODEL

context: decision $x \in S \rightarrow$ random variable $\underset{\sim}{f}_i(x) \in \mathcal{L}^1$

$$\underset{\sim}{f}_i(x) : \omega \mapsto f_i(x,\omega)$$

problem formulation:

minimize $F_0(x) := \mathcal{R}_0(\underset{\sim}{f}_0(x))$ subject to

$$F_i(x) := \mathcal{R}_i(\underset{\sim}{f}_i(x)) \leq 0, \; i = 1, \dots, m$$

$\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_m$ = selected risk functionals

PROPOSITION 1:

If $f_i(x,\omega)$ is convex in x, then

$F_i(x) = \mathcal{R}_i(\underset{\sim}{f}_i(x))$ is convex function of

$\longrightarrow$ convexity is preserved, when present

PROPOSITION 2:

If $f_i(x,\omega) \equiv f_i(x)$ (no uncertainty), the

$F_i(x) = \mathcal{R}_i(\underset{\sim}{f}_i(x)) = f_i(x)$

$\longrightarrow$ certainty is preserved, when present

INITIAL EXAMPLES

seen through risk functionals

1. guessing the future: $\bar{\omega} \in \Omega$ = discrete, finite set

$$f_i(x,\bar{\omega}) = \mathcal{R}(\underset{\sim}{f}_i(x)) \text{ for}$$
$$\mathcal{R}(\underset{\sim}{X}): \underset{\sim}{X} \longrightarrow \underset{\sim}{X}(\bar{\omega})$$
satisfies axioms R1–R5

2. relying on expectations:

$$E_\omega\{f_i(x,\omega)\} = \mathcal{R}(\underset{\sim}{f}_i(x)) \text{ for}$$
$$\mathcal{R}(\underset{\sim}{X}): \underset{\sim}{X} \to E\underset{\sim}{X}$$
satisfies axioms R1–R5

3. worst-case analysis:

$$\sup_\omega f_i(x,\omega) = \mathcal{R}(\underset{\sim}{f}_i(x)) \text{ for}$$
$$\mathcal{R}(\underset{\sim}{X}): \underset{\sim}{X} \to \sup \underset{\sim}{X}$$ "essential" su (maybe ∞)

satisfies axioms R1–R5

4. probabilistic constraints:

$$\text{prob}\{f_i(x,\omega) \le 0\} \ge \pi_i \iff \mathcal{R}_i(\underset{\sim}{f}_i(x)) \le 0$$
for $\mathcal{R}_i(\underset{\sim}{X}) = VaR_{\pi_i}(\underset{\sim}{X})$ "value-at-risk"

satisfies axioms R1, R2, R4, R5, but not R3

convexity lacking, not preserved

VARIANCE EXAMPLE

related to Markovitz model in finance

$\sigma(X)$ = standard deviation

$\lambda_0, \lambda_1, \dots, \lambda_m$ = parameters > 0

minimize $E[\underset{\sim}{f}_0(x)] + \lambda_0 \sigma(\underset{\sim}{f}_0(x))$

subject to $E[\underset{\sim}{f}_i(x)] + \lambda_i \sigma(\underset{\sim}{f}_i(x)) \leq 0$, $i = 1 \dots$

"safeguarding with deviation margins"

(again, constraints and objective could really be treated differently)

This corresponds to choosing:

$$\mathcal{R}_i(X) = EX + \lambda_i \sigma(X)$$

Here, $\mathcal{R}_i$ satisfies axioms R1, R2, R3, R5 but not R4!

