Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model

PETER M. DEMARZO AND YULIY SANNIKOV[*]

Abstract

We derive the optimal dynamic contract in a continuous-time principal-agent setting, and implement it with a capital structure (credit line, long-term debt, and equity) over which the agent controls the payout policy. While the project's volatility and liquidation cost have little impact on the firm's total debt capacity, they increase the use of credit versus debt. Leverage is nonstationary, and declines with past profitability. The firm may hold a compensating cash balance while borrowing (at a higher rate) through the credit line. Surprisingly, the usual conflicts between debt and equity (asset substitution, strategic default) need not arise.

In this paper, we consider a dynamic contracting environment in which a risk-neutral agent or entrepreneur with limited resources manages an investment activity. While the investment is profitable, it is also risky, and in the short run it can generate arbitrarily large operating losses. The agent will need outside financial support to cover such losses and continue the project. The difficulty is that while the probability distribution of the cash flows is publicly known, the agent may distort these cash flows by taking a hidden action that leads to a private benefit. Specifically, the agent may (i) conceal and divert cash flows for his own consumption, and/or (ii) stop providing costly effort, which would reduce the mean of the cash flows. Therefore, from the perspective of the principal or investors that fund the project, there is the concern that a low cash flow realization may be a result of the agent's actions, rather than the project's fundamentals. To provide the agent with appropriate incentives, investors control the agent's wage, and may also withdraw their financial support for the project and force its early termination. We seek to characterize an optimal contract in this framework and relate it to the firm's choice of capital structure.

We develop a method to solve for the optimal contract, given the incentive constraints, in a continuous-time setting and study the properties of the credit line, debt, and equity that implement the contract as in the discrete-time model of DeMarzo and Fishman (2003a). The continuous-time setting offers several advantages. First, it provides a much cleaner characterization of the optimal contract through an ordinary differential equation. Second, it yields a simple determination of the mix of debt and credit. Finally, the continuous-time setting allows us to compute comparative statics and security prices, to analyze conflicts of interest between security holders, and to generalize the model to broader settings.

In the optimal contract, the agent is compensated by holding a fraction of the firm's equity. The remaining equity, debt, and credit line are held by outside investors. The firm draws on the credit line to cover losses, and pays off the credit line when it realizes a profit. Thus, in our model leverage is negatively related with past profitability. Dividends are paid when cash flows exceed debt payments and the credit line is paid off. If debt service payments are not made or the credit line is overdrawn, the firm defaults and the project is terminated. In rare instances in which the firm pays a liquidating dividend to equity holders, only the outside equity is paid. Thus, payments to inside and outside equity differ only at liquidation.

The credit line is a key feature of our implementation of the optimal contract. Empirically, credit lines are an important (and understudied) component of firm financing: Between 1995 and 2004, credit lines accounted for 63% (by dollar volume) of all corporate debt.[1] Our results may shed light both on the choice between credit lines and other forms of borrowing, and the characteristics of the credit line contracts that are used. In our model, it is this access to credit that provides the firm the financial slack needed to operate given the risk of operating losses. The balance on the credit line, and therefore the amount of financial slack, fluctuates with the past performance of the firm. Thus, our model generates a dynamic model of capital structure in which leverage falls with the profitability of the firm.

In our continuous-time setting the project generates cumulative cash flows that follow a Brownian motion with positive drift. Using techniques introduced by Sannikov (2005), we develop a martingale approach to formulate the agent's incentive compatibility constraint. We then characterize the optimal contract through an ordinary differential equation. This characterization, unlike that using the discrete-time Bellman equation, allows for an analytic derivation of the impact of the model parameters on the optimal contract. The methodology we develop is quite powerful, and can be naturally extended to include more complicated moral hazard environments, as well as investment and project selection.

In addition to this methodological contribution, by formulating the model in continuous-time we obtain a number of important new results. First, in the discrete-time setting, public randomization over the decision to terminate the project is sometimes required. We show that this randomization, which is somewhat unnatural, is not required in the continuous-time setting. Indeed, in our model the termination decision is based only on the firm's past performance.

A second feature of our setting is that, because cash flows are normally distributed, arbitrarily large operating losses are possible. In the discrete-time setting, such a project would be unable to obtain financing. We show not only how to finance such a project, but also how, when the risk of loss is severe, the optimal contract may require that the firm hold a compensating balance (a cash deposit that the firm must hold with the lender to maintain the credit line) as a requirement of the credit line. The compensating balance commits outside investors to provide the firm funds, through interest payments, that the firm might not be able to raise ex post. Thus, the compensating balance allows for a larger credit line, which is valuable given the risk of the project, and it provides an inflow of interest

payments to the project that can be used to somewhat offset operating losses. The model therefore provides an explanation for the empirical observation that many firms hold substantial cash balances at low interest rates while simultaneously borrowing at higher rates.

Third, in our capital structure implementation, the agent controls not only the cash flows but also the payout policy of the firm. We show that the agent will optimally choose to pay off the credit line before paying dividends, and, once the credit line has been paid off, to pay dividends rather than hoard cash to generate additional financial slack. In the continuous-time setting, the incentive compatibility of the firm's payout policy reduces to a simple and intuitive constraint on the maximal interest expense that the firm can bear, based on the expected cash flows of the project and the agent's outside opportunity. This constraint implies that the firm's *total* debt capacity is relatively insensitive to the risk of the project and its liquidation cost. However, these factors are primary determinants of the mix of long-term debt and credit that the firm will use. Not surprisingly, firms with higher risk and liquidation costs gain financial flexibility by substituting credit for long-term debt. Note that while this result does not come out of standard theories, it is broadly consistent with the empirical findings of Benmelech (2004) (for 19th century railroads).

In addition to enabling us to compute these and other comparative statics results, our continuous-time framework also allows us to explicitly characterize the market values of the firm's securities. We show how the market value of the firm's equity and debt vary with its credit quality, which is determined by its remaining credit. Moreover, we are able to explore not only the agent's incentives but also those of equity holders. One surprising feature of our model of optimal capital structure is that, despite the firm's use of leverage, equity holders (as well as the agent) have no incentive to increase risk, that is, under our contract, there is no asset substitution problem. In addition, for a wide range of parameters, there is no strategic default problem, that is, equity holders have no incentive to increase dividends and precipitate default, or to contribute new capital and postpone default.[2]

For the bulk of our analysis, we focus on the case in which the agent can conceal and divert cash flows. In Section III, we show that the characterization of the optimal contract is unchanged if the agent makes a hidden effort choice, as in a standard principal-agent model. In Section IV, we endogenize the termination liquidation payoffs by allowing investors to fire and replace the agent and

by allowing the agent to quit to start a new project. We also consider renegotiation and solve for the optimal renegotiation-proof contract.

Our paper is part of a growing literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Phelan and Townsend (1991), and Atkeson (1991) among others (see Ljungqvist and Sargent (2000) for a description of many of these models). As we mention above, this paper builds directly on the model of DeMarzo and Fishman (2003a). Other recent work that develops optimal dynamic agency models of the firm includes Albuquerque and Hopenhayn (2001), Clementi and Hopenhayn (2002), DeMarzo and Fishman (2003b), and Quadrini (2001). However, with the exception of DeMarzo and Fishman (2003a), these papers do not share our focus on an optimal capital structure. In addition, none of these models are formulated in continuous time.[3]

While discrete-time models are adequate conceptually, a continuous-time setting may prove to be simpler and more convenient analytically. An important example is the principal-agent model of Holmstrom and Milgrom (1987), in which the optimal continuous-time contract is shown to be a linear "equity" contract.[4] Several features distinguish our model from theirs, namely, the investor's ability to terminate the project, the agent's consumption while the project is running, the limited wealth of the agent, and the nature of the agency problem. The termination decision is a key feature of our optimal contract, and we demonstrate how this decision can be implemented through bankruptcy.[5]

In contemporaneous work, Biais et al. (2004) consider a dynamic principal-agent problem in which the agent's effort choice is binary. These authors do not formulate the problem in continuous time: rather, they exam the continuous limit of the discrete-time model and focus on the implications for the firm's balance sheet. We show in Section III that their setting is a special case of our model. Tchistyi (2005) develops a continuous-time model that is similar to our setting except that the cash flows follow a binary Markov switching process, that is, cash flows arrive at either a high or low rate, with the switches between states observed only by the agent. The agent's private knowledge of the state introduces a dynamic asymmetric information problem, which Tchistyi shows can be solved by making the interest rate on the credit line increase with the balance.

Of course, there is a large literature on static models of security design. We do not attempt to survey this literature here.[6] That said, our model is loosely related to the continuous-time capital

structure models developed by Leland and Toft (1996), Leland (1998), and others. These papers take the form of the securities as given and derive the effect of capital structure on the incentives of the manager, debt holders, and shareholders, taking into account issues such as the tax benefits of debt, strategic default, and asset substitution. Here, we derive the *optimal* security design and show that the standard agency problems between debt and equity holders may not arise.

I. The Setting and the Optimal Contract

In this section we present a continuous-time formulation of the contracting problem and develop a methodology that can be used to characterize the optimal contract as a solution of a differential equation. We then implement the contract with a capital structure that includes outside equity, long-term debt, and a line of credit. This implementation decentralizes the solution of a standard principal-agent model into separate securities that can be held by dispersed investors, giving the agent a high degree of discretion over the firm's payout policy.

A. The Dynamic Agency Model

The agent manages a project that generates potential cash flows with mean μ and volatility σ

$$dY_t = \mu\, dt + \sigma\, dZ_t,$$

where Z is a standard Brownian motion. For now we assume that the agent is essential to run the project; in Section IV.A we allow the principal to fire the agent and hire a replacement. The agent observes the potential cash flows Y, but the principal does not. The agent reports cash flows $\{\hat{Y}_t; t \geq 0\}$ to the principal, where the difference between Y and $\hat{Y}$ is determined by the agent's hidden actions, which are the source of the agency problem. The principal receives only the reported cash flows $d\hat{Y}_t$ from the agent. The contract then specifies compensation for the agent dI_t, as well as a termination time τ, that are based on the agent's reports.

In this section we model the agency problem by allowing the agent to divert cash flows for his own private benefit; in Section III we show how to adapt the model to the case of hidden effort. The agent receives a fraction $\lambda \in (0,1]$ of the cash flows he diverts; if $\lambda < 1$, there are dead-weight costs of concealing and diverting funds. The agent can also exaggerate cash flows by putting his own money

back into the project. By altering the cash flow process in this way, the agent receives a total flow of income of[7]

$$[dY_t - d\hat{Y}_t]^\lambda + dI_t, \quad \text{where } [dY_t - d\hat{Y}_t]^\lambda \equiv \underbrace{\lambda(dY_t - d\hat{Y}_t)^+}_{\text{diversion}} - \underbrace{(dY_t - d\hat{Y}_t)^-}_{\text{over-reporting}}. \tag{1}$$

The agent is risk neutral and discounts his consumption at rate γ. The agent maintains a private savings account, from which he consumes and into which he deposits his income. The principal cannot observe the balance of the agent's savings account. The agent's balance S_t grows at interest rate $\rho < \gamma$:

$$dS_t = \rho S_t dt + [dY_t - d\hat{Y}_t]^\lambda + dI_t - dC_t, \tag{2}$$

where $dC_t \geq 0$ is the agent's consumption at time t. The agent must maintain a nonnegative balance on his account, that is, $S_t \geq 0$. This resource constraint prevents a solution in which the agent simply owns the project and runs it forever.

Once the contract is terminated, the agent receives payoff $R \geq 0$ from an outside option. Therefore, the agent's total expected payoff from the contract at date 0 is given by[8]

$$W_0 = E\left[\int_0^\tau e^{-\gamma s} dC_s + e^{-\gamma \tau} R \right]. \tag{3}$$

The principal discounts cash flows at rate r, such that $\gamma > r \geq \rho$.[9] Once the contract is terminated, she receives expected liquidation payoff $L \geq 0$. (In Section IV, we consider how the termination payoffs R and L arise, for example, from the principal's ability to fire and replace the agent, or the agent's ability to renegotiate the contract or start a new project). The principal's total expected profit at date 0 is then

$$b_0 = E\left[\int_0^\tau e^{-rs} (d\hat{Y}_s - dI_s) + e^{-r\tau} L \right]. \tag{4}$$

The project requires external capital of $K \geq 0$ to be started. The principal offers to contribute this capital in exchange for a contract (τ, I) that specifies a termination time τ and payments $\{I_t; 0 \leq t \leq \tau\}$ that are based on reports $\hat{Y}$. Formally, I is a $\hat{Y}$-measurable continuous process, and τ is a $\hat{Y}$-measurable stopping time.

In response to a contract (τ, I), the agent chooses a feasible strategy to maximize his expected payoff. A feasible strategy is a pair of processes $(C, \hat{Y})$ adapted to Y such that

(i) $\hat{Y}$ is continuous and, if $\lambda < 1$, $Y_t - \hat{Y}_t$ has bounded variation,[10]

(ii) C_t is nondecreasing, and

(iii)   the savings process, defined by (2), stays nonnegative.

The agent's strategy $(C, \hat{Y})$ is incentive compatible if it maximizes his total expected payoff W_0 given a contract (τ, I). An *incentive compatible contract* refers to a quadruple $(\tau, I, C, \hat{Y})$ that includes the agent's recommended strategies.

Note that we have not explicitly modeled the agent's option to quit and receive the outside option R at any time. Because the agent can always underreport and steal at rate γR until termination, any incentive compatible strategy yields the agent at least R. In contrast, this constraint may bind in a discrete-time setting because of a limit to the amount the agent can steal per period.

The optimal contracting problem is to find an incentive compatible contract $(\tau, I, C, \hat{Y})$ that maximizes the principal's profit subject to delivering the agent an initial required payoff W_0. By varying W_0 we can use this solution to consider different divisions of bargaining power between the agent and the principal. For example, if the agent enjoys all the bargaining power due to competition between principals, then the agent must receive the maximal value of W_0 subject to the constraint that the principal's profit be at least zero.

REMARK. For simplicity, we specify the contract assuming that the agent's income I and the termination time τ are determined by the agent's report, ruling out public randomization. This assumption is without loss of generality: Because the principal's value function turns out to be concave (Proposition 1), we will show that public randomization would not improve the contract.

B. Derivation of the Optimal Contract

We solve the problem of finding an optimal contract in three steps. First, we show that it is sufficient to look for an optimal contract within a smaller class of contracts, namely, contracts in which the agent chooses to report cash flows truthfully and maintain zero savings. Second, we consider a relaxed problem by ignoring the possibility that the agent can save secretly. Third, we show that the contract is fully incentive compatible even when the agent can save secretly.

We begin with a revelation principle type of result:[11]

LEMMA A: *There exists an optimal contract in which the agent i) chooses to tell the truth, and ii) maintains zero savings.*

The intuition for this result is straightforward – it is inefficient for the agent to conceal and divert cash flows $(\lambda \leq 1)$ or to save them $(\rho \leq r)$, as we could improve the contract by having the principal

save and make direct payments to the agent. Thus, we will look for an optimal contract in which truth telling and zero savings are incentive compatible.

