Issuance Costs and Stock Return Volatility*

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Abstract

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1. The Model

The following notation will be maintained throughout the paper. Time is continuous, and labelled by $t \geq 0$. Uncertainty is modelled by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over which is defined a standard Wiener process $W = \{W_t; t \geq 0\}$. We let $\mathcal{F}^W = \{\mathcal{F}^W_t; t \geq 0\}$ be the $\mathbb{P}$–augmentation of the filtration $\{\sigma(W_s; s \leq t); t \geq 0\}$ generated by W.

A firm has a single investment project that generates random cash-flows over time. The cumulative cash-flows $R = \{R_t; t \geq 0\}$ evolve according to an arithmetic Brownian motion with strictly positive drift μ and volatility σ,

$$R_0 = 0, \quad dR_t = \mu dt + \sigma dW_t; \quad t \geq 0. \tag{1}$$

At each date, the project can be continued or liquidated. For simplicity, the liquidation value is set equal to 0. The firm is held by a diffuse basis of risk-neutral equity holders, with limited liability, and has no access to credit. As a result of this, its cash reserves must always remain non-negative. Equity holders discount future payments at a rate $\rho > 0$.

At each date, the firm can retain part of its earnings, or issue new equity. As in Sethi and Taksar (2002) or Løkka and Zervos (2005), issuing equity involves a proportional brokerage commission. Specifically, for each dollar of new equity issued, the firm receives $1/p$ dollars in cash, where $p > 1$ measures the proportional transaction cost. In addition, a distinctive feature of our model is that each issue of equity involves a fixed transaction cost $f > 0$. As a result of this, the firm's issuance strategy can without loss of generality be described by an increasing sequence $(\tau_n)_{n\geq 1}$ of $\mathcal{F}^W$–adapted stopping times representing the dates at which equity is issued, along with a sequence $(i_n)_{n\geq 1}$ of $(\mathcal{F}^W_{\tau_n})_{n\geq 1}$–adapted strictly positive random variables representing the issuance proceeds. At any date $t \geq 0$,

$$I_t = \sum_{n\geq 1} i_n 1_{\{\tau_n \leq t\}} \tag{2}$$

corresponds to the total issuance proceeds up to and including date t, while:

$$F_t = \sum_{n\geq 1} f 1_{\{\tau_n \leq t\}} \tag{3}$$

corresponds to the total fixed transaction costs incurred up to and including date t. We denote by $I = \{I_t; t \geq 0\}$ and $F = \{F_t; t \geq 0\}$ the processes defined by (2)–(3), which are $\mathcal{F}^W$–adapted by construction.

What is not retained from earnings is paid out as dividends. Let $L = \{L_t; t \geq 0\}$ be the cumulative dividend process. It is assumed that L is $\mathcal{F}^W$–adapted and right-continuous, with $L_0 = 0$, and that it is non-decreasing, reflecting the equity holders' limited liability. The cash reserves $M = \{M_t; t \geq 0\}$ of the firm evolve according to:

$$M_{0-} = m, \quad dM_t = rM_t dt + dR_t + \frac{1}{p} dI_t - dF_t - dL_t; \quad t \geq 0, \tag{4}$$

where $r \in [0, \rho)$ is the interest earned on cash, R, I and F are defined by (1)–(3), and $m \geq 0$. Since the firm must hold non-negative cash reserves, (4) represents the dynamics of the cash reserves up to the time:

$$\tau_B = \inf\{t \geq 0 \,|\, M_t < 0\} \tag{5}$$

at which the firm goes eventually bankrupt. Note that we allow τ_B to take infinite values. Given an issuance policy $((\tau_n)_{n\geq 1},(i_n)_{n\geq 1})$, a dividend policy L, and cash reserves $m\geq 0$, the value of the firm can be computed as:

$$v(m;(\tau_n)_{n\geq 1},(i_n)_{n\geq 1},L)=\mathbb{E}^m\left[\int_0^{\tau_B} e^{-\rho t}\,(dL_t-dI_t)\right],$$

where $(\tau_n)_{n\geq 1}$, $(i_n)_{n\geq 1}$, I, L and τ_B are related by (1)–(5). Note that, by construction,

$$\mathbb{E}^m\left[\int_0^{\tau_B} e^{-\rho t}dI_t\right]=\mathbb{E}^m\left[\sum_{n\geq 1} e^{-\rho\tau_n} i_n 1_{\{\tau_n\leq\tau_B\}}\right].$$

We define the corresponding value function as:

$$V^*(m)=\sup_{(\tau_n)_{n\geq 1},(i_n)_{n\geq 1},L}\{v(m;(\tau_n)_{n\geq 1},(i_n)_{n\geq 1},L)\};\quad m\geq 0. \tag{6}$$

It is convenient to extend the value function V^* to $(-\infty,0)$ by setting $V^*(m)=0$ for all $m<0$. This allows us to put no restrictions on the issuance proceeds $(i_n)_{n\geq 1}$ besides that they remain non-negative.

Remark. Our setup embeds the pure dividend distribution model of Jeanblanc-Picqué and Shiryaev (1995) as a special case, in which the proportional cost p or the fixed cost f assume a very large value. In this situation, issuing is not a profitable option for the firm, which is then liquidated as soon as its cash reserves hit 0. It will be also seen that our setup also embeds the model of Løkka and Zervos (2005) as a limit case, in which equity issuance involves a proportional cost but no fixed cost. Note also that, in contrast with these papers, we allow cash reserves to be remunerated.

2. The First-Best Benchmark

Before considering how issuance costs affect the firm's issuance and dividend policy, as well as the dynamics of stock prices, we examine a benchmark case in which such costs are absent, that is $p=1$ and $f=0$. In this first-best environment, the firm's value is simply the sum of its cash reserves m and of the present value of its future cash-flows:

$$\hat{V}(m)=m+\mathbb{E}^m\left[\int_0^{\infty} e^{-\rho t}\,(\mu dt+\sigma dW_t)\right]=m+\frac{\mu}{\rho}. \tag{7}$$