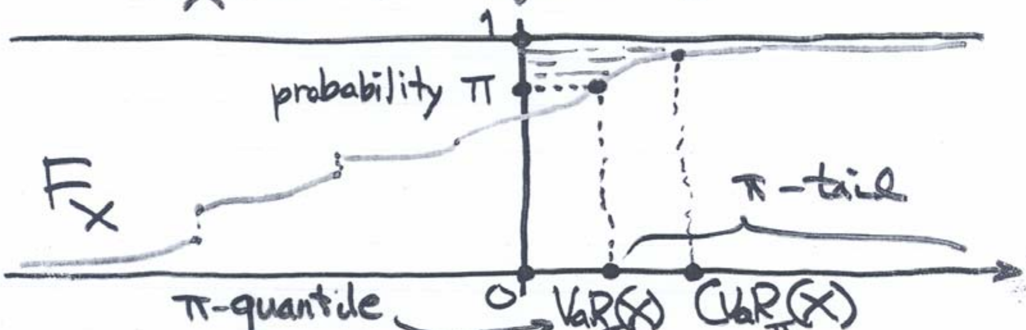
$\hookrightarrow$ such modeling is "incoherent" (more later)

CONDITIONAL VALUE-AT-RISK

Rockafellar & Uryasev, 2000 & 2002

$F_X : (-\infty, \infty) \to [0,1]$ distribution function for random variable $X \in \mathcal{L}^1$

$F_X(z) = \text{prob}\{X \leq z\}$



Definition 1: $CVaR_\pi(X)$ = conditional expectation of X subject to $X \geq VaR_\pi(X)$

needs refinement if F_X has jump at $VaR_\pi(X)$

Definition 2: $CVaR_\pi(X) = \frac{1}{1-\pi} \int_\pi^1 VaR_\tau(X)\, d\tau$

Observation: $CVaR_\pi(X) \leq 0 \Rightarrow \underbrace{\text{prob}\{X \leq 0\} \geq \pi}_{VaR_\pi(X) \leq 0}$

THEOREM $\mathcal{R}(X) = CVaR_\pi(X)$ is a risk functional satisfying R1-R4

coherent, convexity preserving!

CVaR VERSUS VaR

improvement of prob. constraints

constraint comparisons:

- $VaR_{\pi_i}(\underset{\sim}{f}_i(x)) \le 0$ means
  $f_i(x,\omega) \le 0$ with probability π_i
- $CVaR_{\pi_i}(\underset{\sim}{f}_i(x)) \le 0$ means
  $f_i(x,\omega) \le 0$ with probability π_i
  <u>and</u> even in the remaining cases one has $f_i(x,\omega) \le 0$ on average

objective comparisons:

- minimizing $VaR_{\pi_0}(\underset{\sim}{f}_0(x))$ means finding lowest α such that $f_0(x,\omega) \le \alpha$ with probability π
- minimizing $CVaR_{\pi_0}(\underset{\sim}{f}_0(x))$ means finding lowest α such that $f_0(x,\omega) \le 0$ with probability π
  <u>and</u> even in the remaining cases, one has $f_i(x,\omega) \le \alpha$ on average

MINIMIZATION SHORTCUT

$$CVaR_\pi(X) = \min_{z\in\mathbb{R}} \left\{ z + \frac{1}{1-\pi} E_\omega\{\max[0, X(\omega)-z]\} \right\}$$

$$VaR_\pi(X) = \underset{z\in\mathbb{R}}{\operatorname{argmin}} \{ \text{--- - same - ---} \}$$

take lowest point if not unique

this explains the different behaviors!

application to optimization:

$$\begin{cases} \underset{x\in S}{\text{minimize}}: CVaR_{\pi_0}(\underset{\sim}{f}_0(x)) \text{ subject to} \\ \quad CVaR_{\pi_i}(\underset{\sim}{f}_i(x)) \le 0, \; i=1\ldots m \end{cases}$$

$$\begin{cases} \underset{\substack{x\in S \\ z_0, z_1, \ldots z_m \in\mathbb{R}}}{\text{minimize}}: z_0 + \frac{1}{1-\pi_0} E_\omega\{\max[0, f_0(x,\omega)-z_0]\} \\ \quad \text{subject to} \\ \quad z_i + \frac{1}{1-\pi_i} E_\omega\{\max[0, f_i(x,\omega)-z_i]\} \le 0 \\ \quad i=1\ldots m \end{cases}$$

→ converts to linear programming when $\begin{cases} \Omega \text{ is discrete, finite} \\ f_i(x,\omega) \text{ is linear in } x \end{cases}$

→ the values of $CVaR_{\pi_i}(\underset{\sim}{f}_i(x))$ are computed automatically as part of the optimization.