B.1. The Optimal Contract without Savings

Note that if the agent could not save, then he would not be able to overreport cash flows and he would consume all income as it is received. Thus,

$$dC_t = dI_t + \lambda(dY_t - d\hat{Y}_t). \tag{5}$$

We relax the problem by restricting the agent's savings so that (5) holds and allowing the agent to steal only at a bounded rate.[12] After we find an optimal contract for the relaxed problem, we show that it remains incentive compatible even if the agent can save secretly or steal at an unbounded rate.

One challenge when working in a dynamic setting is the complexity of the contract space. Here, the contract can depend on the entire path of reported cash flows $\hat{Y}$. This makes it difficult to evaluate the agent's incentives in a tractable way. Thus, our first task is to find a convenient representation of the agent's incentives. Define the agent's promised value $W_t(\hat{Y})$ after a history of reports ($\hat{Y}_s$, $0 \leq s \leq t$) to be the total expected payoff the agent receives, from transfers and termination utility, if he tells the truth after time t:

$$W_t(\hat{Y}) = E_t\left[\int_t^\tau e^{-\gamma(s-t)}dI_s + e^{-\gamma(\tau-t)}R\right].$$

The following result provides a useful representation of $W_t(\hat{Y})$.

LEMMA B: *At any moment of time $t \leq \tau$, there is a sensitivity $\beta_t(\hat{Y})$ of the agent's continuation value towards his report such that*

$$dW_t = \gamma W_t dt - dI_t + \beta_t(\hat{Y})(d\hat{Y}_t - \mu dt). \tag{6}$$

This sensitivity $\beta_t(\hat{Y})$ is determined by the agent's past reports $\hat{Y}_s$, $0 \leq s \leq t$.

Proof of Lemma B: Note that $W_t(\hat{Y})$ is also the agent's promised value if $\hat{Y}_s$, $0 \leq s \leq t$, were the true cash flows and the agent reported truthfully. Therefore, without loss of generality we can prove (6) for the case in which the agent truthfully reports $\hat{Y} = Y$.[13] In that case,

$$V_t = \int_0^t e^{-\gamma s}dI_s(Y) + e^{-\gamma t}W_t(Y) \tag{7}$$

is a martingale and by the martingale representation theorem there is a process β such that $dV_t = e^{-\gamma t}$ $\beta_t(Y)$ $(dY_t - \mu\, dt)$, where $dY_t - \mu\, dt$ is a multiple of the standard Brownian motion. Differentiating (7) with respect to t we find

$$dV_t = e^{-\gamma t}\beta_t(Y)(Y_t - \mu dt) = e^{-\gamma t}dI_t(Y) - \gamma e^{-\gamma t}W_t(Y)dt + e^{-\gamma t}dW_t(Y),$$

and thus (6) holds. ◆

Informally, the agent has incentives not to steal cash flows if he gets at least λ of promised value for each reported dollar, that is, if $\beta_t \geq \lambda$. If this condition holds for all t then the agent's payoff will always integrate to less than his promised value if he deviates. If this condition fails on a set of positive measure, the agent can obtain at least a little bit more than his promised value if he underreports cash when $\beta_t < \lambda$. We summarize our conclusions in the following lemma.

LEMMA C: *If the agent cannot save, truth-telling is incentive compatible if and only if $\beta_t \geq \lambda$ for all $t \leq \tau$.*

Proof of Lemma C: If the agent steals $dY_t - d\hat{Y}_t$ at time t, he gains immediate income of $\lambda(dY_t - d\hat{Y}_t)$ but loses $\beta_t(dY_t - d\hat{Y}_t)$ in continuation payoff. Therefore, the payoff from reporting strategy $\hat{Y}$ gives the agent the payoff of

$$W_0 + E\left[\int_0^\tau e^{-\gamma t}\lambda(dY_t - d\hat{Y}_t) - \int_0^\tau e^{-\gamma t}\beta_t(dY_t - d\hat{Y}_t)\right], \tag{8}$$

where W_0 denotes the agent's payoff under truth-telling. We see that if $\beta_t \geq \lambda$ for all t then (8) is maximized when the agent chooses $d\hat{Y}_t = dY_t$, since the agent cannot overreport cash flows. If $\beta_t < \lambda$ on a set of positive measure, then the agent is better off underreporting on this set than always telling the truth.[14] ◆

Now we use the dynamic programming approach to determine the most profitable way for the principal to deliver the agent any value W. Here we present an informal argument, which we formalize in the proof of Proposition 1 in the Appendix. Denote by $b(W)$ the principal's value function (the highest profit to the principal that can be obtained from a contract that provides the agent the payoff W). To facilitate our derivation of b, we assume b is concave. In fact, we could always ensure that b is concave by allowing public randomization, but at the end of our intuitive argument we will see that public randomization is not needed in an optimal contract.[15]

Because the principal has the option to provide the agent with W by paying a lump-sum transfer of $dI > 0$ and moving to the optimal contract with payoff $W - dI$,

$$b(W) \geq b(W - dI) - dI . \qquad (9)$$

Equation (9) implies that $b'(W) \geq -1$ for all W; that is, the marginal cost of compensating the agent can never exceed the cost of an immediate transfer. Define W^1 as the lowest value such that $b'(W^1) = -1$. Then it is optimal to pay the agent according to

$$dI = \max(W - W^1, 0). \qquad (10)$$

These transfers, and the option to terminate, keep the agent's promised value between R and W^1. Within this range, equation (6) implies that the agent's promised value evolves according to $dW_t = \gamma W_t dt + \beta_t \sigma \, dZ_t$ when the agent is telling the truth. We need to determine the sensitivity β of the agent's value to reported cash flows. Using Ito's lemma, the principal's expected cash flows and changes in contract value are given by

$$E[dY + db(W)] = \left(\mu + \gamma W b'(W) + \tfrac{1}{2}\beta^2\sigma^2 b''(W) \right) dt.$$

Because at the optimum the principal should earn an instantaneous total return equal to the discount rate, r, we have the following Hamilton-Jacobi-Bellman (HJB) equation for the value function:

$$rb(W) = \max_{\beta \geq \lambda} \quad \mu + \gamma W b'(W) + \tfrac{1}{2}\beta^2\sigma^2 b''(W). \qquad (11)$$

Given the concavity of b, $b''(W) \leq 0$ and thus $\beta = \lambda$ is optimal.[16] Intuitively, because the inefficiency in this model results from early termination, reducing the risk to the agent lowers the probability that the agent's promised value falls to R.

The principal's value function therefore satisfies the following second-order ordinary differential equation:

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W) , \quad R \leq W \leq W^1, \qquad (12)$$

with $b(W) = b(W^1) - (W - W^1)$ for $W > W^1$.

We require three boundary conditions to pin down a solution to this equation and the boundary W^1. The first boundary condition arises because the principal must terminate the contract to hold the agent's value to R, so $b(R) = L$. The second boundary condition is the usual "smooth pasting" condition – the first derivatives must agree at the boundary – and so $b'(W^1) = -1$.[17]

The final boundary condition is the "super contact" condition for the optimality of W^1, which requires that the second derivatives match at the boundary. This condition implies that $b''(W^1) = 0$, or equivalently, using equation (12),

$$rb(W^1) + \gamma W^1 = \mu \, . \tag{13}$$

This boundary condition has a natural interpretation: It is beneficial to postpone payment to the agent by making W^1 larger because doing so reduces the risk of early termination. Postponing payment is sensible until the boundary (13), when the principal and agent's required expected returns exhaust the available expected cash flows.[18] Figure 1 shows an example of the value function.

The following proposition formalizes our findings:

PROPOSITION 1: *The contract that maximizes the principal's profit and delivers the value $W_0 \in [R, W^1]$ to the agent takes the following form: W_t evolves according to*

$$dW_t = \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - \mu\, dt). \tag{14}$$

When $W_t \in [R, W^1)$, $dI_t = 0$. When $W_t = W^1$, payments dI_t cause W_t to reflect at W^1. If $W_0 > W^1$, an immediate payment $W_0 - W^1$ is made. The contract is terminated at time τ, when W_t reaches R. The principal's expected payoff at any point is given by a concave function $b(W_t)$, which satisfies

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W) \tag{15}$$

on the interval [R, W^1], $b'(W) = -1$ for $W \geq W^1$, and boundary conditions $b(R) = L$ and $rb(W^1) = \mu - \gamma W^1$.

B.2. Hidden Savings

Thus far, we restrict the agent from saving and over-reporting strategies. We now show that the contract of Proposition remains incentive compatible even when we relax this restriction. The intuition for the result is that because the marginal benefit to the agent of reporting or consuming cash is constant over time, and further, because private savings grow at rate $\rho < \gamma$, there is no incentive to delay reporting or consumption. In fact, in the proof we show that this result holds even if the agent can save within the firm without paying the diversion cost.

PROPOSITION 2: *Suppose the process $W_t \geq R$ is bounded from above and solves*

$$dW_t = \gamma W_t dt - dI_t dt + \lambda(d\hat{Y}_t - \mu dt) \tag{16}$$

until stopping time $\tau = \min\{t \mid W_t = R\}$. Then the agent earns a payoff of at most W_0 from any feasible strategy in response to a contract (τ, I). Furthermore, the payoff W_0 is attained if the agent reports truthfully and maintains zero savings.

This result confirms that contracts from a broad class, including the optimal contract of Proposition 1, remain incentive compatible even if the agent has access to hidden savings. In the next subsection Proposition 2 will help us characterize incentive-compatible capital structures.

C. Capital Structure Implementation

So far, our results characterize the optimal principal-agent contract. In this section, we show how this contract can be implemented using standard securities that are held by widely dispersed investors or intermediaries. The firm raises initial capital K and possibly additional cash (to fund an initial dividend or cash reserve for the firm) by issuing the securities at time 0.

Because the optimal contract is conditional on the agent's promised payoff W, the implementation we describe will hold regardless of whether the agent designs the securities to maximize his own payoff, or the investors design the securities to maximize the value of the firm. (We discuss alternative distributions of bargaining power between the agent and investors in Section II.A.)

We begin by describing the securities used in the implementation:

Equity. Equity holders receive dividend payments made by the firm. Dividends are paid from the firm's available cash or credit, and are at the discretion of the agent.

Long-term Debt. Long-term debt is a consol bond that pays continuous coupons at rate x. Without loss of generality, we let the coupon rate be r, so that the face value of the debt is $D = x/r$. If the firm defaults on a coupon payment, debt holders force termination of the project.

Credit Line. A revolving credit line provides the firm with available credit up to a limit C^L. Balances on the credit line are charged a fixed interest rate r^c. The firm borrows and repays funds on the credit line at the discretion of the agent. If the balance on the credit line exceeds C^L, the firm defaults and the project is terminated.

We now show that the optimal contract can be implemented using a capital structure based on these securities. While the implementation is not unique (e.g., one could always use the single contract, or strip the debt into zero-coupon bonds), it provides a natural interpretation. It also demonstrates how

the contract can be decentralized into limited liability securities (equity and debt) that can be widely held by investors. Finally, it shows that the optimal contract is consistent with a capital structure in which, in addition to the ability to steal the cash flows, the agent has wide discretion regarding the firm's leverage and payout policy – the agent can choose when to draw on or repay the credit line, how much to pay in dividends, and whether to accumulate cash balances within the firm.

While it is important for pricing the securities, for the implementation it is not necessary to specify the prioritization of the securities over the liquidation payoff L. However, because we compensate the agent with equity, it is important that the agent does not receive part of the liquidation payoff. Thus, we define *inside equity* as identical to equity, but with the provision that it is worthless in the event of termination.[19] (With absolute priority this distinction will often be unnecessary, as debt claims typically exhaust L.)

PROPOSITION 3: *Consider a capital structure in which the agent holds inside equity for fraction λ of the firm, the credit line has interest rate $r^c = \gamma$, and debt satisfies*

$$rD = \mu - \gamma R / \lambda - \gamma C^L. \tag{17}$$

Then it is incentive compatible for the agent to refrain from stealing and to use the project cash flows to pay the debt coupons and credit line before issuing dividends. Once the credit line is fully repaid, all excess cash flows are issued as dividends. Under this capital structure, the agent's expected future payoff W_t is determined by the current draw M_t on the credit line:

$$W_t = R + \lambda \left(C^L - M_t \right). \tag{18}$$

This capital structure implements the optimal contract if, in addition, the credit limit satisfies

$$C^L = \lambda^{-1}(W^1 - R). \tag{19}$$

The intuition for the incentive compatibility of this capital structure is as follows. First, providing the agent fraction λ of the equity eliminates his incentive to divert cash because he can do as well by paying dividends. How can we ensure that the agent does not pay dividends prematurely by, for example, drawing down the credit line immediately and paying a large dividend? Given balance M_t on the credit line, the agent can pay a dividend of $C^L - M_t$ and then default. But (18) implies that the payoff from this deviation would be equal to W_t, the payoff that the agent receives from waiting until the credit line balance is zero before paying dividends. Finally, because the agent earns interest at his

discount rate γ paying off the credit line, but earns interest at rate $r < \gamma$ on accumulated cash, the agent has the incentive to pay dividends once the credit line is repaid.

The role of the long-term debt, defined by (17), is to adjust the profit rate of the firm so that the agent's payoff satisfies equation (18).[20] If the debt were too high, the agent's payoff would be below the amount in (18) and the agent would draw down the credit line immediately. If the debt were too low and the firm's profit rate too high, the agent would build up cash reserves after the credit line was paid off in order to reduce the risk of termination. Thus, if (17) holds, we say that the capital structure is incentive compatible – the agent will not steal and will pay dividends if and only if the credit line is fully repaid.

Under what conditions does this capital structure implement the optimal contract of Section B? The history dependence of the optimal contract is implemented through the credit line, with the balance on the credit line acting as the "memory" device that tracks the agent's payoff W_t. In the optimal contract, the agent is paid in order to keep the promised payoff from exceeding W^1. Here, dividends are paid when the balance on the credit line is $M_t = 0$. To implement the optimal contract, these conditions must coincide. Solving equation (18) for C^L leads to the optimality condition $C^L = \lambda^{-1}(W^1 - R)$.

There is no guarantee that in this capital structure the debt required by equation (17) is positive. If $D < 0$, we interpret the debt as a *compensating balance*, that is, a cash deposit required by the bank issuing the credit line. The firm earns interest on this balance at rate r, and the interest supplements the firm's cash flows. The firm cannot withdraw this cash, and it is seized by creditors in the event of default. We examine the circumstances in which a compensating balance arises in the next section.

The implementation here is very similar to that given for the discrete-time model of DeMarzo and Fishman (2003a).[21] There are three important distinctions, however. First, because cash flows arrive in discrete portions, the termination decision is stochastic in the discrete-time setting (i.e., the principal randomizes when the agent defaults). Second, because cash flows may be arbitrarily negative in a continuous-time setting, the contract may involve a compensating balance requirement as opposed to debt. Lastly, the discrete-time framework does not allow for a simple characterization of the incentive compatibility condition for the capital structure in terms of the primitives of the model, as we do here in equation (17). In particular, when γ is close to r, this condition implies that the *total* debt capacity of the firm,

$$D + C^L = \mu / \gamma - R / \lambda + (1 - r / \gamma) D \approx \mu / \gamma - R / \lambda ,$$

is relatively insensitive to the volatility σ and liquidation value L of the project. The *mix* of debt and credit will depend on these parameters, however, as we explore next.

