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Corporate portfolio management*

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Summary. We solve the optimal portfolio problem in continuous time from the point of view of a corporation, acting on behalf of risk neutral shareholders. Our model fits for example the case of a commercial bank. Risk aversion is generated endogenously by financial frictions, and increases when the value of the firm's assets decrease. We find a remarkably simple investment policy: invest a multiple of the firm's equity into the risky asset, keep the rest as cash reserves, and distribute dividends when the value of the firm exceeds some threshold. As a consequence, the firm locally behaves as a Von Neumann-Morgenstern investor with constant relative risk aversion.

Keywords and Phrases: Corporate investment, Portfolio management, Liquidity management, Corporate risk aversion.

JEL Classification Numbers: G11, G24, G32.

1 Introduction

More than thirty five years ago, Merton [11] provided an elegant solution to the dynamic portfolio problem in continuous time for an individual investor. By using stochastic optimal control techniques, he showed in particular that the basic intuitions provided by the Capital Asset Pricing Model could be extended to a dynamic context, at least when security returns are independently and identically distributed across time. One major insight obtained by Merton was that, due to the continuous times framework, the instantaneous objective function of the investor was essentially quadratic, implying a constant composition of the risky part of the portfolio, the only adjustments being due to possible changes in the investor's intertemporal risk aversion index.

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The objective of this article is to solve Merton's problem for a corporation, instead of an individual investor. The main example we have in mind is that of a bank, financed by deposits and equity, that has to decide on how much to invest in risky loans and securities, and how much to keep as cash reserves. Our model can also be applied, with some precautions, to the case of an industrial firm that has to decide on how much to invest in a risky technology. For tractability reasons, and also to keep in line with the assumptions of the Merton model, we rule out any frictions in the investment technology. We assume indeed that the volume of investment in risky assets can be continuously adjusted at negligible cost, like is the case for a bank that only invests in marketable securities. Our model also applies to bank loans, provided we neglect the costs of origination and securitization. Of course, this is just a first pass at the corporate portfolio management problem, since most investment technologies involve some degree of irreversibility, and adjustments costs have then to be taken into account. However we feel that the case of perfectly adjustable investment is a useful benchmark for further analysis.[1] Its advantage is to lead to an explicit, remarkably simple investment and financial policies: invest a multiple of equity in the risky asset, keep the rest as cash reserves, and distribute dividends whenever the value of assets exceeds some threshold.

The other contribution of this paper is to give a tractable formulation of corporate risk aversion in the presence of financial frictions. A large academic literature has tried to fill the gap between the theoretical benchmark of perfect capital markets [13] and the practical importance of liquidity and risk management have been explored for explaining why widely held firms appear to exhibit some form of risk aversion: managerial risk aversion [17], tax optimization [16], cost of financial frictions ([18], [4]).[2] However, none of these articles provide a simple, tractable, measure of how the risk aversion of a corporation varies with its financial situation.

The model we use, a variant of the model first studied by [7] (see also [14], or [12]), captures financial frictions in a simple way, by assuming that the firm is liquidity constrained: it is not able to issue more debt or equity in the future, and is forced to close when the value of its assets falls below the value of its liabilities. Jeanblanc and Shirayev [7] show that in order to mitigate the risk of closure, the firm's optimal financial policy is to accumulate liquid reserves up to a certain threshold, and only distribute dividends when reserves exceed this threshold. We extend their analysis by introducing flexibility of investment. In our model, the optimal financial policy of the firm has the same flavor: dividends are distributed only when the value of the firm's assets is above a certain threshold. However, due to the flexibility of investment, there is an intermediate region in which the firm does not have to hold any cash reserves: all the assets are optimally invested in the risky technology.

The remainder of the paper is organized as follows:

- The model is presented in Section 2.
- The optimal investment policy is characterized in Section 3.

[1] Leland [8] considers a polar case where the volume of investment is fixed, but assets returns are selected by the firm.

[2] Froot and Stein [5] have applied the Froot, Scharfstein and Stein model to study capital budgeting and capital structure decisions for a financial institution.

- Section 4 discusses the economic implications of our results in terms of corporate risk aversion and liquidity management.
- Section 5 concludes by suggesting possible extensions.
- All mathematical proofs are given in the appendix.

2 The model

We consider a firm characterized at each date t by the following balance sheet:

I_t	D
M_t	X_t

- I_t represents the investment[3] in the productive technology, characterized by stochastic returns.
- M_t represents the amount of cash reserves, remunerated at rate r (the risk free rate).
- D, which is assumed to be fixed, represents the volume of debt that the firm has issued. It is also remunerated at the risk free rate[4] r. In the case of a commercial bank, D can be interpreted as the volume of deposits collected by the bank.
- Finally, X_t represents the book value of equity.

To give a rigorous formulation of the optimization problem that will be solved in this paper, we start with a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which represents the information available at time t. We consider a $\mathcal{F}_t$-Brownian motion $(W_t)_{t\geq 0}$ and assume that the random returns on the risky asset (or the productive technology) evolve according to $\mu dt + \sigma dW_t$. Consequently, the book value of equity of the firm evolves according to the following dynamics:

$$dX_t = I_t\{\mu dt + \sigma dW_t\} + r(M_t - D)dt - dZ_t, \tag{1}$$

where the term between brackets is the return on risky assets, and the second term represents net financial income (or expenses when negative). Finally, Z_t represents the cumulated process of dividends paid to the shareholders of the firm.