STOCHASTIC DOMINANCE
connections with VaR, CVaR

first-order dominance: $\precsim_1$

$$X \precsim_1 Y \iff E[u(X)] \geq E[u(Y)]$$
for all nonincreasing utilities u

second-order dominance: $\precsim_2$

$$X \precsim_2 Y \iff E[u(X)] \geq E[u(Y)]$$
for all [nonincreasing concave] utilities

$$X \leq Y \Rightarrow X \precsim_1 Y \Rightarrow X \precsim_2 Y$$

THEOREM

$$X \precsim_1 Y \iff VaR_\pi(X) \leq VaR_\pi(Y) \text{ for all}$$
$$X \precsim_2 Y \iff CVaR_\pi(X) \leq CVaR_\pi(Y) \text{ for all}$$

consistency of $\mathcal{R}$ with respect to $\precsim_2$:

$\mathcal{R}(X) \leq \mathcal{R}(Y)$ in particular if $X \precsim_2 Y$

otherwise one could have $\mathcal{R}(X) > \mathcal{R}(Y)$ even though $E[u(X)] \geq E[u(Y)]$ for all nonincreasing concave utilities u./

conclusion:

this holds $\iff$ $\mathcal{R}(X)$ depends only on the function $\pi \mapsto CVaR_\pi(X$

satisfied by $\mathcal{R}(X) = CVaR_\pi(X)$, but not by $\mathcal{R}(X) = VaR_\pi(X$

MORE RISK FUNCTIONALS

mixed CVaR:

- $\mathcal{R}(X) = \sum_{k=1}^{r} \lambda_k \, CVaR_{\pi_k}(X)$ (satisfies all axioms)
  $\lambda_k \geq 0, \ \sum_{k=1}^{r} \lambda_k = 1$
- $\mathcal{R}(X) = \int_0^1 CVaR_\pi(X) \, d\lambda(\pi)$ ← weighting measure

  related to "risk profiles" in dual utility theory

general rules:

- $\mathcal{R}_k$ coherent for $k = 1 \dots r$
  $\Longrightarrow \mathcal{R}(X) = \sum_{k=1}^{r} \lambda_k \mathcal{R}_k(X)$ coherent
  $\lambda_k \geq 0, \ \sum_{k=1}^{r} \lambda_k = 1$
- $\mathcal{R}_i$ coherent for $i \in I$ (index set)
  $\Longrightarrow \mathcal{R}(X) = \sup_{i \in I} \mathcal{R}_i(X)$ coherent

limit cases:

- $CVaR_\pi(X) \longrightarrow \sup X$ as $\pi \nearrow 1$
- $CVaR_\pi(X) \longrightarrow EX$ as $\pi \searrow 0$

consistency with stochastic dominance $\succsim_2$:

DUALITY

representation of coherent risk functionals through convex analysis

Definition: risk envelope $\mathcal{Q}$
= a nonempty subset of $\mathcal{L}^\infty = \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$
satisfying
(Q1) $\mathcal{Q}$ is closed and convex
(Q2) every $Q \in \mathcal{Q}$ has $Q \geq 0$, $E[$

interpretation:
each $Q \in \mathcal{Q}$ is a probability density relative to the underlying prob. measure P

$$E[XQ] = \int_\Omega X(\omega) \underbrace{Q(\omega)\, dP(\omega)}_{dP'(\omega)}$$

alternat
prob. mea

THEOREM

$$\left\{\begin{array}{l}\text{risk functionals } \mathcal{R} \\ \text{satisfying R1–R5}\end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{l}\text{risk envelopes } \\ \text{satisfying Q1–}\end{array}\right.$$

$$\mathcal{R}(X) = \sup_{Q \in \mathcal{Q}} E[XQ]$$

$$\mathcal{Q} = \{Q \in \mathcal{L}^\infty \mid E[XQ] \leq \mathcal{R}(X), \forall X \in \mathcal{L}^1\}$$

interpretation:
$\mathcal{R}(X)$ = worst expectation $E_{P'}[X]$
over the alternative probabilit
measures P' corresponding to
the densities $Q \in \mathcal{Q}$