II. Optimal Capital Structure and Security Prices

The capital structure implementation of the optimal contract raises many interesting questions. For instance, what factors determine the amount that the agent borrows? Under what conditions does the agent borrow for initial consumption? When does a compensating balance arise? What is the optimal length of the credit line? How do the market values of the securities involved in the contract depend on the firm's remaining credit? In this section, we exploit the continuous-time machinery to answer these questions and provide new insights.

A. The Debt Choice

A key feature of the optimal capital structure is its use of both fixed long-term debt and a revolving credit line. In this section we develop further intuition for how the amount of long-term debt, the size of the credit line, and the initial draw on the credit line are determined.

To simplify the analysis, we focus on the case $\lambda = 1$ in which there is no cost to diverting cash flows. In this case, the agent holds the equity of the firm and finances the firm solely through debt. While this case might appear restrictive, the following result shows that the optimal debt structure with lower levels of λ can be determined by considering an appropriate change to the termination payoffs.

PROPOSITION 4: *The optimal debt and credit line with agency parameter and termination payoffs (λ, R, L) are the same as with parameters (1, R^λ, L^λ), where*

$$R^\lambda = \tfrac{1}{\lambda} R \quad and \quad L^\lambda = \tfrac{1}{\lambda} L + (1 - \tfrac{1}{\lambda}) \tfrac{\mu}{r} .$$

When $\lambda = 1$, the optimal credit limit is $C^L = W^1 - R$. The optimal level of debt is then determined by (17), which in this case can be written as

$$rD = \mu - \gamma R - \gamma C^L = \mu - \gamma W^1 .$$

Recall also that in the optimal contract, W^1 is determined by the boundary condition (13):

$$rb(W^1) + \gamma W^1 = \mu.$$

Combining these two results implies that the optimal face value of debt is $D = b(W^1)$. Figure 2 provides an example, illustrating the size of the credit line and the face value of debt when the cash flow volatility is low. From the figure, $D > L$, so the debt is risky.

Note that the optimal capital structure for the firm does not depend on the amount of external capital K that is required. However, the initial payoffs of the agent and the investors depend upon K as well as the parties' relative bargaining power. If investors are competitive, the agent's initial payoff is the maximal payoff W_0 such that $b(W_0) = K$ as Figure 2 illustrates. In this example, $W_0 > W^1$. This payoff is achieved by giving the agent an initial cash payment of $W_0 - W^1$, and starting the firm with zero balance on the credit line (providing the agent with future payoff W^1). In other words, the firm issues long-term debt to fund the project and pays an initial dividend of $W_0 - W^1$. The credit line is then used as needed to cover operating losses.

Thus, the firm raises $b(W^1)$ from investors, which is equal to the face value of debt D. However, because the debt is risky ($D > L$), given coupon rate r it must trade at a discount. How does the firm raise the additional capital to make up for this discount? Given the high interest rate γ on the credit line, the lender earns an expected profit from the credit line, and so pays the firm an amount upfront that exactly offsets the initial discount on the long-term debt due to credit risk.

Recall that the optimal credit line results from the following trade-off: A large credit line delays the agent's consumption, but also gives the project more flexibility by delaying termination. Payments on debt are chosen to give the agent incentives to report truthfully. If payments on debt were too burdensome, the agent would draw down the credit line immediately and quit the firm; if they were too small, the agent would delay termination by saving excess cash flows when the credit line is paid off. In Figure 3, we illustrate how these intuitive considerations affect the optimal contract for different levels of volatility. With an increase in volatility, the investors' payoff function drops. Riskier cash flows require more financial flexibility, so the credit line becomes longer. Given the higher interest burden of the longer credit line, the optimal level of debt shrinks.

With medium volatility (the left panel of Figure 3), the face value of debt is below the liquidation value of the firm ($D < L$). Thus, if long-term debt has priority in default, it is now riskless, in which case the firm will raise D through a long-term debt issue. However, in this case we also have $D < K$, so the firm must raise the additional capital needed to initiate the project through an initial draw on the

credit line of $W^1 - W_0$. Because $b' > -1$ on (W_0, W_1), the draw on the credit line exceeds $K - D$. The difference can be interpreted as an initial fee charged by the lender to open the credit line with this initial balance; this fee compensates the lender for the negative net present value of the credit line due to the firm's greater credit risk.

With high volatility (the right panel of Figure 3), the investors' payoff falls further. This very risky project requires a very long credit line. Note that in this case $D = b(W^1) < 0$. We can interpret $D < 0$ as a compensating balance requirement – the firm must hold cash in the bank with a balance equal to $-D$ as a condition of the credit line. Both the required capital K and the compensating balance $-D$ are funded through a large initial draw on the credit line of $W^1 - W_0$. Given this large initial draw, substantial profits must be earned before dividends are paid.

The compensating balance provides the firm additional operating income of rD. This income increases the profitability of the firm, making it incentive compatible for the agent to run the firm rather than consume the credit line and immediately default. Also, by funding the compensating balance upfront, investors are committed to providing the firm with income rD even when the credit line is paid off. This commitment is necessary since investors' continuation payoff at W^1 is negative, which would violate their limited liability. The compensating balance therefore serves to tie the agent and the investors to the firm in an optimal way.

Insert Fig. 3 here

Finally, note that if we increase volatility further in this example, the maximal profit for the principal falls below K. Thus, while such a project has positive net present value, it cannot be financed due to the incentive constraints.

REMARK. While here we assume that the agent owns the firm and the investors are competitive, other possibilities are straightforward. For example, if the current owners choose the capital structure to maximize the firm's value, and the agent is hired from a competitive labor market, the contract would be initiated at the value W^* that maximizes the principal's payoff $b(W^*)$. The optimal capital structure would be unchanged, but the firm would always start with a draw on the credit line. Indeed, the initial leverage of the firm increases with investors' bargaining power. Comparing Figure 2 and Figure 3, while higher volatility decreases $b(W^*)$, the effect on the agent's payoff W^* is not monotonic. Thus, a hired agent might prefer to manage a higher risk project.

B. Comparative Statics

How do the credit line, debt, and the agent's and investors' initial payoffs depend on the parameters of the model? In the discrete-time setting, many of these comparative statics are analytically intractable and can only be computed for a specific example. A key advantage of the continuous-time framework, on the other hand, is that we can use the differential equation that characterizes the optimal contract to compute these comparative statics analytically.

Here we outline a new methodology for explicitly calculating comparatives statics. First, we derive the effect of parameters on the principal's profit. We start with the Hamilton-Jacobi-Bellman equation for the principal's profit for a fixed credit line, which is represented by the interval $[R, W^1]$:

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W).$$

The effect of any parameter θ on the principal's profit can be found by differentiating the HJB equation and its boundary conditions with respect to θ. During differentiation we keep W^1 fixed, which is justified by the envelope theorem. As a result, we get an ordinary differential equation for $\partial b(W)/\partial\theta$ with appropriate boundary conditions. We then apply a generalization of the Feynman-Kac formula to write the solution as an expectation, that is,

$$\frac{\partial b(W)}{\partial\theta} = E\left[\int_0^\tau e^{-rt}\left(\frac{\partial\mu}{\partial\theta} + \frac{\partial\gamma}{\partial\theta}W_t b'(W_t) + \frac{1}{2}\frac{\partial(\lambda^2\sigma^2)}{\partial\theta}b''(W_t)\right)dt + e^{-r\tau}\frac{\partial L}{\partial\theta}\,\Big|\, W_0 = W\right], \qquad (20)$$

where $dW_t = \gamma W_t dt - dI_t + \lambda dZ_t$ as before. Intuitively, equation (20) counts how much profit is gained or lost on the path of W_t due to the modification of parameters. For example,

$$\frac{\partial b(W)}{\partial L} = E\left[e^{-r\tau}\,\big|\,W_0 = W\right],$$

which is the expected discounted value of a dollar at the time of liquidation.

Once we know the effect of parameters on the principal's profit, we deduce their effect on debt and the credit line by differentiating the boundary condition $rb(W^1) + \gamma W^1 = \mu$, and on the agent's starting value by differentiating $b(W_0) = K$ (or $b'(W^*) = 0$ when the principal is a monopolist). For example, the effect of L is found as follows:

$$r\left(\frac{\partial b(W^1)}{\partial L} + \underbrace{b'(W^1)}_{-1}\frac{\partial W^1}{\partial L}\right) + \gamma\frac{\partial W^1}{\partial L} = 0 \;\Rightarrow\; \frac{\partial W^1}{\partial L} = -\frac{r}{\gamma-r}E\left[e^{-r\tau}\,|\,W_0 = W^1\right] < 0.$$

As L increases, the inefficiency of liquidation declines, so a shorter credit line optimally provides less financial flexibility for the project. By similar methods, we can quantify the impact of the model

Insert Table I here

parameters on the main features of an optimal contract. Table I reports the results. The derivations are carried out in the Appendix.

The intuition for the results in Table I is clear. For example, consider the mix of debt and credit. We have already shown that credit decreases as L increases, since liquidation is less inefficient and financial slack is less valuable. If the agent's outside option R increases, the agent becomes more tempted to draw down the credit line and default. The length of the credit line decreases to reduce this temptation, and payments on debt decrease to make it more attractive for the agent to run the project, as opposed to taking the outside option. If the mean of cash flows μ increases, the credit line increases to delay termination and debt increases because the principal can extract more cash flows from the agent. If the agent's discount rate γ increases, then the credit line decreases because it becomes costlier to delay the agent's consumption. On the other hand, the amount of debt could move either way: For small γ, debt increases in γ because the agent is able to borrow more through debt when the credit line is smaller, whereas for large γ, the project becomes less profitable due to the agent's impatience, in which case the agent is able to borrow less through debt. As seen in Section II.A, the credit limit increases and the debt decreases with volatility σ – riskier projects require longer lines of credit and thus the agent is able to borrow less through debt. Finally, the effect of λ is complex. Consider the special case of $R = 0$. For this case the credit line is decreasing in λ: The cost of delaying dividends becomes larger when the impatient agent owns a larger fraction of equity. At the same time, however, debt increases to offset the decreased credit line.

The effect of the parameters on W_0 and $b(W^*)$ is the same since they both reflect the profitability of the project. When L or μ increase, the project becomes more profitable. The project becomes less profitable with an increase in the risk of the project σ^2, the agent's impatience γ, the magnitude of the agency problem λ, or the agent's outside option R. Finally, the effect of the parameters on the agent's starting value W^* when investors have all the bargaining power is determined by the following trade-off: Larger W^* delays termination at a greater cost of paying the agent.

In Figure 4 we conclude by computing the quantitative effect of the parameters on the debt choice of the firm for a specific example Note that in this example, a compensating balance is required if σ is high (to mitigate risk), if R is high or μ is low (to increase the profit rate of the firm and maintain the agent's incentive to stay), or if λ is very low (when the agency problem is small, a smaller threat of

Insert Fig. 4 here

termination is needed, and thus the credit line expands and debt shrinks). Though not visible in the figure, the compensating balance arises also as $\gamma \to r$.

C. Security Market Values

We now consider the market values of the credit line, long-term debt, and equity that implement the optimal contract. To do so, we need to make an assumption regarding the priority of debt in the case of default. Here we assume that long-term debt is senior to the credit line; similar calculations could be performed for different seniority assumptions.[22] With this assumption, the long-term debt holders get $L_D = \min(L, D)$ upon termination. The market value of long-term debt is therefore

$$V_D(M) = E\left[\int_0^\tau e^{-rt} x \, dt + e^{-r\tau} L_D \,\middle|\, M\right].$$

Note that we compute the expected discounted payoff for the debt conditional on the current draw M on the credit line, which measures the firm's "distance to default" in our implementation.

Until termination, the equity holders receive total dividends of $dDiv_t = dI_t/\lambda$, with the agent receiving fraction λ. At termination, the *outside* equity holders receive the remaining part of the liquidation value, $L_E = \max(0, L - D - C^L)/(1-\lambda)$ per share, after the debt and the credit line have been paid off.[23] The per share value of equity to outside equity holders is then

$$V_E(M) = E\left[\int_0^\tau e^{-rt} \, dDiv_t + e^{-r\tau} L_E \,\middle|\, M\right].$$

Finally, the market value of the credit line is

$$V_C(M) = E\left[\int_0^\tau e^{-rt} (dY_t - x \, dt - dDiv_t) + e^{-r\tau} L_C \,\middle|\, M\right],$$

where $L_C = \min(C^L, L - L_D)$. For the optimal capital structure, the aggregate value of the outside securities equals the principal's continuation payoff. That is, from (18),

$$b(R + \lambda(C^L - M)) = V_D(M) + V_C(M) + (1-\lambda) V_E(M).$$

In the Appendix we show how to represent these market values in terms of an ordinary differential equation, so that they may be computed easily. Figure 5 provides an example. In this example, $L < D$, thus the long-term debt is risky. Note that the market value of debt is decreasing towards L as the balance on the credit line increases towards the credit limit. Similarly, the value of equity declines to zero at the point of default. The figure also shows that the initial value of the credit line is positive – the lender earns a profit by charging interest rate $\gamma > r$. However, as the distance to

default diminishes, additional draws on the credit line result in losses for the lender (for each dollar drawn, the value of the credit line goes up by less than one dollar, and eventually declines).

Figure 5 also illustrates several other interesting properties of the security values. Note, for example, that the leverage ratio of the firm is not constant over time. When cash flows are high, the firm will pay off the credit line and its leverage ratio will decline. During times of low profitability, on the other hand, the firm will increase its leverage. Finally, cash flow shocks lead to persistent changes in leverage. These results are broadly consistent with the empirical behavior of leverage.

D. Asset Substitution and Equity Issuance

One surprising observation from Figure 5 is that the value of equity is concave in the credit line balance, which implies that the value of equity would decline if the cash flow volatility were to increase. In fact, we can show:

PROPOSITION 5: *When debt is risky ($L < D + C^L$), for the optimal capital structure the value of equity decreases if cash flow volatility increases. Thus, equity holders would prefer to reduce volatility.*

This result is counter to the usual presumption that risky debt implies that equity holders benefit from an increase in volatility due to their option to default. That is, in our setting, there is no "asset substitution problem" with respect to leverage. Note also that the agent's payoff is linear in the credit line balance, so that the agent is indifferent to changes to volatility.[24]

In Section I.C we demonstrate that the optimal capital structure implies that the firm's payout policy is incentive compatible for the agent; that is, the agent finds it optimal to pay dividends if and only if the credit line is fully repaid. What about the incentives of equity holders? Would they prefer an alternative payout policy? Moreover, could the firm raise new equity capital to delay default? That is, could equity holders benefit from a *strategic* default policy?