Since cash reserves are remunerated at a rate $r<\rho$, it is optimal for the firm to hold no cash reserves beyond date 0. In the absence of financial frictions, the Modigliani and Miller (1958) logic applies, so that we have many degrees of freedom in designing issuance and dividend processes $\hat{I}=\{\hat{I}_t;t\geq 0\}$ and $\hat{L}=\{\hat{L}_t;t\geq 0\}$ that deliver the value (7). We shall assume for simplicity that the total amount of dividends is constant per unit of time. Since the firm distributes all its cash reserves m as a special dividend at date 0, this yields:

$$\hat{L}_t=m1_{\{t=0\}}+lt;\quad t\geq 0, \tag{8}$$

for some arbitrary $l>0$. Consider now the issuance process. Allowing for share repurchases, the requirement that cash reserves be constant and equal to 0 after date 0 yields:

$$d\hat{I}_t=-(\mu-l)dt-\sigma dW_t;\quad t\geq 0. \tag{9}$$

It is immediate that the pair $(\hat{I}, \hat{L})$ defined by (8)–(9) delivers the first-best value (7). We now turn to the dynamics of stock prices in this frictionless market. Let $\hat{S} = \{\hat{S}_t; t \geq 0\}$ be the process describing the ex-dividend price of a share in the firm, and $\hat{N} = \{\hat{N}_t; t \geq 0\}$ the process modelling the number of shares issued by the firm. After date 0, the market capitalization $\hat{N}\hat{S}$ of the firm stays constant at a level μ/ρ. In particular, the dividend per share and per unit of time is $l/\hat{N} = l\rho\hat{S}/\mu$. Therefore, at any date $t \geq 0$, one has:

$$d\hat{I}_t = d(\hat{N}_t\hat{S}_t) - \hat{N}_t d\hat{S}_t = -\hat{N}_t d\hat{S}_t = -\frac{\mu}{\rho}\frac{d\hat{S}_t}{\hat{S}_t}. \tag{10}$$

Along with (9), (10) implies that:

$$\frac{d\hat{S}_t}{\hat{S}_t} = \rho\left(1 - \frac{l}{\mu}\right)dt + \sigma\,\frac{\rho}{\mu}\,dW_t = \rho dt + \sigma\,\frac{\rho}{\mu}\,dW_t - \frac{\rho}{\mu}\,d\hat{L}_t \tag{11}$$

for all $t \geq 0$. This stock price dynamics is similar to that postulated by Black and Scholes (1973) and Merton (1973). It should be noted that, while the dividend distribution process is indeterminate, the constancy of the volatility of stock returns in (11) is a direct implication of the fact that the market capitalization of the firm stays constant over time.

3. The Optimal Issuance and Dividend Policies

In this section, we first derive heuristically a system of variational inequalities for the value function V^*. We then prove that this system has a solution satisfying appropriate regularity conditions. Finally, a verification argument establishes that this solution coincides with V^*, from which the optimal issuance and dividend policies can be inferred.

3.1. A Heuristic Derivation of the Value Function

To derive the system of variational inequalities satisfied by V^*, suppose for the moment that V^* is twice continuously differentiable over $(0, \infty)$ and that for all $m \geq 0$ there exists an optimal policy that attains the supremum in (6). Fix some $m > 0$. The policy that consists in distributing $l \in (0, m)$ worth of dividends, and then immediately executing the optimal policy associated with cash reserves $m - l$ must yield no more than the optimal policy:

$$V^*(m) \geq V^*(m - l) + l.$$

Subtracting $V^*(m - l)$ from both sides of this inequality, dividing through by l and letting l go to 0 yields that:

$$V^{*\prime}(m) \geq 1 \tag{12}$$

for all $m > 0$, as is usual in dividend distribution models. Next, the policy that consists in issuing $i > 0$ worth of equity, and then immediately executing the optimal policy associated with cash reserves $m + i/p - f$ must yield no more than the optimal policy:

$$V^*(m) \geq V^*\left(m + \frac{i}{p} - f\right) - i.$$

Thus, one must have:

$$V^*(m) \geq \sup_{m' \in [m,\infty)} \left\{V^*(m' - f) - p(m' - m)\right\} \tag{13}$$

for all $m > 0$. Finally, consider the policy that consists in abstaining from issuing new equity and from distributing any dividends for $t \wedge \tau_B$ units of time, where $t > 0$, and then executing the optimal policy associated to the resulting cash reserves $m + \int_0^{t \wedge \tau_B} [(\mu + rM_s)ds + \sigma dW_s]$. Again, this policy must yield no more than the optimal policy:

$$V^*(m) \geq \mathbb{E}^m \left[e^{-\rho t \wedge \tau_B} \, V^* \left(m + \int_0^{t \wedge \tau_B} [(\mu + rM_s)ds + \sigma dW_s] \right) \right].$$

Using Itô's Lemma and letting t go to 0 results in:

$$-\rho V^*(m) + \mathcal{L}V^*(m) \leq 0 \tag{14}$$

for all $m > 0$, where the infinitesimal generator $\mathcal{L}$ is defined as:

$$\mathcal{L}u(m) = (\mu + rm)u'(m) + \frac{\sigma^2}{2} u''(m). \tag{15}$$

We shall refer to (12)–(14) as the fundamental system of variational inequalities satisfied by V^*. To move forward, we make the following guess about the optimal strategy. Consider first the issuance policy. Because of the fixed transaction cost associated to equity issuances, it is natural to expect that these should be delayed as much as possible. This suggests that, if any issuance activity takes place at all, it must be at the times when the cash reserves hit 0 so as to avoid bankruptcy. Because of the stationarity of the model, we postulate that the optimal issuance policy then consists in issuing a constant amount worth of equity, or in abstaining from issuing equity altogether, which triggers bankruptcy. As a result of this, the value of the firm when it runs out of cash is:

$$V^*(0) = \left[\max_{i \in [0,\infty)} \left\{ V^* \left(\frac{i}{p} - f \right) - i \right\} \right]^+. \tag{16}$$

Denote by i^* a solution to the maximization problem in (16). It will turn out that i^* is uniquely determined at the optimum. If the firm does choose to issue equity, that is, if $V^*(i^*/p - f) - i^* > 0$, then $m_0^* = i^*/p - f > 0$ represents the post issuance cash reserves of the firm. Consider now the dividend policy. In line with standard dividend distribution models, it is natural to expect dividends to be distributed as soon as cash reserves hit or exceed a boundary $m_1^* > 0$. This implies that, for all $m \geq m_1^*$,

$$V^{*\prime}(m) = 1 \tag{17}$$

Since V^* is postulated to be twice continuously differentiable over $(0, \infty)$, (17) implies that, in addition, the following super contact condition holds at the dividend boundary m_1^*:

$$V^{*\prime\prime}(m_1^*) = 0 \tag{18}$$

When cash reserves lie in $(0, m_1^*)$, no issuance or dividend activity take place, and (14) holds as an equality. It then follows from (15) and (17)–(18) that $V^*(m_1^*) = (\mu + rm_1^*)/\rho$. We are thus led to the problem of finding a function V, along with thresholds $m_0 \geq -f$ and $m_1 > 0$, that solve the following variational system:

$$V(m) = 0; \quad m < 0, \tag{19}$$

$$V(0) = [V(m_0) - p(m_0 + f)]^+; \quad \text{for } m_0 \in \underset{m \in [-f,\infty)}{\arg\max} \{V(m) - p(m + f)\}, \tag{20}$$

$$-\rho V(m) + \mathcal{L}V(m) = 0; \quad 0 < m < m_1, \tag{21}$$

$$V(m) = \frac{\mu + r m_1}{\rho} + m - m_1; \quad m \geq m_1. \tag{22}$$

We shall then proceed as follows. First, we prove that there exists a unique solution V to (19)–(22) that is twice continuously differentiable over $(0, \infty)$. It is then easy to check that V satisfies the variational inequalities (12)–(14) over $(0, \infty)$. One can finally infer from this that V coincides with the value function V^* for problem (6).