A control policy is described by a two-dimensional stochastic process (I_t, Z_t). Each control variable of the firm is $\mathcal{F}_t$ -adapted, meaning that every decision of the firm at date t is made conditionally on the information available at date t. I_t corresponds to the amount invested in the risky technology: we assume that it can be continuously adjusted at no cost. On the other hand, the control I_t is restricted to lie in the interval $[0, X_t + D]$ where $X_t + D = I_t + M_t$ is the total of the balance sheet of the firm. This restriction means in particular that cash reserves M_t

[3] For the ease of exposition, we restrict ourselves to the case of a unique risky asset. The case of multiple risky assets, with i.i.d. returns, is a trivial extension.

[4] We assume that the firm's debt does not bear any default spread. This is justified by the fact that, in our model, the firm never defaults on its debt (see Section 3).

cannot be negative. Z_t is restricted to be a non decreasing process starting from 0, which means that dividends are always non negative, i.e. that shareholders cannot be required to inject new capital in the firm.

In fact, we capture financial frictions in a simple way by assuming that the firm is not allowed to issue more equity or more debt in the future. When X_t hits 0, the firm is liquidated (at no cost) and the debtholders are repaid D. The (random) liquidation date is denoted τ. The dynamics of X_t, which we take as our state variable, is given by:

$$dX_t = I_t\{\mu dt + \sigma dW_t\} + r(X_t - I_t)dt - dZ_t. \tag{2}$$

Shareholders are risk neutral and discount the future at rate ρ. We denote by Π the set of two-dimensional controls satisfying the above restrictions. The value function of a firm starting at $X_0 = x$ is thus:

$$V^*(x) = \max_{(I_t, Z_t) \in \Pi} E\left[\int_0^\tau e^{-\rho t} dZ_t\right]. \tag{3}$$

We assume that $\rho > \mu > r$. This means that in the absence of debt, shareholders would prefer to consume immediately rather than to invest in the risky technology.[5] However, leverage provides shareholders with an opportunity to operate the firm at their profit. To see why, consider for example the case where Z_t is not restricted to be increasing (i.e. shareholders are not cash constrained: they can inject new funds at any time). If the current book value of equity is x, the optimal strategy of the firm would be to distribute x immediately as dividends while investing the debt value D in the risky technology ($I_t = D$ for every $t \geq 0$) and offset profits and losses by payments to or from shareholders. The dividend control process corresponding to this strategy is given by:

$$dZ_t = D\{\mu dt + \sigma dW_t\} - rDdt$$

Shareholder value would then be:

$$V_{FB}(x) = x + \frac{\mu - r}{\rho} D$$

where the notation V_{FB} stands for the "first best" value of the firm. Notice that $V_{FB}(x)$ is composed of two terms: x represents the immediate dividend paid to shareholders, while $\frac{\mu-r}{\rho}D$ represents the expected present value of investing the

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[5] When $\mu > \rho$, the problem does not have a well defined solution: shareholders are better off by reinvesting all the profits in the productive technology and let the book value of equity increase without limit. To see this, take $I_t = X_t$ and $Z_t = \int_0^t \delta X_t dt$ with $0 < \delta < \mu - \rho$. The evolution of the book value of equity is

$$dX_t = (\mu - \delta) X_t dt + \sigma X_t dW_t.$$

X_t follows a geometric Brownian motion, and therefore the liquidation date is infinite. Thus,

$$V(x) \geq \delta x \int_0^\infty e^{(\mu - \rho - \delta)t}\, dt = \infty.$$

amount D (that is borrowed from debtholders and remunerated at rate r) in the productive technology (which has an expected return of μ). Thus if shareholders were not cash constrained, they would maintain a constant investment D in the risky technology, keep no reserves, distribute all gains immediately as dividends, and cover all losses by reinjecting funds when needed.

The problem becomes more interesting when shareholders are cash constrained. Jeanblanc and Shirayev [7] have studied the particular case where $D=0$, $r=0$ and I_t is restricted to be constant. The solution then consists in accumulating cash reserves up to some threshold and distributing dividends above this threshold. We now examine the general case where I_t is flexible, and D and r are positive.

3 Characterizing the optimal policy of the firm

This section characterizes the optimal policy of the firm. Recall that it consists of two elements: the investment process I_t and the cumulated dividend process Z_t. The main difficulty is that I_t is a bounded process (since it is restricted to lie in the interval $[0, X_t + D]$), while Z_t can be any non decreasing process, not necessarily with bounded variations, as illustrated by [7]. The control problem of the firm is thus a mixed singular/regular control problem of the type studied by Fleming and Soner [3] who prove the following result:

Theorem 1. *If the value function V^* defined by (3) is $\mathcal{C}^2$, it satisfies the following variational inequalities:*

$$\forall x > 0, \quad \max\left[\max_{0\le I\le x+D} L(I)V(x),\, 1 - V'(x)\right] = 0, \tag{4}$$

where

$$L(I)V(x) \equiv \{\mu I + r(x - I)\}V'(x) + \frac{\sigma^2}{2} I^2 V''(x) - \rho V(x), \tag{5}$$

with the initial condition

$$V(0) = 0. \tag{6}$$

Theorem 1 gives an analytical characterization of the value function in terms of the Hamilton-Jacobi-Bellman differential equation. However, it is difficult in general to prove that there is an unique solution to (4) and (6) in a classical sense. In order to use the analytical characterization, we have to be guided by economic intuition. The impossibility to obtain external finance leads shareholders to accumulate cash reserves in order to reduce the risk of being forced to liquidate. However, the marginal value of these reserves is likely to decrease (as the level of reserves increases) since liquidation then becomes less likely. This speaks for a concave value function. The concavity of the value function, together with a marginal value bounded below by one, yields that shareholders will distribute dividends when the marginal value of the firm is exactly one. Therefore, we claim the existence of a threshold x_1 above which the firm distributes all the surplus as dividends. This means that

$$V'(x) \equiv 1, \quad V''(x) \equiv 0 \quad \text{for } x > x_1, \tag{7}$$

and we prove the following verification theorem:

Theorem 2. *a) Assume there exists a twice continuously differentiable concave function V and a constant x_1 such that*

$$\forall x \leq x_1 \qquad \max_{0 \leq I \leq x+D} (L(I)V(x)) = 0 \text{ and } V^{'}(x) \geq 1, \tag{8}$$

$$\forall x \geq x_1 \qquad V^{'}(x) = 1 \text{ and } \max_{0 \leq I \leq x+D} (L(I)V(x)) \leq 0 \tag{9}$$

together with the initial condition:

$$V(0) = 0, \tag{10}$$

then $V = V^*$.
b) Furthermore, let I^* be the maximizer in (8) and $\ell_t(x_1)$ the local time[6] *at the level x_1 of the diffusion process*

$$dX_t = I^*(X_t)(\mu dt + \sigma dW_t) + r(X_t - I^*(X_t))dt,$$

then $V^*(x) = E \int_0^{\tau_0} e^{-rs} d\ell_s(x_1)$, where

$$\tau_0 = \inf\{t \geq 0,\ X_t \leq 0\}.$$

We shall construct a concave function, denoted V, satisfying the conditions of Theorem 2. Note that for $x \geq x_1$, we have

$$L(I)V(x) = (\mu - r)I + rxV'(x) - \rho V(x),$$

Because $\mu > r$, $L(I)V(x)$ is increasing with respect to I and thus the optimal investment is $I^*(x) = x + D$. By continuity there exists a second threshold $x_0 \in [0, x_1[$ such that $I^*(x) = (x + D)$ on $[x_0, x_1]$. The first order condition corresponding to the optimal investment choice is:

$$\begin{aligned}(\mu - r)V'(x) + \sigma^2 V''(x)I &= 0 \text{ if } I \in]0, x + D[\\ &\geq 0 \text{ if } I = x + D \\ &\leq 0 \text{ if } I = 0.\end{aligned}$$

Suppose that $I^*(x)$ is interior on $(0, x_0)$. This means that

$$I^*(x) = \frac{\mu - r}{\sigma^2}\left(-\frac{V'}{V''}(x)\right),$$

[6] The interpretation is that X_t is reflected at x_1. The local time $\ell_t(x_1)$ is the process that ensures the reflection. We refer to [9], Section 3.6, for a rigorous presentation of the notion of local time.

a formula reminiscent of [11]. Therefore, V satisfies the nonlinear differential equation:

$$0<x<x_0: \rho V(x)=rxV'(x)-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\frac{V'^2}{V''}(x), \tag{11}$$

together with the boundary condition

$$V(0)=0. \tag{12}$$

A key feature of our model is that the solution of (11), (12) is explicit:

Lemma 1. *Any concave increasing $\mathcal{C}^2$ function V that satisfies (11) and (12) has the following expression:*

$$V(x)=K_1x^{a_1},$$

where $a_1\in(0,1)$ is the smallest solution of

$$ra^2-\left(r+\rho+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)a+\rho=0,$$

and K_1 is a positive constant.

The corresponding investment strategy is thus linear in x:

$$I^*(x)=\left(\frac{\mu-r}{\sigma^2}\right)\frac{x}{1-a_1}\equiv\frac{x}{k}. \tag{13}$$

Since $k<1$ (this will be established in Lemma 2 in the Appendix), x_0 is determined by the equality $\frac{x_0}{k}=x_0+D$, which gives x_0 explicitly:

$$x_0=\frac{kD}{1-k}. \tag{14}$$

Consider now the interval (x_0,x_1), on which $I^*(x)=x+D$. Using (4) and (5) we see that V satisfies, on this interval, a linear differential equation:

$$\rho V(x)=[\mu x+(\mu-r)D]V'(x)+\frac{\sigma^2}{2}(x+D)^2V''(x). \tag{15}$$

On this interval, the solution is not explicit. Nevertheless, by the Cauchy-Lispchitz theorem, we know that for all u, there exists a unique solution V_u of (15) that satisfies the boundary conditions:

$$V_u''(u)=0, \qquad V_u'(u)=1. \tag{16}$$

We are now in a position to state the main result of this paper:

Theorem 3. *There exist an unique real number $x_1>x_0$ and a unique concave increasing $\mathcal{C}^2$ function $V=V_{x_1}$ from $\mathbb{R}_+$ to $\mathbb{R}$ that satisfies (8), (9). It is characterized by the following properties:*

a) $V(x) = K_1 x^{a_1}$ *for* $0 \leq x \leq x_0$.
b) $\rho V(x) = [\mu x + (\mu - r)D]V'(x) + \frac{\sigma^2}{2}(x+D)^2 V''(x)$ *for* $x_0 < x < x_1$.
c) $V(x) = x + \frac{\mu - r}{\rho} D - \frac{\rho - \mu}{\rho} x_1$ *for* $x \geq x_1$,

where K_1 *is uniquely determined.* ■

The idea behind the proof of Theorem 3 (given in the Appendix) is that there exists a unique value of x_1 such that the solution of (15), (16) can be "patched", in a $\mathcal{C}^2$ fashion, with an isoelastic function $V(x) = K_1 x^{a_1}$ defined on the interval $[0, x_0]$.