If the firm increases its payouts by paying additional dividends, for each dollar paid outside equity holders receive $(1-\lambda)$. On the other hand, the increased draw on the credit line changes the value of outside equity by $(1-\lambda) V_E'(M)$. Thus, equity holders prefer that the firm not pay dividends as long as

$$V_E'(M) \leq -1. \tag{21}$$

Alternatively, the firm could pay down the credit line by raising new capital through an equity issue. Each dollar raised increases the value of outside equity by $-(1-\lambda)\,V_E'(M)$. Thus, the firm cannot raise additional equity capital as long as

$$V_E'(M) \geq -1/(1-\lambda). \tag{22}$$

The wedge between equations (21) and (22) results from the fact that the agent receives dividend payments, but does not contribute new equity capital to the firm. We therefore have the following result:

PROPOSITION 6: *When debt is risky* $(L < D + C^L)$*, equation (21) is satisfied and holds with equality at* $M = 0$*. Thus, equity holders would not wish to alter the firm's payout policy. In addition, the firm cannot raise new equity capital if (22) holds for* $M = C^L$*.*

Thus, equity holders have no incentive to alter the firm's dividend policy. To verify that that equity issues will not occur, it is only necessary to check (22) at the default boundary. Numerically, (22) appears to hold as long as λ is not too small (e.g., it holds for the example in Figure 5). In Section IV.B we consider renegotiation-proof contracts, for which we show equation (22) is guaranteed to hold.

III. Hidden Effort

Throughout our analysis so far we concentrate on the setting in which the cash flows are privately observed and the agent may divert them for his own consumption. In this section we consider a standard principal-agent model in which the agent makes a hidden binary effort choice. This model is also studied by Biais et al. (2004) in contemporaneous work. Our main result is that, subject to natural parameter restrictions, the solutions are identical for both models. Thus, all of our results apply to both settings.

In a standard hidden effort model, the principal observes the cash flows. Based on the cash flows, the principal decides how to compensate the agent and whether to continue the project. Thus, there are only two key changes to our model. First, since cash flows are observed, misreporting is not an issue. Second, we assume that at each point in time, the agent can choose to either shirk or work. Depending on this decision, the resulting cash flow process is

$$d\hat{Y}_t = dY_t - a\,dt\ ,\ \text{where } a = \begin{cases} 0 & \text{if the agent works} \\ A & \text{if the agent shirks.} \end{cases}$$

Working is costly for the agent, or equivalently, shirking results in a private benefit. Specifically, we suppose that the agent receives an additional flow of utility equal to $\lambda A \, dt$ if he shirks.[25] With $r < \gamma$ the agent consumes all payments immediately, so that

$$dC_t = dI_t + \lambda a \, dt \,.$$

Again, λ parameterizes the cost of effort and in turn the degree of the moral hazard problem. We assume $\lambda \leq 1$ so that working is efficient.

Our first result establishes the equivalence between this setting and our prior model:

PROPOSITION 7: *The optimal principal-agent contract that implements high effort is the optimal contract of Section I.*

Proof of Proposition 7: The incentive compatibility condition in Lemma C is unchanged: To implement high effort at all times, we must have $\beta_t \geq \lambda \, \sigma$. Proposition 1 shows that our contract is the optimal contract subject to this constraint. ◆

It is not surprising that our original contract is incentive compatible in this setting, since shirking is equivalent to stealing cash flows at a fixed rate. What is surprising is that the additional flexibility that the agent has in the cash flow diversion model does not require a "stricter" contract.

Proposition 7 assumes that implementing high effort at all times is optimal. Because the reduction in cash flows due to shirking is bounded – unlike the case of diversion – it may be optimal to stop providing incentives and to allow the agent to shirk after some histories. Specifically, when the agent shirks his payoff would not need to depend on cash flows, so the agent's promised payoff would evolve according to

$$dW_t = \begin{cases} \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - \mu \, dt) & \text{if } a = 0 \\ \gamma W_t dt - dI_t - \lambda A \, dt & \text{if } a = A. \end{cases}$$

Because the principal's continuation function is concave, this reduction in the volatility of W_t could be beneficial. For that not to be the case, and for high effort to remain optimal, it must be that for all W, the principal's payoff rate from having the agent shirk would be less than that under our existing contract:[26]

$$rb(W) \geq (\mu - A) + (\gamma W - \lambda A)b'(W). \tag{23}$$

The agent and principal's payoff if the agent shirks forever are given by

$$w^s = \lambda A / \gamma \text{ and } b^s = (\mu - A)/r = (\mu - \gamma w^s / \lambda)/r.$$

We then have the following necessary and sufficient condition, as well as a simple sufficient condition, for high effort to remain optimal at all times:

PROPOSITION 8: *Implementing high effort at all times is optimal in the principal-agent setting if and only if $b^s \leq f\left(w^s\right)$, where $f(z) \equiv \min_w b(w) + \frac{\gamma}{r}(z-w)b'(w)$. A simpler sufficient condition is*

$$b^s \leq \frac{\gamma}{r}b\left(w^s\right) + \left(1 - \frac{\gamma}{r}\right)b(W^*). \qquad (24)$$

Given λ, both of these conditions imply a lower bound on A, or equivalently, w^s.

<div style="float:right; border:1px solid black; padding:4px;">Insert Fig. 6 here</div>

We can interpret Proposition 8 as follows. The point (w^s, b^s) represents the agent's and principal's payoffs if the agent shirks forever. Thus, shirking is never optimal if and only if this point lies below the function f. The function f is concave and below b, with equality only at the maximum, as Figure 6 shows. The factor γ/r increases the steepness of f relative to b; when $\gamma = r$, f and b coincide. Proposition 8 puts a lower bound on w^s, or equivalently on A, the magnitude of the cash flow impact of shirking. For example, in Figure 6, if $w^s \geq \underline{w}^s$, then high effort is always optimal. This is the case for (w_1^s, b_1^s).

On the other hand, if A is so small that $w^s < \underline{w}^s$, then the optimal principal-agent contract will involve shirking after some histories. Still, the optimal contracting techniques of this paper may apply. For example, see (w_2^s, b_2^s) in Figure 6. In this case, the optimal contract calls for high effort until (w_2^s, b_2^s) is reached, after which point the agent is paid a fixed wage and shirks forever. Thus, the optimal contract is again as in our model, but with a fixed wage and shirking in place of termination so that $(R, L) = (w_2^s, b_2^s)$.[27]

REMARK. We can also consider a hybrid model in which the agent can both divert cash flows and choose effort. Let λ_d parameterize the benefit the agent receives from diverting cash flows, and let λ_a represent the benefit from shirking. Then we can show that the optimal contract implementing high effort is the optimal contract of Section 0 with $\lambda = \max(\lambda_d, \lambda_a)$. (See Shim (2004) for a discrete-time model of this sort.)

IV. Further Extensions of the Model

In this section we consider various extensions of the basic model. First, we allow the termination payoffs (R, L) to be determined endogenously by either the principal's option to hire a new agent or the

agent's option to start a new project. Second, we consider the construction of an optimal renegotiation-proof contract. Third, we consider the case in which the agent and principal disagree about key parameters of the model, such as the project's profitability or the agent's impatience.

A. Endogenously Determined Termination Payoffs

Thus far, we treat the termination payoffs (R, L) as exogenous. Suppose, however, that they are endogenously determined as in the following two examples.

FIRING AND REPLACING THE AGENT: Suppose the agent can be fired and replaced at cost c_a to the principal (i.e. investors). If the agent is fired, the agent's termination payoff is R and investors, choosing the optimal contract with a new agent, receive termination payoff

$$L = b(W^*) - c_a. \tag{25}$$

INALIENABLE HUMAN CAPITAL: Suppose the agent can quit the firm and start a new firm by raising external capital K from new investors. If the agent quits, the old investors liquidate and receive L, while the agent receives

$$R = e^{-\gamma \Delta t} W_0. \tag{26}$$

where Δt is the time required to start a new firm and W_0 satisfies $b(W_0) = K$.[28]

The optimal contract in either case takes exactly the same form as described in Section I. The only change is that now the boundary condition (25) or (26) replaces $b(R) = L$. Figure 7 illustrates the solution. Because $db(W^*)/dL < 1$, when the agent can be replaced the liquidation value L is decreasing in c_a. From the results of Section 0, the credit line increases and the debt decreases in c_a. This is intuitive, because the project requires more financial flexibility when it is more difficult to replace the agent. Similarly, when the agent can quit and start a new firm, as Δt falls and it becomes easier for the agent to start a new firm, R rises. This leads to a decrease in both the credit line and debt. Note that as $\Delta t \rightarrow \infty$ and starting a new firm becomes impossible, $R \rightarrow 0$, and as $\Delta t \rightarrow 0$ and restarting is costless, $R \rightarrow R^*$, the point at which $b'(R^*) = 0$. (These are but two special cases – other cost structures can be considered, and both settings may operate simultaneously.)

Insert Fig. 7 here

B. Renegotiation

The optimal contracts we derive need not be renegotiation-proof. When $b'(R) > 0$, both the principal and the agent would like to renegotiate termination and restart the contract with the agent's value $W > R$, giving the principal the profit $b(W) > L$. In terms of our implementation, renegotiation corresponds to recapitalizing the firm to avoid default. For example, in Figure 5, the security holders would be willing to "forgive" some of the debt to avoid default.

To be renegotiation-proof, the principal's profit function $b(W)$ must not have positive slope. Renegotiation effectively raises the agent's minimum payoff when running the project to a point R^* such that $b'(R^*) = 0$. This is equivalent to the case in Section IV.A of an agent that can restart the firm immediately ($\Delta t = 0$).

A renegotiation-proof contract under which the principal breaks even exists only if the required external capital $K \leq L$. Until termination the agent's continuation value evolves in the interval $[R^*, W^1]$ as

$$dW_t = \gamma W_t \, dt + \lambda(d\hat{Y}_t - \mu \, dt) - dI_t + dP_t,$$

where processes I and P reflect W_t at endpoints W^1 and R^*, respectively. The project is terminated stochastically whenever W_t is reflected at R^*. The probability that the project continues at time t is

$$\Pr(\tau \geq t) = \exp\left(\frac{-P_t}{R^* - R}\right).$$

Thus, W_t is the agent's true expected future payoff. Indeed, whenever W_t reaches R^* and dP_t is added to the agent's continuation value, the project is terminated with probability $dP_t / (R^* - R)$ to account for this increment to the agent's value.

The implementation of a renegotiation-proof contract involves a credit line and debt as in the optimal contract of Section I.C with R^* in place of R. Since $R^* > R$, both the credit line and debt decrease. Renegotiation-proofness effectively reduces the profitability of the project.[29]

C. Private Benefits of Control

Suppose the agent receives private benefits of control from running the project. Specifically, suppose that prior to termination the agent earns additional utility at rate $\gamma \omega$. With this private benefit, the agent's continuation value evolves according to

$$dW_t = \gamma(W_t - \omega) \, dt - dI_t + \lambda(d\hat{Y}_t - \mu \, dt)$$

How does this alter the form of the optimal contract? Interestingly, as the following result shows, this is equivalent to reducing the agent's outside opportunity by ω.

PROPOSITION 9: *Suppose the agent earns private benefits at rate $\gamma\omega$ while running the project. Then the optimal contract is the same as the optimal contract without private benefits and termination payoff $\hat{R} = R - \omega$. Under this contract, given a value of the state variable $\hat{W}_t$, the agent's total payoff including private benefits is $W_t = \hat{W}_t + \omega$.*

Using our comparative statics results for R from Section II.B, increasing the agent's private benefits increases the optimal credit limit and amount of debt. Intuitively, the potential threat of losing the private benefits in termination enhances the agent's incentives and hence increases the debt capacity of the firm. Moreover, because $\hat{W}_0$ rises as $\hat{R}$ falls with ω, the agent's total payoff rises by more than a dollar for each dollar of private benefits, all else equal, due to the "commitment effect" of private benefits.

D. Parameter Uncertainty and Cash Reserves

As with other optimal contracting settings, the form of our optimal contract depends upon the parameters that define the project and the agent's preferences. For example, consider the agent's impatience parameter γ. While $\gamma = r$ may be a natural (or at least neutral) assumption, we argue here that it is more robust to consider optimal contracts in which $\gamma > r$. First, note from Figure 4 that the optimal debt-credit mix is very sensitive as $\gamma \to r$, but is much less sensitive when γ exceeds r by more than 0.5%. Indeed, it is generally the case that $\partial D/\partial\gamma = +\infty$ when $\gamma = r$. Second, the following result implies that to design a contract, the principal has a strong incentive to overestimate, rather than underestimate, γ.

PROPOSITION 10: *Suppose that the principal offers a contract designed for an agent with discount rate γ. If the agent's true discount rate is $\gamma' < \gamma$, then the principal's payoff is the same as if $\gamma' = \gamma$, while the agent earns a payoff greater than W_0 by accumulating a positive cash reserve prior to paying dividends. If the agent's true discount factor γ' is greater than γ, then the agent will draw the entire credit line and default immediately. The agent earns W_0, whereas the principal earns $L - (W_0 - R)$.*

Thus, if there is some uncertainty regarding the agent's impatience, when using our contract investors will set γ at the high end of the range.[30] While they could do better by designing a contract to

"screen" for γ, the benefits of doing so would be minor. This result also offers an explanation for the use of cash reserves in our setting.

A similar argument applies to other parameters in the model. The critical incentive compatibility condition (17) depends also on the project's profit rate μ, the agent's outside option R, and the ability to divert cash flows λ. If the level of debt D is set below the quantity specified in (17), the firm will accumulate cash reserves before paying dividends, with only a modest impact on the investor's payoff. If D exceeds the amount in (17), the agent will divert profits immediately. Thus, in the presence of parameter uncertainty, we expect the firm's payout policy to include the buildup of cash reserves.

V. Conclusion

We analyze a situation in which an agent (entrepreneur) needs to raise external capital to (i) start up a profitable project, (ii) cover future operating losses that may occur, and (iii) consume. In our setting, the agent can divert cash flows from the project for personal consumption without the principal's (investor's) knowledge. To enforce payments, the investors can threaten to withhold future funding and terminate the project. We analyze an optimal contract between the investors and the agent in this setting.

An optimal contract involves a credit line, debt and equity. Debt, outside equity, and possibly the credit line provide the funds for start-up capital and the agent's initial consumption. For the duration of the project, the credit line provides the flexibility to cover possible operating losses. The agent has incentives to pay interest and not consume from the credit line because in the case of default, he has to surrender the project to investors. The agent holds an equity stake and has discretion over the payment of dividends. The agent's equity stake is sufficiently large that he does not divert excess cash flows for personal consumption, but pays them out as dividends appropriately.

The continuous-time setting of our paper has several advantages. First, the features of an optimal contract are cleaner. Unlike in discrete time, an optimal contract in continuous time does not require stochastic termination. Second, a continuous-time model provides a convenient characterization of the optimal contract through an ordinary differential equation. With this characterization we can say a great deal about how the optimal capital structure is determined by the specific features of the project. Also,

we are able to compute the values of the securities that are involved in the implementation of an optimal contract, and we show that typical conflicts of interest between debt and equity holders do not arise. Finally, we can easily analyze extensions. For example, we show how our contract also solves a standard principal-agent problem with costly effort. Other extensions are considered; in many cases the solution only involves finding the appropriate boundary conditions for the differential equation that defines an optimal contract.

Our results raise several interesting questions for future research. For example, how can the contract be designed to elicit information regarding the agent's impatience? How does the agency problem considered here affect project selection, investment, and the scope of the firm? In a trading context, can a model like the one developed here, in which inefficient termination is necessary to provide incentives, provide a rationale for "limits to arbitrage"? These and other questions are the subject of ongoing research.