$Q = \frac{dP'}{dP}$

RISK ENVELOPE EXAMPLES

1. relying on expectations:

$$\mathcal{R}(X) = EX \iff \mathcal{Q} = \{1\}$$

$$\mathcal{R}(X) = E_{P'}[X] \iff \mathcal{Q} = \{\frac{dP'}{dP}\}$$

2. worst-case analysis:

$$\mathcal{R}(X) = \sup X \iff \mathcal{Q} = \{\text{all } Q \geq 0 \text{ with } EQ$$

all prob. densities

3. conditional-value-at-risk:

$$\mathcal{R}(X) = CVaR_\pi(X) \iff \mathcal{Q} = \{Q \mid 0 \leq Q \leq \frac{1}{\pi}, EQ =$$

some general rules:

$$\mathcal{R} = \lambda_1 \mathcal{R}_1 + \cdots + \lambda_m \mathcal{R}_m \iff \mathcal{Q} = \lambda_1 \mathcal{Q}_1 + \cdots + \lambda_m \mathcal{Q}_m$$

$$\lambda_k \geq 0, \ \lambda_1 + \cdots + \lambda_m = 1$$

$$\mathcal{R} = \max\{\mathcal{R}_1, \ldots, \mathcal{R}_m\} \iff \mathcal{Q} = \mathrm{conv}\{\mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_m$$

$\longrightarrow$ subgradients, dual problems, characterizations of optimality

GENERALIZED DEVIATIONS

a different aspect of risk

deviation functionals $\mathcal{D}: \mathcal{L}^1 \to [0,\infty]$

$\mathcal{D}(X)$ = an assessment of the uncertainty in the random variable X

axioms for generalized deviation:

(D1) $\mathcal{D}(C)=0$ for constants C

(D2) $\mathcal{D}(\lambda X)=\lambda\mathcal{D}(X)$ for $\lambda>0$

(D3) $\mathcal{D}(X_1+X_2)\le\mathcal{D}(X_1)+\mathcal{D}(X_2)$

(D4) $\mathcal{D}(X)>0$ for nonconstant X

(D5) $\{X \mid \mathcal{D}(X)\le 1\}$ is closed in $\mathcal{L}^1$

classical example: $\mathcal{D}(X)=\sigma(X)$ or $\lambda\sigma(X)$

THEOREM

the relations $\begin{cases}\mathcal{D}(X)=\mathcal{R}(X-EX)\\ \mathcal{R}(X)=EX+\mathcal{D}(X)\end{cases}$

give a one-to-one correspondence between

- $\mathcal{D}$ satisfying (D1)–(D5) and also
  $\mathcal{D}(X)\le \sup X - EX$
- $\mathcal{R}$ satisfying (R1)–(R5) and also
  $\mathcal{R}(X)\le \sup X$

fails for $\mathcal{D}(X)=\lambda\sigma(X)$
(incoherency)

GENERALIZED ERRORS

error-functionals $\mathcal{E} : \mathcal{L}^2 \to [0, \infty)$

$\mathcal{E}(X) =$ error relative to $X \neq 0$

axioms for generalized error:

(E1) $\mathcal{E}(0) = 0$, $\mathcal{E}(X) > 0$ for $X \neq C$

(E2) $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$ for $\lambda > 0$

(E3) $\mathcal{E}(X_1 + X_2) \leq \mathcal{E}(X_1) + \mathcal{E}(X_2)$

associated deviation:

$$\mathcal{D}(X) = \min_C \mathcal{E}(X - C)$$

associated "statistic":

$$\mathcal{S}(X) = \operatorname{argmin}_C \mathcal{E}(X - C)$$

examples:

$\mathcal{E}(X) = [E X^2]^{1/2}$, $\mathcal{D}(X) = \sigma(X)$, $\mathcal{S}(X) =$

$\mathcal{E}(X) =$ K-B error, $\mathcal{D}(X) = CVaR_\alpha(X -$

$\mathcal{S}(X) = VaR_\alpha(X)$

etc.

$\longrightarrow$ "generalized regression"

SOME REFERENCES

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www.ise.ufl.edu/uryasev