4 Economic implications

In this section, we derive several economic implications from Theorem 3. In Subsection 4.1 we establish that the optimal investment of the firm essentially consists in investing a multiple of the firm's equity into the risky asset. In Subsection 4.2 we prove that the firm's optimal financial policy consists in distributing dividends whenever the book value of equity exceeds a certain threshold x_1. Finally, Subsection 4.3 studies the determinants of the firm's risk aversion.

4.1 Investment policy

We have established that the optimal investment policy of the firm was relatively simple:

$$I^*(x) = \min\left(\frac{x}{k}, x + D\right). \tag{17}$$

That is, it consists of investing a multiple $\frac{1}{k}$ of the firm's equity X_t, up to a point x_0 where this exhausts cash reserves. Above x_0, the firm invests everything in the risky technology. k can be interpreted as a minimum capital ratio $\left(\frac{\text{equity}}{\text{investment}}\right)$ as in bank solvency regulations. However, this minimum capital ratio is not imposed by regulators, it corresponds to the optimal investment policy for shareholders. Whenever $x \leq x_0$, the minimum capital ratio is binding. When $x > x_0$, the capital ratio of the firm is above the minimum value k.

The following proposition characterizes the comparative statics properties of the minimal capital ratio k, as a function of the cost of capital $\rho - r$, the volatility σ of assets and the expected excess return $\mu - r$ on these assets.

Proposition 1. *Other things being equal, the target capital ratio* k *optimally chosen by the firm satisfies the following properties:*

a) It decreases with the cost of capital $(\rho - r)$.
b) It increases with the volatility of assets σ.
c) It is single peaked with respect to the expected excess return on assets $(\mu - r)$.

Properties a) and b) are intuitive, but c) is more surprising, since one could have expected a capital ratio that decreases with the profitability of the technology, measured by the expected excess return on assets. Proposition 1 c) confirms that very profitable firms should invest a lot (i.e., k is small when $\mu - r$ is large) but, surprisingly, the same is true when μ is close to zero. The reason is that, when μ is close to r, the risk aversion coefficient of the firm, namely $1 - a_1$, converges to zero (see Proposition 2). Moreover it does so faster than $\mu - r$, which explains why k itself converges to zero.

4.2 Financial policy

Shareholder value $V(x)$ is a concave function of the book value of equity x. This means that financial frictions (captured here by the assumption that the firm cannot obtain external finance) generate a simple form of risk aversion. Above x_1, interpreted as the optimal level of equity for the firm, the firm becomes risk neutral: the marginal value of additional cash is one, and dividends can be distributed.

We were not able to establish the comparative statics properties of the target level of capital x_1, as a function of the model's parameters. The reason is that x_1 is only determined implicitly as the solution to a free boundary problem. However, we conjecture that x_1 has the same properties as k (see Proposition 1), namely that it should decrease with the cost of capital $(\rho - r)$, increase with the volatility of assets σ and be single peaked with respect to $(\mu - r)$. In a companion paper [15], we prove these properties for the fixed investment model of [7]. They remain to be established for the flexible investment model studied here.

The following figure illustrates the value function and the investment policy.

4.3 Corporate risk aversion

As we already mentioned, one of the implications of our results is that a liquidity constrained firm behaves (in the region where its investment policy is not constrained) like a Von Neumann-Morgenstern investor with constant Relative Risk Aversion. The next proposition establishes the way in which the risk aversion coefficient of the firm varies with the cost of capital, the volatility of assets, and the profitability of the firm.

Proposition 2. *Other things being equal, the risk aversion coefficient of the firm satisfies the following properties:*

a) It decreases with the cost of capital $(\rho - r)$.
b) It decreases with the volatility of assets σ.
c) It increases with the expected excess return on assets $(\mu - r)$.

Proposition 2 thus endogenizes the (implicit) risk aversion coefficient of the firm as a function of three parameters: the cost of capital, $\rho - r$, the volatility σ and the expected excess return $\mu - r$ on the risky technology. The fact that risk aversion