Appendix: Proofs

Proof of Lemma A: Consider any incentive compatible contract (τ, I, C, $\hat{Y}$). To prove the proposition, we show that there is a new incentive compatible contract, which gives the same payoff to the agent and the same or greater payoff to the principal, under which the agent reports cash flows truthfully and maintains zero savings. This contract is ($\tau'(Y) = \tau(\hat{Y}(Y))$, $I'(Y) = C(Y)$, C, Y). Note that the agent's consumption is the same as under the old contract, so he earns the same payoff. Let us show that the agent's strategy is incentive compatible and that the principal earns the same or greater payoff.

Under the new contract the agent cannot improve his payoff by a deviation (C', $\hat{Y}'$), because any feasible consumption C' is feasible under the old contract as well. If C' is feasible under the new contract, then the agent always has nonnegative savings if he reports $\hat{Y}(\hat{Y}'(Y))$ and consumes C' under the old contract. Indeed,

$$\underbrace{\int_0^t e^{\rho(t-s)}(dI_t(\hat{Y}(\hat{Y}')) - dC_t') + \int_0^t e^{\rho(t-s)}[dY_t - d\hat{Y}_t(\hat{Y}')]^\lambda}_{\text{savings under old contract if the agent consumes } C' \text{ and reports } \hat{Y}(\hat{Y}')} \geq$$

$$\underbrace{\int_0^t e^{\rho(t-s)}(dI_t(\hat{Y}(\hat{Y}')) - dC_t(\hat{Y}')) + \int_0^t e^{\rho(t-s)}[d\hat{Y}_t' - d\hat{Y}_t(\hat{Y}')]^\lambda}_{\text{savings under old contract if the agent consumes } C, \text{ reports } \hat{Y}, \text{ and } \hat{Y}' \text{ are true cash flows}} +$$

$$\underbrace{\int_0^t e^{\rho(t-s)}(dC_t(\hat{Y}') - dC_t') + \int_0^t e^{\rho(t-s)}[dY_t - d\hat{Y}_t']^\lambda}_{\text{savings under new contract under deviation } (C', \hat{Y}')} \geq 0.$$

To show that the principal is at least as well-off as before, note that the new contract avoids the inefficiency due to both stealing and inefficient savings (at rate $\rho < r$) by the agent. Therefore, the principal's profit improves by

$$E\left[\int_0^\tau e^{-rt}\left((1-\lambda)(dY_t - d\hat{Y}_t)^+ + (r-\rho)S_t dt\right)\right],$$

where S denotes the agent's savings under the old contract. ◆

Proof of Proposition 1: First, let us verify that function b defined in the proposition is concave. Note that $b'(W) \geq -1$ and $rb(W) < \mu - \gamma W$ imply $b'' < 0$. Therefore, to the left of W^1, with boundary conditions $b'(W^1) = -1$ and $rb(W^1) = \mu - \gamma W^1$, function b enters the region where it is concave. Moreover, it stays concave because a concave function can never exit this region (this can be seen geometrically).

Next, let us prove that b represents the principal's optimal profit, which is achieved by the contract outlined in the proposition. Define

$$G_t \equiv \int_0^t e^{-rs}(dY_s - dI_s) + e^{-rt}b(W_t).$$

Under an arbitrary incentive compatible contract, W_t evolves according to (6). Then, from Ito's lemma,

$$e^{rt}dG_t = \underbrace{\left(\mu + \gamma W_t b'(W_t) + \tfrac{1}{2}\beta_t^2\sigma^2 b''(W_t) - rb(W_t)\right)}_{\leq 0}dt \underbrace{-(1 + b'(W_t))}_{\leq 0}dI_t + (1 + \beta_t b'(W_t))\sigma dZ_t.$$

From (15) and the fact that $b'(W_t) \geq -1$, G_t is a supermartingale. It is a martingale if and only if $\beta_t = \lambda$, $W_t \leq W^1$ for $t > 0$, and I_t is increasing only when $W_t \geq W^1$.

We can now evaluate the principal's payoff for an arbitrary incentive compatible contract. Note that $b(W_\tau) = L$. For all $t < \infty$,

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L\right] = E\left[G_{t\wedge\tau} + 1_{t\leq\tau}\left(\int_t^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L - e^{-rt}b(W_t)\right)\right]$$

$$= \underbrace{E[G_{t\wedge\tau}]}_{\leq G_0 = b(W_0)} + e^{-rt}E\left[1_{t\leq\tau}\left(\underbrace{E_t\left[\int_t^\tau e^{-r(s-t)}(dY_s - dI_s) + e^{-r(\tau-t)}L\right]}_{\leq \mu/r - W_t = \text{ first best}} - b(W_t)\right)\right].$$

Now, since $b'(W) \geq -1$, $\mu/r - W - b(W) \leq \mu/r - R - L$. Therefore, letting $t \to \infty$,

$$E\left[\int_0^\tau e^{-rs}(dY_s - dI_s) + e^{-r\tau}L\right] \leq b(W_0).$$

Finally, for a contract that satisfies the conditions of the proposition, G_t is a martingale until time τ because $b'(W_t)$ stays bounded. Therefore, the payoff $b(W_0)$ is achieved with equality. ◆

REMARK. It is easy to modify this proof to show that the principal cannot improve her profit by adding additional randomization. Such randomization would add an extra term to the expression for dG_t, but the process G_t would still be a supermartingale since $b(W)$ is a concave function.

Proof of Proposition 2: Recall that the rate of return on savings is $\rho \leq r$. We consider the case $\rho = r$, in which savings is most attractive, without loss of generality. We also generalize the setting to allow the agent to save within the firm and on his own account (this will be useful in our implementation of the optimal contract). Denote the savings within the firm by S_t^f, which evolve according to

$$dS_t^f = rS_t^f \, dt + (dY_t - d\hat{Y}_t) - dQ_t.$$

Here, dQ_t represents the agent's diversion of the cash flows to his own account, which evolves as

$$dS_t = rS_t \, dt + [dQ_t]^\lambda + dI_t - dC_t.$$

Note that the agent bears the cost of diversion when moving funds outside the firm. Both accounts must maintain nonnegative balances. We show that for an arbitrary feasible strategy $(C, \hat{Y})$ of the agent,

$$\hat{V}_t = \int_0^t e^{-\gamma s} \, dC_s + e^{-\gamma t}\left(S_t + \lambda S_t^f + W_t\right)$$

is a supermartingale. Now,

$$e^{\gamma t}d\hat{V}_t = dC_t + dS_t - \gamma S_t \, dt + \lambda(dS_t^f - \gamma S_t^f \, dt) + dW_t - \gamma W_t \, dt.$$

Using (16) and the definitions of dS_t and dS_t^f,

$$e^{\gamma t}d\hat{V}_t = [dQ_t]^\lambda - \lambda dQ_t - (\gamma - r)(S_t + \lambda S_t^f) \, dt + \lambda(dY_t - \mu \, dt)$$
$$= -(1-\lambda)dQ_t^- - (\gamma - r)(S_t + \lambda S_t^f) \, dt + \lambda\sigma \, dZ_t.$$

Because $\lambda \leq 1$, dQ_t^- is nondecreasing, $\gamma > r$, and the savings balances are nonnegative, $\hat{V}$ is a supermartingale until time τ because W_t is bounded from below. If W_t is bounded from above and there is no savings, $S_t = S_t^f = 0$, and the agent reports truthfully so that $d\hat{Y}_t = dY_t$ and $dQ_t = 0$, then $\hat{V}$ is a martingale. Thus,

$$W_0 = \hat{V}_0 \geq E\left[\hat{V}_\tau\right] = E\left[\int_0^\tau e^{-\gamma s} \, dC_s + e^{-\gamma \tau}\left(S_\tau + \lambda S_\tau^f + R\right)\right]$$

with equality if the agent maintains zero savings and reports truthfully. This is true even if $Y_t - \hat{Y}_t$ is not Lipschitz-continuous. ◆

Proof of Proposition 3: Let Div_t be an increasing process that represents the cumulative dividends paid by the firm. Then the credit line balance evolves according to

$$dM_t = \gamma M_t \, dt + x \, dt + dDiv_t - d\hat{Y}_t,$$

where we can assume $dDiv_t$ and $d\hat{Y}_t$ are such that $M_t \geq 0$. Defining W_t from (18), and using, from (17), $\lambda x = \lambda rD = \lambda \mu - \gamma(R + \lambda C^L)$, we have

$$dW_t = -\lambda dM_t = -\lambda \gamma M_t dt - \lambda x dt - \lambda dDiv_t + \lambda d\hat{Y}_t = \gamma W_t dt - \lambda dDiv_t + \lambda(d\hat{Y}_t - \mu dt)$$

Letting $dI_t = \lambda \, dDiv_t$, the incentive compatibility result of Proposition 3 follows from Proposition 2. Optimality follows from (19), since then $M_t = 0$ implies $W_t = W^1$. ◆

Proof of Proposition 4: Let b^λ be the value function for parameters $(1, R^\lambda, L^\lambda)$, which satisfies

$$rb^\lambda(W) = \mu + \gamma W b^{\lambda\prime}(W) + \tfrac{1}{2}\sigma^2 b^{\lambda\prime\prime}(W), \qquad b^\lambda(R^\lambda) = L^\lambda, \qquad b^{\lambda\prime}(W^{\lambda,1}) = -1, \qquad \text{and}$$

$rb^\lambda(W^{\lambda,1}) = \mu - \gamma W^{\lambda,1}$. Define

$$b(W) = \lambda b^\lambda(W/\lambda) + (1-\lambda)(\mu/r).$$

Then it is straightforward to check that $b(W)$ satisfies (12) with boundary conditions $b(R) = L$, $b'(W^1) = -1$, and $rb(W^1) = \mu - \gamma W^1$, where $W^1 = \lambda W^{\lambda,1}$. Hence, b is the value function for parameters *(λ, R, L)*, and W^1 is the dividend boundary for parameters (λ, R, L) if and only if W^1/λ is the dividend boundary for $(1, R^\lambda, L^\lambda)$. Thus, from (17) and (19), the optimal debt structure is unchanged. ◆

A. Market Values of Securities

We need Lemma D to compute the market values of the securities and to generate the comparative statics:

LEMMA D: *Suppose W_t evolves as according to*

$$dW_t = \gamma W_t dt - dI_t + \lambda(d\hat{Y}_t - \mu \, dt)$$

in the interval [R, W¹] until time τ, when Wₜ reaches R, where Iₜ is a nondecreasing process that reflects Wₜ at W¹. Let k be a real number and g:[R, W¹] →ℜ be a bounded function. Then the same function G:[R, W¹] →ℜ both solves

$$rG(W) = g(W) + \gamma W \, G'(W) + 1/2 \, \lambda^2 \sigma^2 G''(W),$$ (A1)

with boundary conditions G(R) = L and G′(W¹)= - k, and satisfies

$$G(W_0) = E\left[\int_0^\tau e^{-rt} g(W_t) dt - k \int_0^\tau e^{-rt} dI_t + e^{-r\tau} L \right].$$ (A2)

Proof of Lemma D: Suppose that G solves (A1), and let us show that it satisfies (A2). Define

$$H_t = \int_0^\tau e^{-rt} g(W_t) dt - k \int_0^\tau e^{-rt} dI_t + e^{-r\tau} G(W_t).$$

Then, using Ito's lemma,

$$e^{rt} dH_t = \left(g(W_t) + \gamma W_t G'(W_t) + \frac{1}{2} \lambda^2 \sigma^2 G''(W_t) - rG(W_t) \right) dt - (k + G'(W_t)) dI_t + G'(W_t) \lambda \sigma dZ_t.$$

From equation (A1), condition G′(W¹)= -k, and the fact that I increases only when $W_t = W^1$, H is a martingale. Because G is bounded, H is a martingale until time τ, so that

$$G(W_0) = H_0 = E[H_\tau] = E\left[\int_0^\tau e^{-rt} g(W_t) dt - k \int_0^\tau e^{-rt} dI_t + e^{-r\tau} L \right]. \quad \blacklozenge$$

We now have that the values of the credit line, debt, and equity can be expressed in terms of the functions

$$G_\tau(W) = E\left[e^{-r\tau} \mid W_0 = W \right] \qquad \text{and} \qquad G_I(W) = E\left[\int_0^\tau e^{-rt} dI_t \mid W_0 = W \right].$$

By Lemma , both of these functions solve the differential equation

$$rG(W) = \gamma W \, G'(W) + 1/2 \, \lambda^2 \sigma^2 G''(W),$$ (A3)

with boundary conditions $G_\tau(R) = 1$, $G_\tau(W^1) = 0$ and $G_I(R) = 0$, $G_I(W^1) = 1$. Functions G_τ and G_I can be easily computed. To evaluate the market values of the securities, we also use the fact that

$$E\left[\int_0^\tau e^{-rt} dt \mid W_0 = W \right] = \frac{1 - G_\tau(W)}{r}.$$

Then, the market values for the credit line, debt, and equity are

$$V_C(M) = E\left[\int_0^\tau e^{-rt} \left(dY_t - x dt - \frac{dI_t}{\lambda} \right) + e^{-r\tau} L_C \mid W_0 = W \right] = \frac{\gamma W^1}{\lambda} \frac{1 - G_\tau(W)}{r} - \frac{G_r(W)}{\lambda} + L_C G_\tau(W)$$

$$V_D(M) = E\left[\int_0^\tau e^{-rt} x dt + e^{-r\tau} L_D \mid W_0 = W \right] = x \frac{1 - G_\tau(W)}{r} + L_D G_\tau(W), \text{ and}$$

$$V_E(M) = E\left[\int_0^\tau e^{-rt}\frac{dI_t}{\lambda} + e^{-r\tau}L_E \mid W_0 = W\right] = \frac{G_I(W)}{\lambda} + L_E G_\tau(W),$$

respectively, where $W=W^1 - \lambda M$. If $\lambda = 1$, no funds remain after debt and credit line holders are paid off at time τ.