3.2. Solving the Variational Inequalities

We solve (19)–(22) as follows. First fix some $m_1 > 0$, and consider the following boundary value problem over $[0, m_1]$:

$$-\rho V(m) + \mathcal{L}V(m) = 0; \quad 0 \leq m \leq m_1, \tag{23}$$

$$V'(m_1) = 1, \tag{24}$$

$$V''(m_1) = 0. \tag{25}$$

Standard existence results for linear second-order differential equations yield that (23)–(25) has a unique solution over $[0, m_1]$, which we denote by V_{m_1}. By construction, this solution satisfies $V_{m_1}(m_1) = (\mu + r m_1)/\rho$. Extending linearly V_{m_1} to $[m_1, \infty)$ as in (22), we obtain a twice continuously differentiable function over $[0, \infty)$, which we denote again by V_{m_1}. The following lemma establishes key monotonicity and concavity properties of V_{m_1}.

Lemma 1. $V'_{m_1} > 1$ *and* $V''_{m_1} < 0$ *over* $[0, m_1)$.

Now observe that if there exists a solution V to (19)–(22) that is twice continuously differentiable over $(0, \infty)$, then by construction, V must coincide with some V_{m_1} over $[0, \infty)$ for an appropriate choice of m_1. This choice is in turn dictated by the boundary condition (20) that V must satisfy at 0. It is therefore crucial to examine the behavior of V_{m_1} and V'_{m_1} at 0. One has the following result.

Lemma 2. $V_{m_1}(0)$ *is a strictly decreasing and concave function of* m_1, *and* $V'_{m_1}(0)$ *is a strictly increasing and convex function of* m_1.

Since $\lim_{m_1 \downarrow 0} V_{m_1}(0) = \mu/\rho > 0$ and $\lim_{m_1 \downarrow 0} V'_{m_1}(0) = 1 < p$, it follows from Lemma 2 that there exists a unique $\hat{m}_1 > 0$ such that $V_{\hat{m}_1}(0) = 0$, and that there exists a unique $\tilde{m}_1 > 0$ such that $V'_{\tilde{m}_1}(0) = p$. It is easy to verify that $\hat{m}_1 > \tilde{m}_1$ if and only if $V'_{\hat{m}_1}(0) > p$.

Lemma 1 along with the fact that $V'_{m_1}(m_1) = 1$ further implies that if $m_1 \geq \tilde{m}_1$, there exists a unique $m_p(m_1) \in [0, m_1)$ such that $V'_{m_1}(m_p(m_1)) = p$. This corresponds to the unique maximum over $[0, \infty)$ of the function $m \mapsto V_{m_1}(m) - p(m + f)$. Note that, by construction, $m_p(\tilde{m}_1) = 0$. There are now two cases to consider.

Case 1. Suppose first that:

$$\hat{m}_1 \leq \tilde{m}_1 \quad \text{or} \quad V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \leq 0. \tag{26}$$

Define a function $\hat{V}$ by:

$$\hat{V}(m) = \begin{cases} 0 & m < 0, \\ \\ V_{\hat{m}_1}(m) & m \geq 0. \end{cases} \tag{27}$$

Note that, by construction, $\hat{V}(0) = 0$. Furthermore, condition (26) implies that the function $m \mapsto \hat{V}(m) - p(m + f)$ reaches its maximum over $[-f, \infty)$ at $\hat{m}_0 = -f$. It is then easy to check that $(V, m_0, m_1) = (\hat{V}, \hat{m}_0, \hat{m}_1)$ solves the variational system (19)–(22).

Case 2. Suppose next that:

$$\hat{m}_1 > \tilde{m}_1 \quad \text{and} \quad V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0. \tag{28}$$

One then has the following lemma.

Lemma 3. *If (26) holds, there exists a unique* $\bar{m}_1 \in (\tilde{m}_1, \hat{m}_1)$ *such that:*

$$V_{\bar{m}_1}(0) = V_{\bar{m}_1}(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]. \tag{29}$$

Define a function $\bar{V}$ by:

$$\bar{V}(m) = \begin{cases} 0 & m < 0, \\ \\ V_{\bar{m}_1}(m) & m \geq 0. \end{cases} \tag{30}$$

Note that Lemma 2 together with the fact that $\bar{m}_1 < \hat{m}_1$ implies that $\bar{V}(0) > 0$. Furthermore, since $\bar{m}_1 > \tilde{m}_1$, the function $m \mapsto \bar{V}(m) - p(m + f)$ reaches its maximum over $[-f, \infty)$ at $\bar{m}_0 = m_p(\bar{m}_1)$. It is then easy to check that $(V, m_0, m_1) = (\bar{V}, \bar{m}_0, \bar{m}_1)$ solves the variational system (19)–(22).

The following proposition summarizes our findings.

Proposition 1. *There exists a unique solution* V *to the variational system (19)–(22) that is twice continuously differentiable over* $(0, \infty)$. *Moreover,* V *satisfies the variational inequalities (12)–(14) over* $(0, \infty)$.

3.3. The Verification Argument

In this subsection, we establish that the solution V to (19)–(22) coincides with the value function V^* for problem (6). Our first result is that V is an upper bound for V^*.

Lemma 4. *For any admissible issuance and dividend policy* $((\tau_n)_{n \geq 1}, (i_n)_{n \geq 1}, L)$,

$$V(m) \geq v(m; (\tau_n)_{n \geq 1}, (i_n)_{n \geq 1}, L); \quad m \geq 0.$$

We now construct an admissible policy whose value coincides with V. Given Lemma 4, this establishes that $V^* = V$, and thereby provides the optimal issuance and dividend policy. Define $m_0^* = m_0^+$ and $m_1^* = m_1$, where m_0 and m_1 are given by (19)–(22). To construct the optimal policy, consider the following version of Skorokhod's problem:

$$M_t^* = m + \int_0^t (\mu + rM_s^*)\, ds + \sigma W_t + \sum_{n\geq 1} m_0^* 1_{\{T_n^* \leq t\}} - L_t^*, \tag{31}$$

$$M_t^* \leq m_1^*, \tag{32}$$

$$L_t^* = \int_0^\infty 1_{\{M_s^* = m_1^*\}}\, ds, \tag{33}$$

for all $t \in [0, \tau_B^*]$, where $\tau_B^* = \inf\{t \geq 0 \,|\, M_t^* < 0\}$ and the sequence of stopping times $(T_n^*)_{n\geq 1}$ is recursively defined by:

$$T_n^* = \inf\{t \geq T_{n-1}^* \,|\, M_{t-}^* = 0\}; \quad n \geq 1 \tag{34}$$

where $T_0^* = 0$. Standard results on Skorokhod's problem (Tanaka (1979)) along with the strong Markov property imply that there exists a pathwise unique solution $(M^*, L^*) = \{(M_t^*, L_t^*); t \geq 0\}$ to (31)–(34). Condition (33) requires that L^* increases only when M^* hits the boundary m_1^*, while (31)–(32) express that this causes M^* to be reflected back at m_1^*. As for the behavior of M^* at 0, two cases can arise. If (26) holds, $m_0^* = (-f)^+ = 0$, so that $\tau_B^* = T_1^*$ $\mathbb{P}$–almost surely. This corresponds to a situation in which the project is liquidated as soon as the firm runs out of cash. By contrast, if (28) holds, $m_0^* = m_p(m_1^*) > 0$. In that case, the process M^* discontinuously jumps to m_0^* each time it hits 0, so that $\tau_B^* = \infty$ $\mathbb{P}$–almost surely. This corresponds to a situation in which an amount $i^* = p(m_0^* + f)$ of new equity is issued when the firm runs out of cash. One has the following result.

Proposition 2. *The value function V^* for problem (6) coincides with the unique solution V to the variational system (19)–(22) that is twice continuously differentiable over $(0, \infty)$. The optimal issuance and dividend policy is given by $((\tau_n^*)_{n\geq 1}, (i_n^*)_{n\geq 1}, L^*)$, where:*

$$\tau_n^* = \infty, \quad i_n^* = 0; \quad n \geq 1 \quad \textit{if condition (26) holds,}$$

$$\tau_n^* = T_n^*, \quad i_n^* = i^*; \quad n \geq 1 \quad \textit{if condition (28) holds.}$$

According to Proposition 1, the firm's optimal dividend policy consists in retaining all its earnings until accumulated cash reserves exceed the threshold m_1^*. When this arises, the firm pays all the excess over m_1^* as dividends. Regarding the firm's issuance policy, two situations can arise. If condition (26) holds, which intuitively arises when the issuance costs p and f are high, the firm never resorts to outside financing. The model is then essentially equivalent to that of Jeanblanc-Picqué and Shiryaev (1995), the only difference being that we allow for cash remuneration. By contrast, if issuance costs are low and condition (26) holds, the firm avoids liquidation by issuing new equity when its cash reserves are depleted. Although the firm is never liquidated, its value $V^*(m)$ falls short of the first-best value $\hat{V}(m) = m + \mu/\rho$ because of the presence of transaction costs. The concavity of V^* over $[0, \infty)$ reflects that the value of firm reacts less to changes in the level of cash reserves when past performance has been high. This is because high accumulated cash reserves allow the firm to postpone the time at which it will have to raise new equity and incur the corresponding issuance costs. By contrast, following unfavorable cash-flow realizations, cash reserves are low, and the value of the firm reacts strongly to performance and ensuing changes in cash reserves.

Two limiting cases of our analysis are worth mentioning. If $p = 1$, equity issuances involve no proportional cost. It is then easy to see that, if f is small enough so as to ensure that condition (28) is fulfilled, the dividend boundary m_1^* coincides with the post issuance level of cash reserves of the firm, $m_0^* = m_1^*$. The intuition is that since issuances involve only a fixed cost f, it is optimal for the firm to raise as much equity as possible from the market. In that case, equity issuances are tied to dividend distribution: following an equity issuance and a favorable cash-flow realization, the firm immediately distributes the excess of cash over m_1^* as dividends. By contrast, if f tends to 0, the lump sum amounts of equity issued tend to 0, and in the limit we have $V^{*\prime}(0) = p$ as in the model of Løkka and Zervos (2005). In that case, the optimal issuance policy is no longer described by an impulse control as in Proposition 2, with discontinuous jumps in the cash reserves at 0, but rather by a singular control similar to the dividend process. In practice, equity issuances are rarely followed by dividend distributions, and firms undertake equity adjustments in lumpy and infrequent issues (Bazdresch (2005)). This is consistent with a combination of fixed and proportional issuance costs such as the one we have postulated.

The characterization of the value function V^* provided in Proposition 1 allows us to study the impact of an increase in issuance costs on the sensitivity of the value of the firm to changes in its cash reserves. To focus on an interesting case, we assume in the following result that issuance costs remain low enough so as to guarantee that condition (28) holds and hence that the firm does resort to outside financing at the optimum.

Corollary 1. *The elasticity of the value of the firm with respect to its cash reserves,*

$$\epsilon^*(m) = \frac{mV^{*\prime}(m)}{V^*(m)}; \quad m \geq 0, \tag{35}$$

is an increasing function of the issuance costs p and f.

The proof of this result proceeds as follows. An increase in issuance costs obviously results in a fall in the firm's value, which mechanically raises the elasticity (35). This fall in value is tied to an increase in the dividend boundary m_1^*, reflecting the intuitive fact that as issuance costs increase, the firm must accumulate more liquidities before distributing dividends. Using the non-crossing property of the solutions to (23)–(25), it is then easy to establish that this implies that the marginal value of cash increases with issuance costs, which further raises the elasticity (35).

For a given firm, the concavity of V^* guarantees that the elasticity $\epsilon^*(m)$ is a decreasing function of the level m of its cash reserves. What Corollary 1 establishes is that this effect is magnified by issuance costs. Intuitively, the percentage change in firm value per percentage change in cash reserves is larger when issuance costs are relatively high, because allowing the firm to postpone a costly new equity issuance is more valuable in this situation. Conversely, the holding of liquid assets is less important when the firm has access to cheap outside financing. A testable implication of this is that firms' valuations should be more responsive to changes in their cash reserves on markets with high transaction costs. Alternatively, a reduction in transaction costs triggered by a capital market liberalization should reduce the responsiveness of firms' valuations to changes in their cash reserves.

4. Stock Prices

We are now ready to derive the implications of our theory for stock prices. To focus on the case where the firm does resort occasionally to outside financing, we suppose thereafter that condition (28) holds. We denote by $S^* = \{S^*_t; t \geq 0\}$ the process describing the ex-dividend price of a share in the firm, and by $N^* = \{N^*_t; t \geq 0\}$ the process modelling the number of shares issued by the firm. Thus at any date $t \geq 0$, S^*_t does not include dividends distributed at date t, while N^*_t includes new shares issued at date t. We assume that N^* is a non-decreasing process and we adopt the normalization $N^*_{0-} = 1$. A key observation is that issuance and payout decisions critically depend on the amount of liquidities accumulated by the firm. As a result of this, the stock price and the number of outstanding shares are contingent on the current level of cash reserves. At any date $t \geq 0$, the value of the firm satisfies:

$$V^*(M^*_t) = N^*_t S^*_t. \tag{36}$$

Now turn to the optimal issuance process I^*. At any date $t \geq 0$,

$$dI^*_t = d[V^*(M^*_t)] - N^*_t dS^*_t = d(N^*_t S^*_t) - N^*_t dS^*_t = S^*_t dN^*_t, \tag{37}$$

where the first equality reflects that part of the change in the value of the firm due to new equity issuance is absorbed by existing shareholders, and the third inequality follows from the fact that N^* is an increasing process, so that $d\langle N^*, S^*\rangle_t = 0$. The following lemma holds.