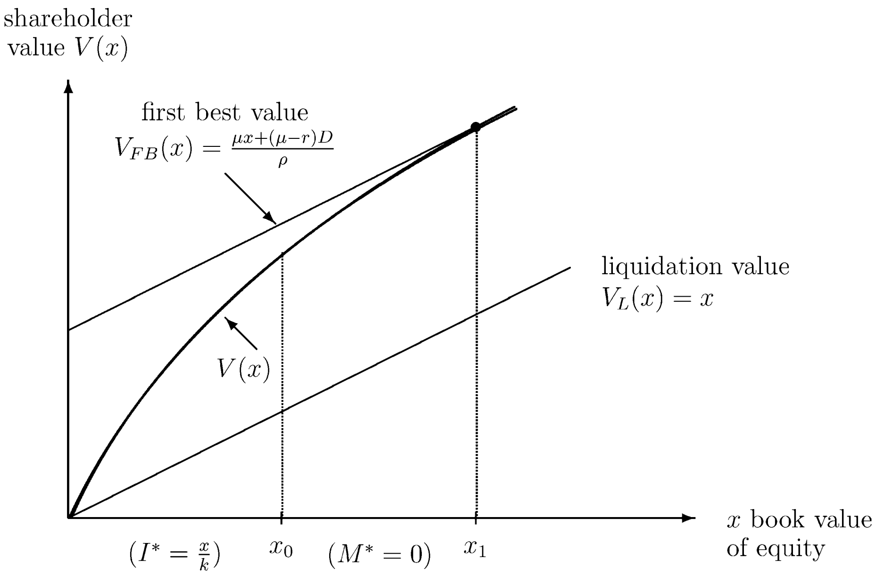

Figure 1. Shareholder value $V(x)$ as a function of x, the book value of equity. I^* is the investment policy. M^* is the level of cash reserves

decreases with the cost of capital (Proposition 2 a) is not surprising: The higher the remuneration demanded by shareholders, the higher the risk taken by the firm (remember however that the firm's investment policy is such that the firm never fails). The variation of the firm's (implicit) risk aversion associated with changes in the net profitability $(\mu - r)$ and the volatility σ of the risky asset has to be viewed in relation with Proposition 1. Remember that in the region $[0, x_0]$ where the firm chooses to keep some liquid reserves (and thus investment I in the risky asset is not maximum) the capital ratio of the firm is equal to:

$$k = \frac{1 - a_1}{\frac{\mu - r}{\sigma^2}},$$

where $(1 - a_1)$ is the (relative) risk aversion coefficient of the firm, and $\frac{\mu - r}{\sigma^2}$ is the Sharpe ratio of the risky asset. Comparing Propositions 1 b (and 2 b) shows that k increases with σ, but less rapidly that σ^2. Similarly, comparing Propositions 1 c (and 2 c) shows that $(1 - a_1)$ increases with $(\mu - r)$ but not at a constant rate, since k is single peaked with respect to this variable.

5 Conclusion

This paper has studied the corporate version of the continuous-time portfolio problem solved by Merton [11] for an individual investor. We have considered a corporation, owned by risk neutral shareholders, that can continuously adjust, at a negligible cost, the composition of its assets portfolio. Even if shareholders are risk neutral, they are liquidity constrained. Thus the corporation exhibits risk aversion, generated endogenously by the possibility of closure whenever the value of the firm's assets falls below the value of liabilities, assumed here to be constant.

Shareholder value is maximized when the firm adjusts its share of risky assets proportionally to its net wealth, effectively eliminating the risk of failure. Thus the firm exhibits constant relative risk aversion, up to the point where all its assets are invested in the risky technology.

Several directions of extension seems fruitful. First it would be interesting to introduce frictions in the investment technology: we have in mind a situation where some fraction of the investment in risky assets is fixed, while the remaining fraction can be adjusted, possibly at a cost. Consider for example a bank who has to decide whether or not to securitize a fraction of its loans, and replace them by marketable assets. Several interesting questions appear, notably which loans to securitize and also when to do so.

Another useful extension would be to introduce liability risk, either in the form of random withdrawals (for a bank) or indemnity risk (for an insurer). This would add an hedging motive to the choice of assets by the firm, as in [10]. Finally, it would be important to introduce the possibility of costly external financing, by allowing the firm to issue new debt or equity when needed. Froot, Scharfstein and Stein [4] do this by assuming a convex cost of external financing. Another possibility would be to endogenize financial frictions by introducing problems of moral hazard or cash flow verifiability, like for example in [1]. In such a model, the liability structure of the firm would be endogenized, and the liquidation decision would form part of an optimal contract between the owners of the firm and outside financiers.

Appendix: Mathematical proofs

Proof of Theorem 2. The following proof is an adaptation of the proof of Proposition 3.2 p. 171 by Hojgaard and Taksar (1999). Fix a policy (I_t, Z_t) and write the process $Z_t = Z_t^c + Z_t^d$ where Z_t^c is the continuous part of Z_t and Z_t^d is the pure discontinuous part of Z_t. Let:

$$dX_t = I_t(\mu dt + \sigma dW_t) + r(X_t - I_t)dt - dZ_t,$$

be the dynamic of equity under the policy (I_t, Z_t), and let us define:

$$\tau_\varepsilon = \inf\{t \geq 0,\ X_t \leq \varepsilon\}.$$

Using the generalized Ito formula (see Dellacherie and Meyer Theorem VIII.27), we can write:

$$\begin{aligned} e^{-r(t\wedge\tau_\varepsilon)}V(X_{t\wedge\tau_\varepsilon}) = V(x) &+ \int_0^{t\wedge\tau_\varepsilon} e^{-\rho s} L(I_s)V(X_s)\,ds \\ &+ \int_0^{t\wedge\tau_\varepsilon} e^{-\rho s}V'(X_s)\,\sigma dW_t - \int_0^{t\wedge\tau_\varepsilon} e^{-\rho s}V'(X_s)\,dZ_s^c \\ &+ \sum_{s\leq t\wedge\tau_\varepsilon} e^{-\rho s}(V(X_s) - V(X_{s-})), \end{aligned}$$