LEMMA E: *If $\lambda = 1$, then in the optimal contract, $L < D + C^L$.*

Proof of Lemma E: When $\lambda = 1$, $D + C^L = b(W^1) + W^1 - R$. Since $b'(W) > -1$ for $W \in (R,W^1)$, $b(W^1) + W^1 > b(R) + R = L + R$. Thus, $D + C^L > L$. ◆

Proof of Proposition 5: When $L < D$, $L_E = 0$. To demonstrate that equity holders prefer less volatility, we need to prove that G_I is concave. From the stochastic representation, we see that G_I is an increasing function. From (A3),

$$1/2\ \lambda^2\sigma^2 G_I''(R) = -\gamma R G_I'(R) < 0.$$

Suppose that G_I were not concave somewhere on $[R,W^1]$, and let $V = \inf\{G_I''(W) > 0\}$. Then $V>R$ and $G_I''(V) = 0$ by continuity of G_I''. But then from (A3),

$$1/2\ \lambda^2\sigma^2 G'''(V)=(r-\gamma)\ G'(V) -\gamma W\ G''(V)= (r-\gamma)\ G'(V) < 0,$$

so $G_I''(V+\varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, contradiction. ◆

Proof of Proposition 6: Note that $L_E = 0$ when debt is risky. First, equation (21) holds with equality at $M = 0$ because from (A3), $G_I'(W^1)=-1$. Furthermore, (21) holds for $M > 0$ because G_I is concave (see the proof of Proposition 5). Also, from the concavity of G_I it follows that if $V_E'(C^L) \geq -1/(1-\lambda)$, then $V_E'(M) > -1/(1-\lambda)$ for all $M<C^L$, and the firm cannot raise equity capital. ◆

B. Comparative Statics Results

LEMMA F: *Suppose θ is one of parameters L, μ, γ, σ^2 or λ and denote by $b_\theta(W)$ the optimal continuation function for that parameter value. Then*

$$\frac{\partial b_\theta(W)}{\partial\theta} = E\left[\int_0^\tau e^{-rt}\left(\frac{\partial\mu}{\partial\theta} + \frac{\partial\gamma}{\partial\theta}W_t b'_\theta(W_t) + \frac{1}{2}\frac{\partial\lambda^2\sigma^2}{\partial\theta}b''_\theta(W_t)\right)dt + e^{-r\tau}\frac{\partial L}{\partial\theta}\mid W_0 = W\right].$$

Proof of Lemma F: Consider a value of W^1 and a corresponding incentive compatible contract of Proposition 5, that is one in which process I reflects W_t at W^1. Then the principal's profit under this contract is

$$b_{\theta,W^1}(W) = E\left[\int_0^\tau e^{-rt}\mu dt - \int_0^\tau e^{-rt}dI_t + e^{-r\tau}L\mid W_0 = W\right].$$

By Lemma D, $b_{\theta,W^1}(W)$ solves equation

$$rb_{\theta,W^1}(W) = \mu + \gamma W b'_{\theta,W^1}(W) + \tfrac{1}{2}\lambda^2\sigma^2 b''_{\theta,W^1}(W) \tag{A4}$$

with boundary conditions $b_{\theta,W^1}(R) = L$ and $b'_{\theta,W^1}(W^1) = -1$. Denote by $W^1(\theta)$ the choice of W^1 that maximizes the principal's profit $b_{\theta,W^1}(W_0)$ for a given value of parameter θ. Then $b_\theta(W) = b_{\theta,W^1(\theta)}(W)$.

By the Envelope Theorem,

$$\frac{\partial b_\theta(W)}{\partial \theta} = \left.\frac{\partial b_{\theta,W^1}(W)}{\partial \theta}\right|_{W^1=W^1(\theta)}. \tag{A5}$$

Differentiating (A4) with respect to θ at $W^1 = W^1(\theta)$ and using (A5) we find that $\dfrac{\partial b_\theta(W)}{\partial \theta}$ satisfies the

equation

$$r\frac{\partial b_\theta(W)}{\partial \theta} = \frac{\partial \mu}{\partial \theta} + \frac{\partial \gamma}{\partial \theta} W b'_\theta(W) + \gamma W \frac{\partial}{\partial W}\frac{\partial b_\theta(W)}{\partial \theta} + \frac{1}{2}\frac{\partial \lambda^2\sigma^2}{\partial \theta} b''_\theta(W) + \frac{1}{2}\lambda^2\sigma^2 \frac{\partial^2}{\partial W^2}\frac{\partial b_\theta(W)}{\partial \theta}$$

with boundary conditions $\dfrac{\partial b_\theta(R)}{\partial \theta} = \dfrac{\partial L}{\partial \theta}$ and $\dfrac{\partial}{\partial W}\dfrac{\partial b_\theta(W^1)}{\partial \theta} = 0.$ The lemma then follows from Lemma

D. ◆

COROLLARY: *From Lemma F we obtain that*

$$\frac{\partial b(W)}{\partial L} = G_\tau(W), \qquad \frac{\partial b(W)}{\partial \gamma} = G_1(W), \qquad \frac{\partial b(W)}{\partial \mu} = \frac{1 - G_\tau(W)}{r},$$

$$\frac{\partial b(W)}{\partial \sigma^2} = \frac{\lambda^2}{2} G_2(W), \quad and \quad \frac{\partial b(W)}{\partial \lambda} = \lambda \sigma^2 G_2(W), \tag{A6}$$

where G_τ is defined by (A3) and G_1 and G_2 are given by

$$G_1(W) = E\left[\int_0^\tau e^{-rt} W_t b'(W_t)dt \mid W_0 = W\right] \quad and \quad G_2(W) = E\left[\int_0^\tau e^{-rt} b''(W_t)dt \mid W_0 = W\right]. \tag{A7}$$

Additionally, because the principal's profit remains the same if the agent's outside option increases by dR and liquidation value decreases by $b'(R)dR$, the effect of a change in R on the principal's profit is

$$\frac{\partial b(W)}{\partial R} = -b'(R)G_\tau(W).$$

To find the effect of the parameters on W^1, W_0, and W^*, we need to differentiate $rb_\theta(W^1) + \gamma W^1 = \mu$, $b_\theta(W_0(\theta)) = K$ and $b_\theta'(W^*(\theta)) = 0$ with respect to θ and use the corollary of Lemma F. As a result, we obtain the comparative statics shown in Table II, which is an expanded version of Table I.

We do not include a row for λ, but it is easy to see that λ and σ have the same effects on W_0, W^*, and $b(W^*)$, and the effect of λ on C^L and D can be found using Proposition 4:

Insert Table II

$$\frac{\partial C^L}{\lambda} = \frac{R}{\lambda^3} - \left(\frac{\mu}{r} - L + Rb'(R)\right)\frac{rG_\tau(W^1)}{\lambda^3(\gamma - r)} ? 0 \qquad \text{and} \qquad \frac{\partial D}{\partial \lambda} = \left(\frac{\mu}{r} - L + b'(R)R\right)\frac{\gamma G_\tau(W^1)}{\lambda^3(\gamma - r)} > 0$$.

Most of the signs in this table are obvious, except for a few entries in parentheses, which we justify below. The following lemma allows us to compare the principal's profit for different γs and to sign two entries that involve $G_1(W)$.

LEMMA G: *Let $\lambda = 1$. Suppose that the principal offers a contract designed for an agent with discount rate γ to an agent whose true discount rate is $\gamma' < \gamma$. Then this agent would derive utility greater than W_0, and the principal would receive profit of exactly $b(W_0)$.*

Proof of Lemma G: Let us investigate how an agent with discount rate γ' responds to a contract created for an agent with discount rate γ. Then, W_t can be perceived as a balance on a high-interest savings account:

$$dW_t = \gamma W_t dt + (d\hat{Y}_t - \mu \, dt),$$

where $d\hat{Y}_t - \mu \, dt$ are deposits. This account has a cap of W^1 and a minimum balance of R. The agent consumes

$$dC_t = dY_t - d\hat{Y}_t - dQ_t,$$

where dQ_t are deposits into the low-interest savings account with balance

$$dS_t = \rho S_t \, dt + dQ_t.$$

With these two accounts, it is optimal to never have a positive balance on the low-interest account, unless the high-interest account is full (i.e., $W_t = W^1$). Since the high-interest account earns a greater return than the agent's own discount rate, it is optimal to deposit all cash flows into the high-interest account and not consume when $W_t < W^1$, in which case the agent receives a payoff greater than W_t.

Let us show that the principal still earns $b(W_t)$ when the agent follows such a strategy. The agent deposits all cash flows into the credit line when $W_t < W^1$ (just like an agent with discount factor γ), but he may save cash rather than consume when $W_t = W^1$. This modification in the agent's strategy does not alter the principal's profit because at $W_t = W^1$, the agent would pay the principal $\mu - \gamma W^1 = rb(W^1)$, which is exactly what the principal needs to realize a profit of $b(W^1)$. ◆

Note that the contract in Lemma G is not optimal for agent γ': An optimal contract would give the principal higher profit for the same value of the agent. Therefore, for every point $(W, b_\gamma(W))$ with

$W \geq W^*(\gamma)$, there is a point $(W', b_\gamma(W')) > (W, b_\gamma(W))$. We conclude that $b_\gamma(W)$ must be increasing as γ falls for all $W \geq W^*(\gamma)$, so $G_1(W) < 0$. This conclusion holds even if $\lambda < 1$, because the profit function for parameters λ and σ is identical to the profit function for parameters $\lambda' = 1$ and $\sigma' = \lambda\sigma$.

COROLLARY: $\quad -\dfrac{G_1(W_0)}{b'(W_0)} < 0 \ and \ G_1(W^*) < 0.$

The following Lemma allows us to determine the sign of $dW^*/d\gamma$.

LEMMA H: $G_1'(W) < 0$ whenever $G_1(W) < 0$. Therefore, $G_1'(W) < 0$ on $[W^*, W^1]$.

Proof of Lemma H: We will prove the lemma in two steps. Suppose that $G_1(W) < 0$. First, we will show that if $G_1(W) \geq Wb'(W)$, then $G_1(W') < G_1(W)$ for all $W' > W$. Second, we will show that if $G_1(W) \leq Wb'(W)$, then $G_1(W') > G_1(W)$ for all $W' < W$. Therefore, G_1 must be decreasing whenever it is negative.

If $0 > G_1(W) \geq Wb'(W)$, then $W > W^*$ and $wb'(w)$ is decreasing for $w \in [W, W^1]$. If $W' > W$,

$$G_1(W') = E\left[\int_0^{\hat\tau} e^{-rt} W_t b'(W_t)dt + e^{-r\hat\tau}G_1(W) \,|\, W_0 = W'\right],$$

where $\hat\tau$ is the first time W_t hits W. Because $W_t b'(W_t) < Wb'(W) \leq G_1(W)$, it follows that $G_1(W') < G_1(W)$.

If $0 > G_1(W)$ and $G_1(W) \leq Wb'(W)$, then $wb'(w) > G_1(W)$ for all $w < W$ because $wb'(w)$ is decreasing on the range $[W^*, W^1]$ and nonnegative on the range $[R, W^*]$. If $W' < W$,

$$G_1(W') = E\left[\int_0^{\hat\tau} e^{-rt} W_t b'(W_t)dt + e^{-r\hat\tau}G_1(W) \,|\, W_0 = W'\right] > G_1(W),$$

where $\hat\tau$ is the first time W_t hits W, and $W_t b(W_t)$ is interpreted to be zero in the first integral if $t > \tau$. It follows from Lemma G that $G_1(W) < 0$ when $W \geq W^*$. Therefore, $G_1'(W) \leq 0$ on $[W^*, W^1]$. ◆

For the remaining two entries of Table II, we need to relate $b'(W)$ and $G_\tau(W)$.

LEMMA I: *The following inequality holds for all $W < W^1$:*

$$b'(W) < \frac{(\gamma - r)G_\tau(W)}{rG_\tau(W^1)} - \frac{\gamma}{r}. \tag{A8}$$

Proof of Lemma I: Differentiating equation (12) with respect to W we find that $b'(W)$ satisfies

$$(r - \gamma)b'(W) = \gamma Wb''(W) + \frac{\sigma^2}{2}b'''(W) \tag{A9}$$

with boundary conditions $b'(W^1) = -1$ and $b''(W^1) = 0$. Denote the right-hand side of (A8) by $g(W) - \gamma/r$. From (A7), we know that $g(W)$ satisfies

$$rg(W) = \gamma W g'(W) + \frac{\sigma^2}{2} g''(W) \Rightarrow$$

$$(r-\gamma)(g(W) - \frac{\gamma}{r}) + (r-\gamma)\frac{\gamma}{r} + \gamma g(W) = \gamma W g'(W) + \frac{\sigma^2}{2} g''(W), \tag{A10}$$

with boundary conditions $g(W^1)=(\gamma\text{-}r)/r$ and $g'(W^1)=0$. Denote $f(W) = g(W)\text{-}\gamma/r - b'(W)$. To prove the lemma, we need to show that $f(W)>0$ for all $W<W^1$. Since $f(W^1)=0$, this property follows if we show that $f'(W)<0$ for all $W<W^1$. Subtracting (A9) from (A10),

$$\frac{\sigma^2}{2} f''(W) = (r-\gamma)f(W) + (r-\gamma)\frac{\gamma}{r} + \gamma g(W) - \gamma W f'(W) \tag{A11}$$

with boundary conditions $f(W^1)=0$ and $f'(W^1)=0$. From (A11) we find that

$$\frac{\sigma^2}{2} f''(W^1) = (r-\gamma)\frac{\gamma}{r} + \gamma \frac{\gamma-r}{r} = 0,$$

$$\frac{\sigma^2}{2} f'''(W^1) = (r-2\gamma)f'(W^1) + \gamma g'(W^1) + \gamma W^1 f''(W^1) = 0 \text{, and}$$

$$\frac{\sigma^2}{2} f^{(4)}(W^1) = (r-3\gamma)f''(W^1) + \gamma g''(W^1) + \gamma W^1 f'''(W^1) > 0.$$

Therefore, $f'(W)<0$ for $W<W^1$ in a small neighborhood of W^1. If $f'(W)<0$ fails for some $W<W^1$, there has to be a largest point V at which it fails. Then $f'(V)=0$ and $f(W)$ is positive and decreasing on $[V,W^1)$. But then from (A11)

$$\frac{\sigma^2}{2} f''(V) = (r-\gamma)f(V) + (r-\gamma)\frac{\gamma}{r} + \gamma g(V) > 0, \text{ since } g(V) > \frac{\gamma-r}{r}.$$

We conclude that $f'(V+\varepsilon)>0$, which contradicts our definition of V as the largest point at which $f'(V)\geq0$. We conclude that $f'(W)<0$ and $f(W)>0$ for $W<W^1$, so (A8) holds. $\blacklozenge$

Now we can sign the remaining two fields in Table II.

COROLLARY: *Applying (A8) at W=R, we have*

$$\frac{rb'(R)G_\tau(W^1)}{\gamma - r} - 1 < -\frac{\gamma G_\tau(W^1)}{\gamma - r} < 0 \qquad and \qquad 1 - \frac{\gamma G_\tau(W^1)}{\gamma - r} > \frac{rb'(R)G_\tau(W^1)}{\gamma - r} > 0.$$

C. Hidden Effort and Extensions

Proof of Proposition 8: Let $w^s = \lambda A/\gamma$ and $b^s = (\mu\text{--}A)/r$. We can rewrite (23) as $b^s \leq b(W) + \frac{\gamma}{r}(w^s - W)b'(W)$. This must hold for all W, leading to the condition

$$b^s \leq f(w^s) = \min_W b(W) + \frac{\gamma}{r}(w^s - W)b'(W). \tag{A12}$$

To prove that condition (24) of Proposition 8 guarantees (A12), it is sufficient to show that for all w,

$$b(w^s) - \frac{\gamma-r}{r}(b(W^*) - b(w^s)) \leq b(W) + \frac{\gamma}{r}(w^s - W)b'(W). \tag{A13}$$

Because b is concave and $\gamma > r$, if $(w^s - W)b'(W) > 0$,
$$b(w^s) \le b(W) + \left(w^s - W\right)b'(W) \le b(W) + \frac{\gamma}{r}\left(w^s - W\right)b'(W),$$

which implies (A13) for W not between w^s and W^*. For W between w^s and W^*, note that
$$b(w^s) - \frac{\gamma-r}{r}(b(W^*) - b(w^s)) \le b(w^s) - \frac{\gamma-r}{r}(b(W) - b(w^s))$$
$$= b(W) + \frac{\gamma}{r}(b(w^s) - b(W)) \le b(W) + \frac{\gamma}{r}(w^s - W)b'(W)$$

so that (A13) again holds, verifying the sufficiency of condition (24). Note that $f'(w^s) = \gamma/r\, b'(W) \ge -\gamma/r$, whereas $\partial b^s / \partial w^s = -(\gamma/r)/\lambda$. Thus, both (A12) and (24) imply a lower bound on w^s (or equivalently, A).