Lemma 5. *For each* $n \geq 1$, $S^*_{\tau^*_n} = S^*_{\tau^*_n -}$.

Lemma 5 expresses the fact that the stock price does not jump at the optimal equity issuance dates. This is because the issuance process is predictable in our model: the firm raises new equity as it runs out of cash, an event that is observable by all participants to the market. The fact that the stock price does not react to new equity issuances follows then simply from the absence of arbitrage opportunities. In particular, equity issuances do not convey bad news about the profitability of the firm, unlike what typically happens when firms have private information about future profitability (Myers and Majluf (1984)).

We are now ready to derive the dynamics of the processes N^* and S^*. Our first result is a direct implication of the fact that, when an equity issuance occurs, the value of the firm discontinuously jumps from $V^*(0)$ to $V^*(m^*_0)$, while the stock price itself is unaffected according to Lemma 5.

Proposition 3. *The process* N^* *modelling the number of outstanding shares is ponctual and defined by*:

$$N^*_t = \begin{cases} 1 & t \in [0, \tau^*_1), \\ \left[\dfrac{V^*(m^*_0)}{V^*(0)}\right]^n & t \in [\tau^*_n, \tau^*_{n+1}). \end{cases} \tag{38}$$

According to (38), each time new equity is raised, the ratio of new shares to outstanding shares is constant and equal to $[V^*(m^*_0) - V^*(0)]/V^*(0)$, which corresponds to a constant dilution factor. The number of shares is constant between two consecutive issuance dates. Thus, for all $n \geq 0$ and $t \in [\tau_n^*, \tau_{n+1}^*)$, one has $dS_t^* = d[V^*(M_t^*)]/N_{\tau_n^*}^*$. Using Itô's Lemma along with (21), together with the facts that the dividend process L^* increases only at m_1^* and that $V^*(m_1^*) = (\mu + rm_1^*)/\rho$ and $V^{*\prime}(m_1^*) = 1$, it is easy to derive the following result.

Proposition 4. *Between two consecutive issuance dates τ_n^* and τ_{n+1}^*, the stock price process S^* evolves according to*:

$$\frac{dS_t^*}{S_t^*} = \rho dt + \sigma\, \frac{V^{*\prime}((V^*)^{-1}(N_{\tau_n^*}^* S_t^*))}{N_{\tau_n^*}^* S_t^*}\, dW_t - \frac{\rho}{\mu + rm_1^*}\, dL_t^*. \tag{39}$$

Along with the characterization of the value function V^* provided in Section 3, this result implies that the dynamics of the stock price S^* differs in three important ways from the log-normal specification postulated by Black and Scholes (1973) and Merton (1973), and derived in equation (11) in the first-best benchmark. First, the stock price is reflected back each time dividends are distributed, which occurs when the process $V^*(M^*) = N^*S^*$ hits m_1^*. As a result of this, the stock price cannot take arbitrarily large values. Second, since the function V^* is strictly increasing and strictly concave over $[0, \infty)$, the volatility $\sigma V^{*\prime}((V^*)^{-1}(N^*S^*))/(N^*S^*)$ of stock returns is a decreasing function of S^*, so that changes in the volatility of stock returns are negatively correlated with stock price movements. That is, between two consecutive issuance dates, volatility tends to rise in response to bad news, and to fall in response to good news. Therefore our model predicts heteroscedasticity in stock prices, as documented for instance by Black (1976), Christie (1982) and Nelson (1991). While this "leverage effect" that ties stock returns and volatility changes cannot be attributed to financial leverage, as our firm is 100% equity financed, one can argue following Black (1976) that the firm has "operating leverage" as it must occasionally resort to costly outside financing to continue its activity. When earnings fall, the likelihood that these expenses will have to be incurred in the near future raises. As the value of the firm declines, it becomes more volatile, as small changes in earnings result in large changes in the difference between earnings and anticipated financial costs. Finally, the last difference between the stock price process (39) and the standard log-normal specification is that the dynamics of stock prices is path dependent, as it is modified by the successive equity issuances. As more stocks are issued, the number of outstanding shares N^* increases, which modifies both the stock price threshold m_1^*/N^* at which dividends are distributed and the volatility $\sigma V^{*\prime}((V^*)^{-1}(N^*S^*))/(N^*S^*)$ of stock returns. As more equity issuances take place, both the stock prices and the volatility on their return tend to fall. Another testable implication of our model is that the value of the firm is always more volatile than the cash-flows, $\sigma V^{*\prime}((V^*)^{-1}(N^*S^*)) \geq \sigma$, with equality only at the dividend boundary.

It should be noted that, while the stock price processes in the first-best benchmark and in the presence of issuance costs are qualitatively very different, there is nevertheless some formal analogy between (11) and (39). Indeed, in the absence of issuance costs, the value of the firm as given by (7) has a slope equal to 1, the market capitalization of the firm stays constant at μ/ρ, and the firm holds no cash reserves, which can be heuristically expressed by saying that the dividend boundary is equal to 0. Substituting formally in (39), one retrieve the second half of formula (11). Of course, this abstracts from the qualitative differences in the dividend process, which is absolutely continuous in (11) and singular in (39).

It is instructive to compare the stock price process (39) with that arising in the dynamic agency models of Biais, Mariotti, Plantin and Rochet (2004) or DeMarzo and Sannikov (2004). Much like in our framework, these models predict that stock return volatility tends to increase in response to bad performance. However, the mechanism that leads to this result is different. Agency costs typically make it optimal to liquidate the project as soon as the firm runs short of cash. This is what generates a concavity of the firm value and of the stock price in the level of liquidities that the firm has accumulated. In the implementation of the optimal contract, it is never optimal to raise new funds as the firm becomes illiquid. By contrast, time-varying volatility arises in our model precisely because there are costs to raise new funds from the market.

In line with Corollary 1, it is easy to characterize the impact of an increase in issuance costs on the volatility of stock returns. Again, we assume that condition (28) holds so that the firm does resort to outside financing at the optimum.

Corollary 2. *The volatility of stock returns as a function of the firm's valuation*,

$$\sigma^*(v) = \sigma \frac{V^{*\prime}((V^*)^{-1}(v))}{v}; \quad v \geq V^*(0), \tag{40}$$

is an increasing function of the issuance costs p and f.