Because V satisfies (8) and (9) the second term of the right hand side is negative. Since V is concave and increasing, $0 \leq V'(X_s) \leq V'(\varepsilon)$ and thus the third term is a centered square integrable martingale. Taking expectations, we get

$$E\left(e^{-r(t\wedge\tau_\varepsilon)}V(X_{t\wedge\tau_\varepsilon})\right) \le V(x) - E\int_0^{t\wedge\tau_\varepsilon} e^{-\rho s}V'(X_s)\,dZ_s^c$$
$$+E\sum_{s\le t\wedge\tau_\varepsilon} e^{-\rho s}(V(X_s)-V(X_{s-})).$$

By concavity and since $V'(x)\ge 1$, we get $V(X_s)-V(X_{s-})\le X_s - X_{s-} = -(Z_s - Z_{s-})$. Therefore,

$$V(x)\ge E\left(e^{-\rho(t\wedge\tau_\varepsilon)}V(X_{t\wedge\tau_\varepsilon})\right)+E\int_0^{t\wedge\tau_\varepsilon} e^{-\rho s}\,dZ_s.$$

The end of the proof consists in getting rid of the first term of the right hand side which needs several steps.

Step 1: We shall first give an uniform bound for the second moment of the controlled diffusion

$$dX_t = (rX_t + (\mu - r)I_t)dt + I_t\sigma dW_t.$$

Since $0\le I_t\le D+X_t$ we can write using Cauchy-Schwarz inequality

$$|X_t|^2 \le 2\left(x^2+\left(\int_0^t rX_s+(\mu-r)I_s ds\right)^2+\left(\int_0^t \sigma I_s dW_s\right)^2\right)$$
$$\le 2\left(x^2+\int_0^t \mu^2 X_s^2 ds+(\mu-r)^2D^2t+\left(\int_0^t \sigma I_s dW_s\right)^2\right).$$

Now take $T>0$ fixed, Doob's inequality yields

$$\mathbb{E}\left[\sup_{t\le T}|X_t|^2\right]\le\left(2x^2+(\mu-r)^2D^2T+\mathbb{E}\int_0^T \mu^2X_s^2ds+4\int_0^T\sigma^2(X_s^2+D^2)ds\right).$$

Let

$$f(T)=\mathbb{E}\left[\sup_{t\le T}|X_t|^2\right]$$

we deduce that

$$f(T)\le 2\left(x^2+((\mu-r)^2+4\sigma^2)D^2T+\int_0^T(\mu^2+4\sigma^2)f(s)ds\right).$$

Thus, Gronwall lemma yields for all $T>0$,

$$\mathbb{E}\left[\sup_{t\le T}|X_t|^2\right]<+\infty.$$

Step 2: The process $(\int_0^t \sigma I_s dW_s)$ is a square integrable martingale for $t \geq 0$. Indeed,

$$\mathbb{E}\int_0^t \sigma^2 I_s^2 ds \leq \sigma^2 \mathbb{E}\int_0^t (X_s^2 + D) ds < \infty,$$

according to Step 1. Thus, taking expectations in

$$X_t = x + \int_0^t (rX_s + (\mu - r)I_s) ds + \int_0^t \sigma I_s dW_s,$$

we get:

$$\mathbb{E}(X_t) \leq x + \mu \int_0^t \mathbb{E}(X_s) ds + (\mu - r) Dt.$$

Once again, Gronwall lemma yields:

$$\mathbb{E}(X_t) \leq (x + (\mu - r)Dt)(1 + e^{\mu t}).$$

Because $\rho > \mu$, we finally

$$\lim_{t \to \infty} \mathbb{E}\left[e^{-\rho t} X_t\right] = 0.$$

We are now in a position to conclude. Because τ_ε converges to τ_0 as ε decreases to 0 and $V(0) = 0$, we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E}\left[e^{-\rho t \wedge \tau_\varepsilon} V(X_{t \wedge \tau_\varepsilon})\right] &= \mathbb{E}\left[e^{-\rho t \wedge \tau_0} V(X_{t \wedge \tau_0})\right] \\ &= \mathbb{E}\left[e^{-\rho t} V(X_t) 1\!\!1_{\{\tau_0 > t\}}\right]. \end{aligned}$$

By concavity of V, we have that $V(x) \leq K(1 + x)$ for some constant K. Thus

$$\mathbb{E}\left[e^{-\rho t} V(X_t) 1\!\!1_{\{\tau_0 > t\}}\right] \leq K\mathbb{E}(e^{-\rho t}(X_t + 1)).$$

Using Step 2, we conclude by letting $t \to \infty$ that

$$\lim_{t \to \infty} \lim_{\varepsilon \to 0} E\left(e^{-\rho(t \wedge \tau_\varepsilon)} V(X_{t \wedge \tau_\varepsilon})\right) = 0.$$

Let I^* be the maximizer of (8) and $\ell_t(x_1)$ the local time defined in Theorem 2. It is a direct consequence of Proposition 3.2 p. 171 in [6] that the control $(I^*(X_t), \ell_t(x_1))$ is optimal.