Finally, we note the following properties of f: Setting $W = w^s$ in (A12) implies $f(w) \le b(w)$. Also, since f is the lower envelope of linear functions it is concave. Finally, (24) implies that $f(W^*) = b(W^*)$.

◆

Proof of Proposiiton 9: Let b be the optimal continuation function given boundary condition $b(R-\omega) = L$. Define $b^*(W) = b(W - \omega)$. Then $b^*(R) = L$ and
$$rb^*(W) = rb(W - \omega) = \mu + \gamma(W - \omega)b'(W - \omega) + \tfrac{1}{2}\lambda^2\sigma^2 b''(W - \omega)$$
$$= \mu + \gamma(W - \omega)b^{*'}(W) + \tfrac{1}{2}\lambda^2\sigma^2 b^{*''}(W).$$

Finally, $b'(W^1) = -1$ implies $b^{*'}(W^1 + \omega) = -1$ and $b^{*''}(W^1 + \omega) = 0$. Thus, by the same arguments as in the proof of Proposition 1, b^* is the optimal continuation function for the setting with private benefits. ◆

Proof of Proposition 10: The first result holds by Lemma G. Next, suppose the agent's true discount factor γ' is greater than γ. The process
$$\hat{V}_t = \int_0^t e^{-\gamma's}dC_s + e^{-\gamma't}(S_t + W_t)$$

is a strict supermartingale. Indeed,
$$e^{\gamma t}d\hat{V}_t = -(1 - \lambda)(dY_t - d\hat{Y}_t)^- - (\gamma' - \gamma)W_t dt - (\gamma' - \rho)S_t dt + \lambda\sigma dZ_t,$$

so $\hat{V}$ has a negative drift. Since W_t and S_t are bounded from below, $\hat{V}$ is a strict supermartingale until time τ. If the agent draws the entire credit line and defaults at time 0, then he gets a payoff of W_0. If he follows any other strategy, then $\tau > 0$ and the agent's payoff is
$$E\left[\int_0^\tau e^{-\gamma's}dC_s + e^{-\gamma'\tau}(S_\tau + W_\tau)\right] = E\left[\hat{V}_\tau\right] < \hat{V}_0 = W_0.$$

Therefore, the agent will draw the entire credit line immediately if $\gamma' > \gamma$. ◆

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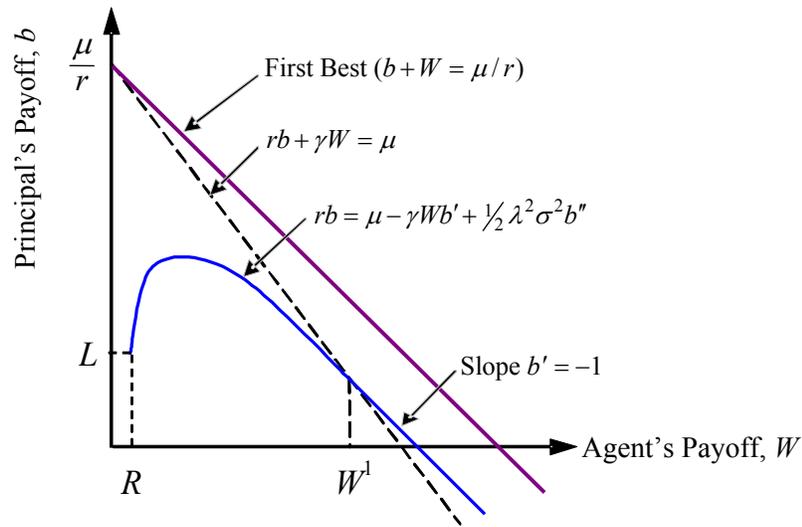
Figure 1. The principal's value function $b(W)$. The principal's value function starts at (L, R), and obeys the differential equation (15) until the point W^1, and then continues with slope -1.
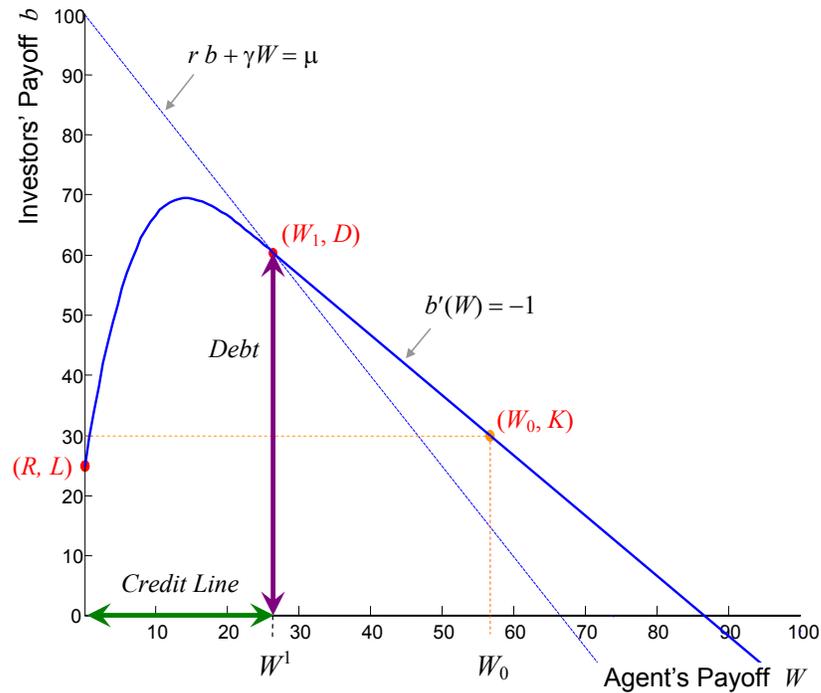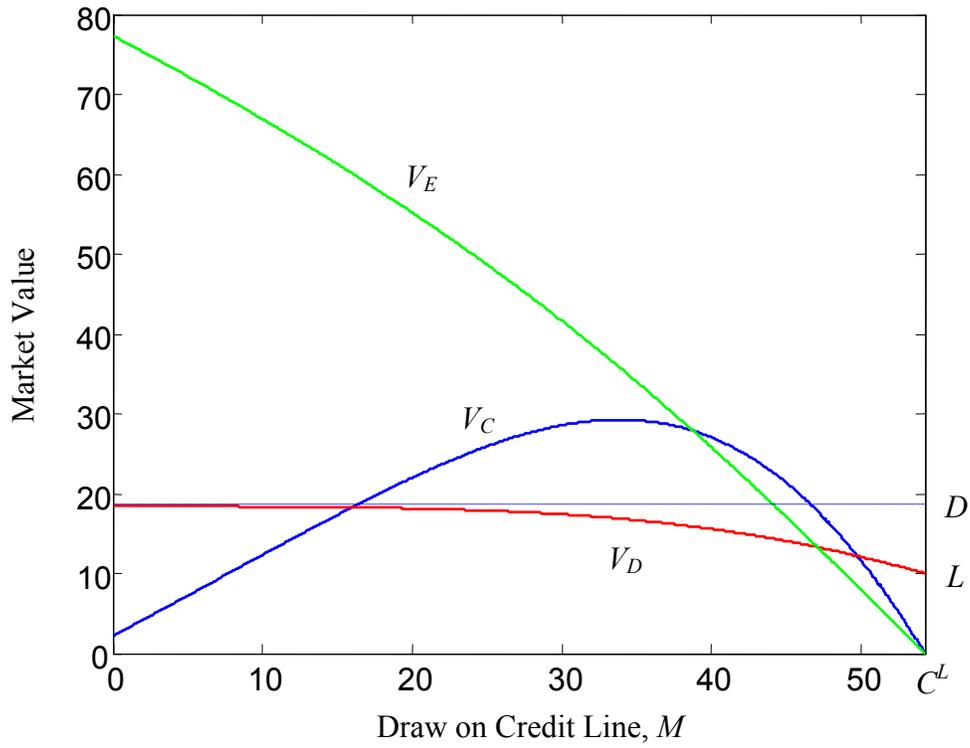

Figure 2. The Optimal Contract with Low Volatility. For $L = 25$, $R = 0$, $\mu = 10$, $\sigma = 5$, $r = 10\%$, $\gamma = 15\%$, $\lambda = 1$, $K = 30$.
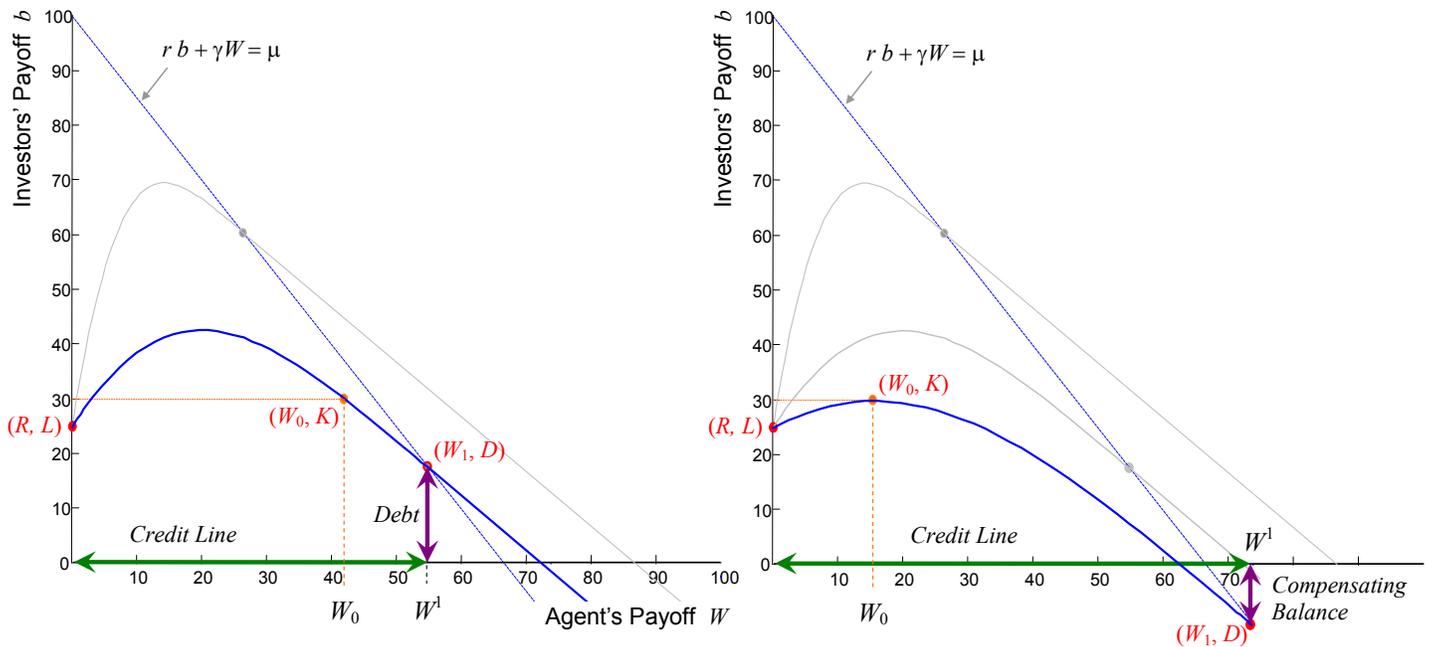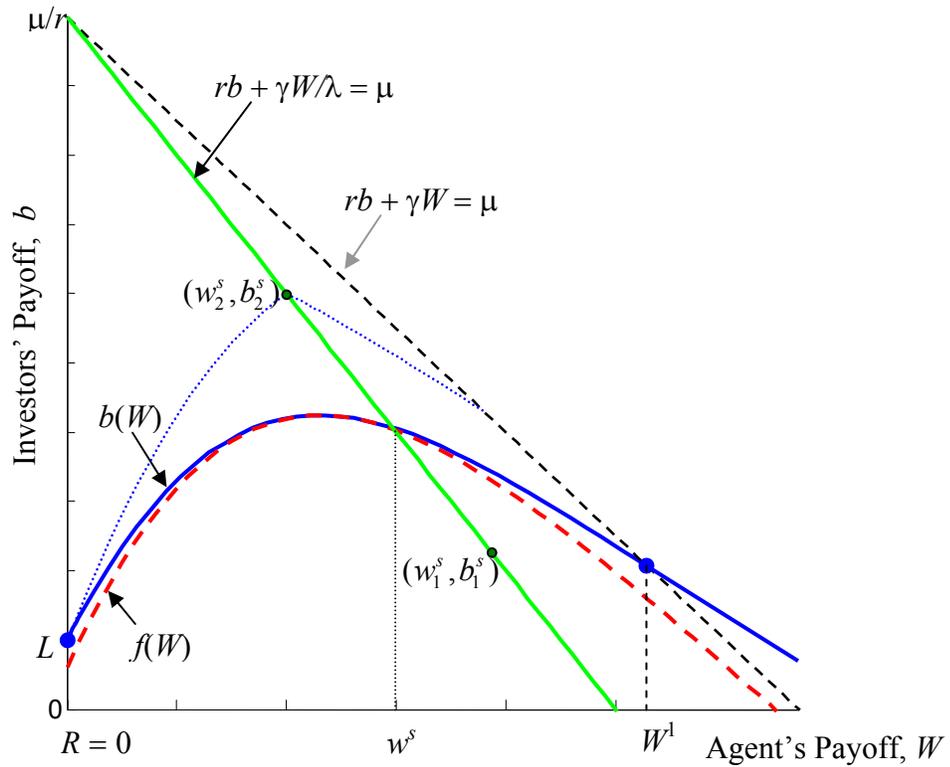
**Figure 3. The optimal contract with medium and high volatility.** For $\sigma = 12.5$ and $\sigma = 19.07$.
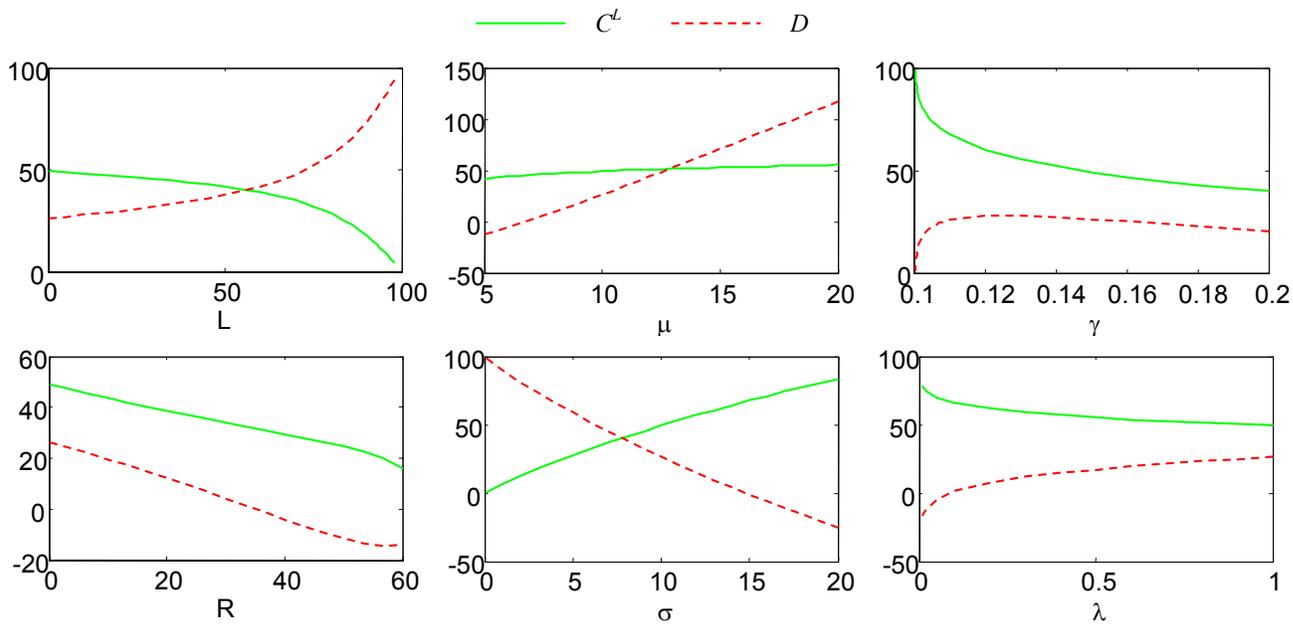
Figure 4. Comparative statics. Base case: $L = 0$, $R = 0$, $\mu = 10$, $\sigma = 10$, $r = 10\%$, $\gamma = 15\%$, $\lambda = 1$.