The proof follows from the fact that V^* is a decreasing function of p and f, while $V^{*\prime}$ is an increasing function of p and f. A testable implication of this result is that a reduction in transaction costs triggered by a capital market liberalization should lead to a fall in the volatility of stock returns.

Appendix

Proof of Lemma 1. Since V_{m_1} is smooth over $[0, m_1)$, differentiating (23) and using the definition (15) of $\mathcal{L}$ yields that $-(\rho - r)V'_{m_1} + \mathcal{L}V'_{m_1} = 0$ over $[0, m_1)$. Using this along with (24)–(25), we obtain that $V'''_{m_1 -}(m_1) = 2(\rho - r)/\sigma^2 > 0$. Since $V''_{m_1}(m_1) = 0$ and $V'_{m_1}(m_1) = 1$, it follows that $V''_{m_1} < 0$ and thus $V'_{m_1} > 1$ over an interval $(m_1 - \varepsilon, m_1)$ for $\varepsilon > 0$. Now suppose by way of contradiction that $V'_{m_1}(m) \leq 1$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} = \sup\{m \in [0, m_1 - \varepsilon] \,|\, V'_{m_1}(m) \leq 1\} < m_1$. Then $V'_{m_1}(\tilde{m}) = 1$ and $V'_{m_1} > 1$ over $(\tilde{m}, m_1)$, so that $V_{m_1}(m_1) - V_{m_1}(m) > m_1 - m$ for all $m \in (\tilde{m}, m_1)$. Since $V_{m_1}(m_1) = (\mu + rm_1)/\rho$, this implies that for any such m,

$$\begin{aligned} V''_{m_1}(m) &= \frac{2}{\sigma^2}\left[\rho V_{m_1}(m) - (\mu + rm)V'_{m_1}(m)\right] \\ &< \frac{2}{\sigma^2}\left\{\rho[m - m_1 + V_{m_1}(m_1)] - (\mu + rm)\right\} \\ &= \frac{2}{\sigma^2}(\rho - r)(m - m_1) \\ &< 0, \end{aligned}$$

which contradicts the fact that $V'_{m_1}(\tilde{m}) = V'_{m_1}(m_1) = 1$. Therefore $V'_{m_1} > 1$ over $[0, m_1)$, from which it follows as above that $V''_{m_1} < 0$ over $[0, m_1)$. Hence the result. ■

Proof of Lemma 2. Consider the solutions H_0 and H_1 to the linear second-order differential equation $-\rho H + \mathcal{L}H = 0$ over $[0, \infty)$ that are characterized by the initial conditions $H_0(0) = 1$, $H'_0(0) = 0$, $H_1(0) = 0$ and $H'_1(0) = 1$. We first show that H'_0 and H'_1 are strictly positive over $(0, \infty)$. Consider H'_0. Since $H_0(0) = 1$ and $H'_0(0) = 0$, one has $H''_0(0) = 2\rho/\sigma^2 > 0$, so that $H'_0 > 0$ over an interval $(0, \varepsilon)$ for $\varepsilon > 0$. Suppose that $\tilde{m} = \inf\{m \geq \varepsilon \,|\, H'_0(m) \leq 0\} < \infty$. Then $H'_0(\tilde{m}) = 0$ and $H''_0(\tilde{m}) \leq 0$. Since $-\rho H_0 + \mathcal{L}H_0 = 0$, it follows that $H_0(\tilde{m}) \leq 0$, which stands in contradiction with the facts that $H_0(0) = 1$ and that H_0 is strictly increasing over $[0, \tilde{m}]$. Thus $H'_0 > 0$ over $(0, \infty)$, as claimed. The proof for H'_1 is similar, and is therefore omitted. Note that both H_0 and H_1 remain strictly positive over $(0, \infty)$. Next, let $W_{H_0,H_1} = H_0 H'_1 - H_1 H'_0$ be the Wronskian of H_0 and H_1. One has $W_{H_0,H_1}(0) = 1$ and:

$$\begin{aligned} W'_{H_0,H_1}(m) &= H_0(m)H''_1(m) - H_1(m)H''_0(m) \\ &= \frac{2}{\sigma^2}\left\{H_0(m)[\rho H_1(m) - (\mu + rm)H'_1(m)] - H_1(m)[\rho H_0(m) - (\mu + rm)H'_0(m)]\right\} \\ &= -\frac{2(\mu + rm)}{\sigma^2}\, W_{H_0,H_1}(m) \end{aligned}$$

for all $m \geq 0$, from which Abel's identity follows by integration:

$$W_{H_0,H_1}(m) = \exp\left(-\frac{2\mu m + rm^2}{\sigma^2}\right) \tag{41}$$

for all $m \geq 0$. Since $W_{H_0,H_1} > 0$, H_0 and H_1 are linearly independent. As a result of this, (H_0, H_1) is a basis of the 2-dimensional space of solutions to the equation $-\rho H + \mathcal{L}H = 0$. It follows in particular that for any $m_1 > 0$, one can represent V_{m_1} as:

$$V_{m_1} = V_{m_1}(0)H_0 + V'_{m_1}(0)H_1$$

over $[0, m_1]$. Using the boundary conditions $V_{m_1}(m_1) = (\mu + rm_1)/\rho$ and $V'_{m_1}(m_1) = 1$, one can solve for $V_{m_1}(0)$ and $V'_{m_1}(0)$ as follows:

$$V_{m_1}(0) = \frac{H_1'(m_1)(\mu + rm_1)/\rho - H_1(m_1)}{W_{H_0,H_1}(m_1)}, \tag{42}$$

$$V'_{m_1}(0) = \frac{H_0(m_1) - H_0'(m_1)(\mu + rm_1)/\rho}{W_{H_0,H_1}(m_1)}. \tag{43}$$

Using the explicit expression (41) for W_{H_0,H_1} along with the fact that H_0 and H_1 are solutions to $-\rho H + \mathcal{L}H = 0$, it is easy to verify from (42)–(43) that:

$$\begin{aligned}
\frac{dV_{m_1}(0)}{dm_1} &= -\left(1 - \frac{r}{\rho}\right) \exp\left(\frac{2\mu m_1 + rm_1^2}{\sigma^2}\right) H_1'(m_1), \\
\frac{d^2V_{m_1}(0)}{dm_1^2} &= -\frac{2}{\sigma^2}(\rho - r) \exp\left(\frac{2\mu m_1 + rm_1^2}{\sigma^2}\right) H_1(m_1), \\
\frac{dV'_{m_1}(0)}{dm_1} &= \left(1 - \frac{r}{\rho}\right) \exp\left(\frac{2\mu m_1 + rm_1^2}{\sigma^2}\right) H_0'(m_1). \\
\frac{d^2V'_{m_1}(0)}{dm_1^2} &= \frac{2}{\sigma^2}(\rho - r) \exp\left(\frac{2\mu m_1 + rm_1^2}{\sigma^2}\right) H_0(m_1).
\end{aligned}$$