Proof of Lemma 1. Let V be a solution of equation (11) on the interval $[0, x_0]$, and define

$$u(x) = x\frac{V'(x)}{V(x)} \qquad \text{for} \qquad x \text{ in }]0, x_0].$$

We have that:

$$V'(x) = \frac{u(x)V(x)}{x},$$

and thus:

$$V''(x)=\left[\frac{u(x)V(x)}{x}\right]'=\frac{xu'(x)-u(x)}{x^2}V(x)+\frac{u(x)V'(x)}{x}.$$

Using again the fact that $V'(x)=\frac{u(x)V(x)}{x}$, we can write:

$$V''(x)=\left[\frac{xu'(x)-u(x)+u^2(x)}{x^2}\right]V(x),$$

and thus:

$$\frac{V'^2(x)}{V''(x)}=\frac{u^2(x)}{xu'(x)-u(x)+u^2(x)}V(x). \tag{A1}$$

Thus if we divide equation (11) by $V(x)$, we find that:

$$\rho=ru(x)-\frac{tu^2(x)}{xu'(x)+u^2(x)-u(x)}, \tag{A2}$$

where

$$t\equiv\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2.$$

This is equivalent to:

$$(ru(x)-\rho)[xu'(x)+u^2(x)-u(x)]=tu^2(x),$$

or:

$$\begin{aligned} xu'(x)[ru(x)-\rho] &= -u(ru^2-(\rho+r+t)u+\rho) \\ &= -ru(x)[u(x)-a_1][u(x)-a_2], \end{aligned} \tag{A3}$$

where a_1 and a_2 (with $0<a_1<1<\frac{\rho}{r}<a_2$) are the solutions of

$$ra^2-(\rho+r+t)a+\rho=0.$$

By separating variables and integrating, we can see that the solutions of (A1) are all such that

$$[|u(x)-a_1|]^{\alpha_1}\,[|u(x)-a_2|]^{\alpha_2}=\lambda\frac{u(x)}{x}, \tag{A4}$$

where

$$\alpha_1=\frac{a_2-1}{a_2-a_1}, \qquad \alpha_2=\frac{1-a_1}{a_2-a_1}$$

and λ is a constant. λ is determined by the initial condition $V(0)=0$, which implies that

$$\lim_{x\to 0}\frac{u(x)}{x}=\lim_{x\to 0}\frac{V'(x)}{V(x)}=+\infty.$$

Moreover, concavity of V implies by (A1) and (A2) that $u(x) \leq \frac{\rho}{r}$. Thus when we take the limit of (A4) for $x \to 0$, we see that the left-hand side remains bounded (since $\alpha_1 > 0$ and $\alpha_2 > 0$), while $\frac{u(x)}{x} \to \infty$. This implies that $\lambda = 0$ and thus that $u(x) \equiv a_1$ or a_2. a_2 is excluded since $a_2 > \frac{\rho}{r}$ (this would contradict the concavity of V). Thus $u(x) \equiv a_1$ is the only possibility. This gives

$$V(x) = K_1 x^{a_1},$$

where K_1 is a positive constant, and the proof of Lemma 1 is complete. ■

Lemma 2. *The target capital ratio*

$$\frac{x}{I^*(x)} = k = \frac{1 - a_1}{\frac{\mu - r}{\sigma^2}}$$

is less than 1.

Proof of Lemma 2. Remember that a_1 is the smallest root of

$$\varphi(a) = ra^2 - (r + \rho + t)a + \rho = 0.$$

We want to establish that $k < 1$, which is equivalent to $a_1 > 1 - \frac{\mu - r}{\sigma^2}$. Thus we have to prove that

$$\delta \equiv \varphi\left(1 - \frac{\mu - r}{\sigma^2}\right) > 0.$$

Now:

$$\begin{aligned} \delta &= r\left[1 - \frac{\mu - r}{\sigma^2}\right]^2 - [r + \rho + t]\left[1 - \frac{\mu - r}{\sigma^2}\right] + \rho \\ &= -t + \frac{\mu - r}{\sigma^2}\left[-2r + r + \rho + t + \frac{r}{\sigma^2}(\mu - r)\right]. \end{aligned}$$

But $t = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$. Thus:

$$\delta = \frac{\mu - r}{2\sigma^2}\left[-\mu + r - 2r + 2\rho + 2t + \frac{2r}{\sigma^2}(\mu - r)\right],$$

or:

$$\delta = \frac{\mu - r}{2\sigma^2}\left[2\rho - \mu - r + 2t + \frac{2r}{\sigma^2}(\mu - r)\right].$$

Since $\rho > \mu > r$, $\delta > 0$ and the proof is complete. ■

Proof of Theorem 3. We already know that for all $x_1 > x_0$, there exists a unique function $V_{x_1}(x)$ that satisfies b) on (x_0, x_1), and such that:

$$V'_{x_1}(x_1) = 1, \qquad V''_{x_1}(x_1) = 0.$$

If this function is extended to $[x_1, \infty[$ as in c), it is indeed $\mathcal{C}^2$ at x_1. Notice that

$$\rho V_{x_1}(x_1) = \mu x_1 + (\mu - r)D.$$

We have to prove the existence and uniqueness of a couple (K_1, x_1) such that, if V_{x_1} is extended to $[0, x_0]$ as in a), it is also $\mathcal{C}^2$ at x_0. Two conditions need to be satisfied:

$$V_{x_1}(x_0) = K_1 x_0^{a_1}, \qquad V'_{x_1}(x_0) = a_1 K_1 x_0^{a_1 - 1}.$$

Clearly, $K_1 = \frac{V_{x_1}(x_0)}{x_0^{a_1-1}}$. This leaves one last equation for finding the last unknown x_1:

$$V'_{x_1}(x_0) = a_1 \frac{V_{x_1}(x_0)}{x_0}. \tag{A5}$$

We now prove that this equation has a unique solution $x_1 > x_0$. We first need to establish the following lemma:

Lemma 3. *For all $x < u$, the mapping $u \to V_u(x)$ is decreasing, and the mapping $u \to V'_u(x)$ is increasing.*

Proof of Lemma 3. Let $\theta(x) \equiv V_u(x) - V_v(x)$ for any $x < u < v$. By definition:

$$\theta(u) = \frac{\mu u + (u - r)D}{\rho} - V_v(u) \geq 0.$$

Moreover:

$$\theta'(u) = V'_u(u) - V'_v(u) = 1 - V'_v(u) \leq 0,$$

and

$$\theta''(u) = V''_u(u) - V''_v(u) = -V''_v(u) \geq 0.$$

Let $F = \{x \in (0, u) \text{ such that } \theta'(x) = 0\}$. We are going to show by contradiction that F is empty. Suppose indeed $F \neq \emptyset$ and take $y = \sup F$. By construction $\theta'(y) = 0$ and $\theta'(x) < 0$ for all x in (y, u). Thus $\theta''(y) \leq 0$ and also $\theta(y) > \theta(u) \geq 0$.

But now equation (15) implies:

$$\rho\theta(y) = \frac{\sigma^2}{2}(y + D)\theta''(y) \leq 0,$$

which gives a contradiction.

We have thus established that $\theta'(x) \leq 0$ for every x in $(0, u)$. Since $\theta(u) \geq 0$, we also have $\theta(x) \geq 0$ for all x in $(0, u)$. This ends the proof of Lemma 3. ∎

Consider now condition (A5) as x_1 varies on $[x_0, \infty[$. Lemma 3 shows that $x_1 \to V_{x_1}(x_0)$ is decreasing. Thus the right-hand side of (A5) decreases in x_1. Moreover

$$V_{x_1}(x_1) - V_{x_1}(x_0) = \int_{x_0}^{x_1} V'_{x_1}(u)du \geq (x_1 - x_0)$$

since $V'_{x_1}(u) \geq V'_{x_1}(x_1) = 1$. Therefore

$$V_{x_1}(x_0) \leq V_{x_1}(x_1) - (x_1 - x_0) = \frac{\mu x_1 + (\mu - r)D}{\rho} - (x_1 - x_0).$$

Since $\mu < 0$, this converges to $-\infty$ when $x_1 \to \infty$. Lemma 3 also shows that $x_1 \to V'_{x_1}(x_0)$ increases in x_1. Uniqueness of the solution to (A5) is thus guaranteed. Existence of a solution $x_1 > x_0$ is then established if we can prove that

$$V'_{x_0}(x_0) < a_1 \frac{V_{x_0}(x_0)}{x_0}.$$

But $V'_{x_0}(x_0) = 1$, while $V_{x_0}(x_0) = \frac{\mu x_0 + (\mu - r)D}{\rho}$.
Thus we are left to prove that

$$H \equiv a_1 \left[\mu + (\mu - r)\frac{D}{x_0} \right] - \rho > 0.$$

But

$$\frac{D}{x_0} = \frac{1}{k} - 1 = \frac{\mu - r}{\sigma^2(1 - a_1)} - \rho.$$

Thus

$$\begin{aligned} H &= a_1 \left[\mu + \left(\frac{\mu - r}{\sigma} \right)^2 \frac{1}{1 - a_1} - \mu + r \right] - \rho \\ &= \frac{1}{1 - a_1} \left[r a_1 (1 - a_1) + 2t a_1 - \rho + \rho a_1 \right]. \end{aligned}$$

Remember that, by definition of a_1:

$$r a_1^2 + \rho = (r + \rho + t) a_1.$$

Thus

$$H = \frac{1}{1 - a_1} \left[(r + 2t + \rho) a_1 - (r + \rho + t) a_1 \right] = \frac{t a_1}{1 - a_1} > 0.$$

■

Proof of Proposition 1. By definition,

$$k = \frac{(1-a_1)\sigma^2}{\mu - r},$$

where a_1 is the smallest root of the quadratic equation:

$$ra^2 - \left[r + \rho + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right] a + \rho = 0.$$

After straightforward computations, we obtain:

$$k = \frac{(\mu - r)}{\rho - r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 + \sqrt{\left[\rho - r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right]^2 + 4tr}}.$$

Therefore it is clear that k decreases with the cost of capital $(\rho - r)$ and increases with the volatility of assets σ. To see that it is single-peaked with respect to $\mu - r$, it suffices to notice that

$$\frac{1}{k} = \frac{\rho - r}{\mu - r} + \frac{\mu - r}{2\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{\rho - r}{\mu - r} + \frac{\mu - r}{2\sigma^2}\right)^2}$$

which is an increasing function of

$$y = \frac{\rho - r}{\mu - r} + \frac{\mu - r}{2\sigma^2}.$$

Since y is itself a U-shaped function of $\mu - r$, k is single peaked in $\mu - r$, and the proof of Proposition 1 is complete.

Proof of Proposition 2. As we saw in the proof of Proposition 1, the risk aversion coefficient of the firm can be written as:

$$1 - a_1 = \frac{2}{\frac{\rho - r}{t} + 1 + \sqrt{\frac{4r}{t} + \left(\frac{\rho - r}{t} + 1\right)^2}},$$

where $t = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2$. Thus $1 - a_1$ is a decreasing function of $(\rho - r)$ and an increasing function of t. The proof of Proposition 2 a results immediately. ■

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