Figure 5. Market values of securities. For $\mu=10$, $\sigma=10$, $\lambda=50\%$, $r=10\%$, $\gamma=15\%$, $L=10$, $R=0$.

Figure 6. The optimal contract with hidden effort. If A is sufficiently large that (w^s, b^s) is below the curve $f(W)$, then high effort is optimal and the optimal contract is the same as in the cash flow diversion model of Section I.
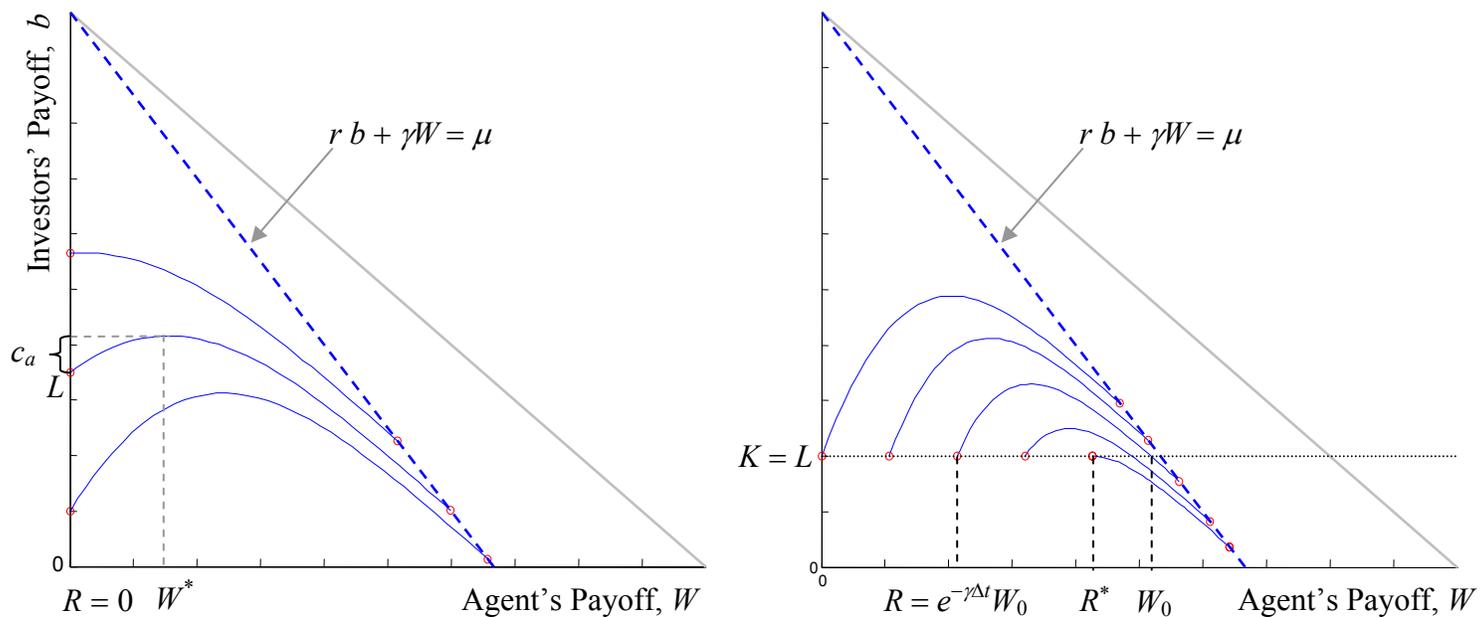
Figure 7. Determining *L* or *R* endogenously. The left panel considers the case in which the agent can be fired and replaced at cost c_a, so that $L = b(W*) - c_a$. The right panel considers the case in which the agent can quit and raise capital K (in the example, $K = L$) to start a new firm with delay Δt, so that $R = e^{-\gamma \Delta t} W_0$.

Table I

Comparative Statics for the Optimal Contract

	$dC^L/$	$dD/$	$dW_0/$	$dW^*/$	$db(W^*)/$
dL	−	+	+	−	+
dR^{31}	−	−	−	+	−
$d\gamma$	−	±	−	−	−
$d\mu$	+	+ (if $\lambda=0$)	+	+	+
$d\sigma^2$	+	−	−	±	−
$d\lambda$	− (if $R=0$)	+	−	±	−

Table II

Explicit Comparative Statics Calculations

	$dC^L\big/ \ d\lambda^{-1}(W^1-R)\big/ =$	$dD\big/ \ dr^{-1}(\mu-\gamma W^1/\lambda)\big/ =$	$dW_0 /$	$dW^* /$	$db(W^*)/$
dL	$-\dfrac{rG_\tau(W^1)}{\lambda(\gamma-r)}<0$	$\dfrac{\gamma G_\tau(W^1)}{\lambda(\gamma-r)}>0$	$-\dfrac{G_\tau(W_0)}{b'(W_0)}>0$	$-\dfrac{G_\tau{}'(W^*)}{b''(W^*)}<0$	$G_\tau(W^*)>0$
dR	$\left(\dfrac{rb'(R)G_\tau(W^1)}{\lambda(\gamma-r)}-\dfrac{1}{\lambda}<0\right)$	$-\dfrac{\gamma b'(R)G_\tau(W^1)}{\lambda(\gamma-r)}<0$	$\dfrac{b'(R)G_\tau(W_0)}{b'(W_0)}<0$	$b'(R)\dfrac{G_\tau{}'(W^*)}{b''(W^*)}>0$	$-b'(R)G_\tau(W^*)<0$
$d\gamma$	$-\dfrac{W^1+rG_1(W^1)}{\lambda(\gamma-r)}<0$	$\dfrac{W^1+\gamma G_1(W^1)}{\lambda(\gamma-r)}?0$	$\left(-\dfrac{G_1(W_0)}{b'(W_0)}<0\right)$	$\left(-\dfrac{G_1{}'(W^*)}{b''(W^*)}<0\right)$	$\left(G_1(W^*)<0\right)$
$d\mu$	$\dfrac{G_\tau(W^1)}{\lambda(\gamma-r)}>0$	$\left(\dfrac{1}{r}-\dfrac{\gamma G_\tau(W^1)}{r\lambda(\gamma-r)}?0\right)^{32}$	$-\dfrac{1-G_\tau(W_0)}{rb'(W_0)}>0$	$-\dfrac{G_\tau{}'(W^*)}{rb''(W^*)}>0$	$\dfrac{1-G_\tau(W^*)}{r}>0$
$d\sigma^2$	$-\dfrac{r\lambda G_2(W^1)}{2(\gamma-r)}>0$	$\dfrac{\gamma\lambda G_2(W^1)}{2(\gamma-r)}<0$	$-\dfrac{\lambda^2 G_2(W_0)}{2b'(W_0)}<0$	$-\dfrac{\lambda^2 G_2{}'(W^*)}{2b''(W^*)}?0$	$\dfrac{\lambda^2}{2}G_2(W^*)<0$

Footnotes

[1] Data from the Loan Pricing Corporation. Public debt (including convertibles) accounts for 15%, and standard term loans for 22%, of corporate borrowing for this period.

[2] See Acharya et al. (2002) for an analysis of the impact of these options held by equity holders on credit spreads and firm value.

[3] Radner (1986) demonstrates a folk-theorem result for repeated principal-agent problems. Though the play is continuous in our setting, because of the volatility of the cash flows the first-best cannot be attained.

[4] Schattler and Sung (1993) develop a rigorous mathematical framework for this problem in continuous time, and Sung (1995) allows the agent to control volatility as well. See also Bolton and Harris (2001), Ou-yang (2003), Detemple, Govindaraj, and Loewenstein (2001), Cadenillas, Cvitannic, and Zapatero (2003), Sannikov (2003), and Williams (2004) for further generalization and analysis of the HM setting.

[5] Spear and Wang (2003) also analyze the decision of when to fire an agent in a discrete-time model. They do not consider the implementation of the decision through standard securities.

[6] For models based on cash flow diversion, see, for example, Townsend (1979), Diamond (1984), and Bolton and Scharfstein (1990). See also Innes (1990) for optimal security design in a standard principal-agent setting. Dewatripont and Tirole (1994) discuss the role of capital structure (including inside and outside equity) in a two-period model in which investors learn noncontractible information regarding the manager's performance in the first period, and the firm's choice of capital structure provides incentives for appropriate external intervention.

[7] Equation (1) implies that the agent pays a proportional cost $(1-\lambda)$ to conceal funds, even if the funds are ultimately put back into the firm. We could instead assume that the cost is only paid if the funds are diverted for the agent's consumption. This change would not alter the results in any way (see Proposition 2).

[8] We can ignore consumption beyond date τ because $\gamma \geq r$ implies that it is optimal for the agent to consume all savings at termination (i.e., $S_\tau = 0$).

[9] Typically, the intertemporal marginal rate of substitution for a borrowing-constrained agent is greater than the market interest rate r. To capture this detail in a risk-neutral setting, we assume $\gamma > r$. The case $\gamma = r$ requires

either a finite horizon or a bound on the project's per-period operating losses, otherwise it would be optimal for the agent to postpone consumption "forever." See Section IV.D for a further justification of this point.

[10] Bounded variation ensures that $[Y_t - \hat{Y}_t]^\lambda$ is well defined. With unbounded variation of $Y_t - \hat{Y}_t$, the agent would steal and overreport a dollar infinitely many times, earning an income of minus infinity (which would be infeasible).

[11] See the Appendix for proofs that are not in the text.

[12] Formally, $Y_t - \hat{Y}_t$ is Lipschitz-continuous (see also footnote 13).

[13] By Lipschitz continuity of $Y_t - \hat{Y}_t$, the probability measures over the paths of Y and $\hat{Y}$ are equivalent.

[14] For example, the agent can report $d\hat{Y}_t = dY_t - dt$ when $\beta < \lambda$ and tell the truth when $\beta \geq \lambda$. Because the probability measures over paths of Y and $\hat{Y}$ are equivalent, $\beta(\hat{Y}) < \lambda$ on set of positive measure and the agent will gain from this deviation.

[15] Given the linearity of the incentive compatibility condition, public randomization would only be useful for allowing stochastic termination of the contract.

[16] The proof shows that $b(W)$ is strictly concave for $W \leq W^1$ (see also footnote 18), so that $\beta = \lambda$ is the unique optimum.

[17] Roughly speaking, if there were a kink at W^1, $b''(W^1) = -\infty$ and (12) could not be satisfied.

[18] A similar argument shows that public randomization is not useful. For an optimal contract, $rb(W) \geq \mu + \gamma W b'(W) + \frac{1}{2}\lambda^2 \sigma^2 b''(W)$, since it is always possible to run the project and delay payment. If public randomization were necessary to convexify $b(W)$, we would have $b''(W)=0$ where it is used. But then $b'(W) \geq -1$ would imply that $rb(W) + \gamma W > \mu$. Thus, randomization is not beneficial for $W < W^1$.

[19] Inside equity could correspond to a stock grant to the agent combined with a zero interest loan due upon termination that equals or exceeds the liquidation value of the equity.

[20] One can rewrite (17) as $\lambda (\mu - rD - \gamma C^L) = \gamma R$, which states that the agent's share of the firm's profit rate (after interest payments) matches the agent's outside option when the credit line is exhausted.

[21] An alternative implementation is given in Shim (2004) and Biais et al. (2004) for a specialized setting. Rather than a credit line, they suppose that the firm retains a cash reserve and that the coupon payment on the debt varies contractually with the level of the cash reserves.

[22] Recall that only the aggregate payments to investors matter for incentives; the division of the payments across securities is only relevant for pricing.

[23] Lemma E in the Appendix shows that $L < D + C^L$ when $\lambda = 1$ and there are no outside equity holders. In that case, we can set $L_E = 0$ to compute the "shadow price" of outside equity.

[24] Leland (1994) notes that covenants that force default as soon as asset values fall below the face value of debt eliminate the asset substitution problem. Here, there is no asset substitution despite the fact that debt may be risky.

[25] While we assume the effort choice is binary, nothing would change if it were continuous, as long as the marginal cost to the agent of increasing the drift remained constant at λ.

[26] Formally, (23) is needed in the proof of Proposition 1 for G_t to remain a supermartingale for either effort choice.

[27] This result holds when A is small enough that shirking yields investors the highest possible payoff. For intermediate values of A, an optimal contract calls for shirking only temporarily and a more complicated contract than the one described in this paper will be necessary to achieve optimality.

[28] This setting is similar to Hart and Moore's (1994) notion of "inalienable human capital" and its relationship to optimal debt structure.

[29] Gromb (1999) considers renegotiation-proofness in a related discrete-time model. While not providing a complete characterization, he does show that in an infinite-horizon stationary setting, the maximum external capital that the firm can raise is the liquidation value L. Note also that we can relax the renegotiation constraint by assuming costs of renegotiation and adapting the approach in Section IV.A.

[30] For example, suppose investors hire the agent, $r = 10\%$ and $\gamma \in [10\%, 11\%]$, with all other parameters as in Figure 2. By choosing the contract for $\gamma = 11\%$, investors lose at most about 2½% of the payoff they could have attained by choosing γ correctly. But if they choose a contract for $\gamma < 11\%$, and the true γ is higher, investors lose about 90% of their payoff. With a uniform prior for γ, the contract for $\gamma = 11\%$ is best for the investors.

[31] These are for the case when the project's value to investors can exceed L, which implies that $b'(R) > 0$.

[32] This expression is positive if $\lambda = 1$.