The result then follows immediately from the fact that $\rho > r$ and that H_0, H_0', H_1 and H_1' are strictly positive over $\mathbb{R}_{++}$. ■

Proof of Lemma 3. Equation (29) can be rewritten as $\varphi(\bar{m}_1) = 0$, where:

$$\varphi(m_1) = V_{m_1}(m_p(m_1)) - V_{m_1}(0) - p[m_p(m_1) + f].$$

If $\hat{m}_1 > \tilde{m}_1$, the function φ is well defined and continuous over $[\tilde{m}_1, \hat{m}_1]$, while $\varphi(\tilde{m}_1) = -pf < 0$ and $\varphi(\hat{m}_1) = V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$ if the second half of condition (28) holds. Thus φ has at least a zero over $(\tilde{m}_1, \hat{m}_1)$. To prove that it is unique, we show that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$. Using the Envelope Theorem to evaluate the derivative of φ, this amounts to:

$$\frac{\partial W}{\partial m_1}(m_p(m_1), m_1) > \frac{\partial W}{\partial m_1}(0, m_1)$$

for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, where $W(m, m_1) = V_{m_1}(m)$ for all $(m, m_1) \in [0, \infty) \times (\tilde{m}_1, \hat{m}_1)$. Since $m_p(m_1) \in (0, m_1)$ for all $m_1 \in (\tilde{m}_1, \hat{m}_1)$, all that needs to be established is that for any such m_1, $(\partial W/\partial m_1)(\cdot, m_1)$ is strictly increasing over $[0, m_1]$. From (23)–(25), it is easy to check that $(\partial W/\partial m_1)(\cdot, m_1)$ solves the following boundary value problem over $[0, m_1]$:

$$-\rho \frac{\partial W}{\partial m_1}(m, m_1) + \mathcal{L}\frac{\partial W}{\partial m_1}(m, m_1) = 0; \quad 0 \leq m \leq m_1, \tag{44}$$

$$\frac{\partial^2 W}{\partial m \partial m_1}(m_1, m_1) = 0, \tag{45}$$

$$\frac{\partial^3 W}{\partial^2 m \partial m_1}(m_1, m_1) = -\frac{2}{\sigma^2}(\rho - r). \tag{46}$$

We are interested in the sign of $(\partial^2 W/\partial m \partial m_1)(m, m_1)$ for $m \in [0, m_1)$. As $(\partial^2 W/\partial m \partial m_1)(m_1, m_1) = 0$ and $(\partial^3 W/\partial^2 m \partial m_1)(m_1, m_1) < 0$, $(\partial^2 W/\partial m \partial m_1)(\cdot, m_1) > 0$ over an interval $(m_1 - \varepsilon, m_1)$ for $\varepsilon > 0$.

Now suppose by way of contradiction that $(\partial^2 W/\partial m\partial m_1)(m, m_1) \leq 0$ for some $m \in [0, m_1 - \varepsilon]$, and let $\tilde{m} = \inf\{m \in [0, m_1 - \varepsilon] \mid (\partial^2 W/\partial m\partial m_1)(m, m_1) \leq 0\}$. Then $(\partial^2 W/\partial m\partial m_1)(\tilde{m}, m_1) = 0$ and $(\partial^2 W/\partial m\partial m_1)(m, m_1) > 0$ for all $m \in (\tilde{m}, m_1)$, so that $(\partial W/\partial m_1)(m, m_1) < 0$ for all $m \in (\tilde{m}, m_1)$ as $(\partial W/\partial m_1)(m_1, m_1) = -(\rho - r)/\rho < 0$ by (44)–(46). This implies that for any such m,

$$\frac{\partial^3 W}{\partial^2 m\partial m_1}(m, m_1) = \frac{2}{\sigma^2}\left[\rho\,\frac{\partial W}{\partial m_1}(m, m_1) - (\mu + rm)\,\frac{\partial^2 W}{\partial m\partial m_1}(m, m_1)\right] < 0,$$

which contradicts the fact that $(\partial^2 W/\partial m\partial m_1)(\tilde{m}, m_1) = (\partial^2 W/\partial m\partial m_1)(m_1, m_1) = 0$. Therefore $(\partial^2 W/\partial m\partial m_1)(\cdot, m_1) > 0$ over $[0, m_1)$, and the result follows. Note for further reference that the above argument shows that $(\partial W/\partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$. ■

Proof of Proposition 1. We first establish uniqueness. As explained in the text, any solution V to (19)–(22) that is twice continuously differentiable over $(0, \infty)$ must coincide with some V_{m_1} over $[0, \infty)$. Since $V(0)$ must be non-negative by (20), one must have $m_1 \leq \hat{m}_1$. Suppose first that $\hat{m}_1 \leq \tilde{m}_1$, and that $m_1 < \hat{m}_1$. Then $V(0) = V_{m_1}(0) > 0$. But since $m_1 < \tilde{m}_1$, one has $V'_+(0) = V'_{m_1}(0) < p$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ is either attained at $-f$, for a value of 0, or at 0, for a value of $V(0) - pf$. In either case, this is inconsistent with condition (20). It follows that $m_1 = \hat{m}_1$, and thus $V = \hat{V}$ as given by (27). Suppose next that $\hat{m}_1 > \tilde{m}_1$. The above argument can be used to show that necessarily $m_1 > \tilde{m}_1$. Two cases must be distinguished. If $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] > 0$, then Lemma 3 establishes the uniqueness of a value $\bar{m}_1$ of $m_1 \in (\tilde{m}_1, \hat{m}_1)$ consistent with condition (20). It follows that $m_1 = \bar{m}_1$, and thus $V = \bar{V}$ as given by (30). Suppose finally that $V_{\hat{m}_1}(m_p(\hat{m}_1)) - p[m_p(\hat{m}_1) + f] \leq 0$. Defining φ as in the proof of Lemma 3, and using the fact that φ is strictly increasing over $(\tilde{m}_1, \hat{m}_1)$, we obtain that φ has no zeros over $(\tilde{m}_1, \hat{m}_1)$. Thus condition (20) cannot be satisfied for $m_0 = m_p(m_1)$ and $m_1 \in (\tilde{m}_1, \hat{m}_1)$. It follows that the maximum of the mapping $m \mapsto V(m) - p(m + f)$ over $[f, \infty)$ must be attained at $-f$, for a value of 0. The only choice of m_1 that is then consistent with (20) is $m_1 = \hat{m}_1$, and thus $V = \hat{V}$ as given by (27).

We now verify that our solution V to (19)–(22) satisfies the variational inequalities (12)–(14) over $(0, \infty)$. Inequality (12) follows from (22) and Lemma 1, while inequality (14) follows from (21)–(22) along with the fact that $\rho > r$. As for (13), two cases must be distinguished. Suppose first that $\hat{m}_1 \leq \tilde{m}_1$, and hence $V'_+(0) \leq p$. For any $m \geq 0$, the mapping $m' \mapsto V(m' - f) - p(m' - m)$ is then strictly decreasing over $[m, \infty)$, and thus (13) holds as $V(m) \geq V(m - f)$ for any such m. Suppose next that $\hat{m}_1 > \tilde{m}_1$, and hence $V'_+(0) > p$. If $m \geq m_p(m_1) + f$, the same reasoning as above applies and (13) holds. If $m_p(m_1) + f > m \geq 0$, the maximum of the mapping $m' \mapsto V(m' - f) - p(m' - m)$ over $[m, \infty)$ is attained at $m_p(m_1) + f$, and we must therefore check that:

$$V(m) - pm \geq V(m_p(m_1)) - p[m_p(m_1) + f] \tag{47}$$

for any such m. The mapping $m \mapsto V(m) - pm$ is strictly increasing over $[0, m_p(m_1)]$, and strictly decreasing over $[m_p(m_1), m_p(m_1) + f]$. Thus we need only to check that (47) holds at $m = 0$ and at $m = m_p(m_1) + f$. The latter point is immediate. For the former, two cases must be distinguished. If (26) holds, then $m_1 = \hat{m}_1$, and (47) holds at $m = 0$ since the right-hand side is non-positive while the left-hand side is equal to 0 as $\hat{V}(0) = 0$. If (28) holds, then $m_1 = \bar{m}_1$, and (47) holds as an equality at $m = 0$ as $\bar{V}(0) = \bar{V}(m_p(\bar{m}_1)) - p[m_p(\bar{m}_1) + f]$. The result follows. ■

Proof of Lemma 4. Fix an admissible policy $((\tau_n)_{n\geq 1}, (i_n)_{n\geq 1}, L)$, from which processes I, F and M and the bankruptcy date τ_B can be obtained as in (2)–(5). Let us decompose the process L as

$L_t = L_t^c + \Delta L_t$ for all $t \geq 0$, where L^c is the pure continuous part of L. The generalized Itô's formula (Dellacherie and Meyer (1982, Theorem VIII.27)) yields:

$$
\begin{aligned}
e^{-\rho T\wedge\tau_B} V(M_{T\wedge\tau_B}) \;=\; V(m) &+ \int_0^{T\wedge\tau_B-} e^{-\rho t}[-\rho V(M_t) + \mathcal{L}V(M_t)]\,dt \\
&+ \sigma \int_0^{T\wedge\tau_B-} e^{-\rho t} V'(M_t)\,dW_t - \int_0^{T\wedge\tau_B-} e^{-\rho t} V'(M_t)\,dL_t^c \\
&+ \sum_{t\in[0,T\wedge\tau_B]} e^{-\rho t}[V(M_t) - V(M_{t-})]
\end{aligned}
\tag{48}
$$

for all $T \geq 0$. Since V satisfies (12) by Proposition 1, it follows that for each $t \in [0, T\wedge\tau_B]$,

$$
\begin{aligned}
V(M_t) - V(M_{t-}) \;&=\; V\left(M_{t-} + \frac{\Delta I_t}{p} - \Delta F_t - \Delta L_t\right) - V(M_{t-}) \\
&\leq V\left(M_{t-} + \frac{\Delta I_t}{p} - \Delta F_t\right) - \Delta L_t - V(M_{t-}).
\end{aligned}
$$

Plugging into (48) and using again inequality (12) yields:

$$
\begin{aligned}
e^{-\rho T\wedge\tau_B} V(M_{T\wedge\tau_B}) \;\leq\; V(m) &+ \int_0^{T\wedge\tau_B-} e^{-\rho t}[-\rho V(M_t) + \mathcal{L}V(M_t)]\,dt \\
&+ \sigma \int_0^{T\wedge\tau_B-} e^{-\rho t} V'(M_t)\,dW_t - \int_0^{T\wedge\tau_B-} e^{-\rho t}\,dL_t \\
&+ \sum_{n\geq 1} e^{-\rho\tau_n} i_n 1_{\{\tau_n\leq T\wedge\tau_B\}} \\
&+ \sum_{n\geq 1} e^{-\rho\tau_n}\left[V\left(M_{\tau_n-} + \frac{i_n}{p} - f\right) - i_n - V(M_{\tau_n-})\right] 1_{\{\tau_n\leq T\wedge\tau_B\}}.
\end{aligned}
\tag{49}
$$

Since V' is bounded over $(0,\infty)$, the third term of the left hand side of (49) is a square integrable martingale. Using inequalities (13) and (14) along with the fact that V is non-negative by construction, we can take expectations in (49) to obtain:

$$
V(m) \geq \mathbb{E}^m\left[\int_0^{T\wedge\tau_B-} e^{-\rho t}(dL_t - dI_t)\right],
\tag{50}
$$

from which the result follows by letting T go to ∞. ∎

Proof of Proposition 2. Assume that (28) holds, so that $\tau_B = \infty$ $\mathbb{P}$–almost surely, and suppose without loss of generality that $m \in [0, m_1^*]$. The process M^* has paths that are continuous except at the dates $(\tau_n^*)_{n\geq 1}$ at which new equity is issued, in which case $V(M^*_{\tau_n^*}) - V(M^*_{\tau_n^*-}) = V(m_0^*) - V(0) = i^*$ by construction. Proceeding as in the proof of Proposition 1, we obtain that:

$$
\begin{aligned}
\mathbb{E}^m\left[e^{-\rho T} V(M_T)\right] \;&=\; V(m) - \mathbb{E}^m\left[\int_0^{T-} e^{-\rho t} V'(M_t^*)\,dL_t^*\right] + \mathbb{E}^m\left[\sum_{n\geq 1} e^{-\rho\tau_n^*} i^* 1_{\{\tau_n^*\leq T\}}\right] \\
&= V(m) - \mathbb{E}^m\left[\int_0^{T-} e^{-\rho t}\,(dL_t^* - dI_t^*)\right]
\end{aligned}
\tag{51}
$$

for all $T \geq 0$, where the process $I^* = \{I_t^*; t \geq 0\}$ is defined as in (2) with $i_n^* = i^*$ for all $n \geq 1$, and the second inequality follows from (33) along with the fact that $V'(m_1^*) = 1$. To conclude the proof, we need only to check that $\lim_{T \to \infty} \mathbb{E}^m[e^{-\rho T} V(M_T)] = 0$ in (51). Since V is non-negative with bounded derivatives, one has:

$$0 \leq e^{-\rho T} V(M_T) \leq e^{-\rho T} C(1 + M_T^*) \leq e^{-\rho T} C(1 + m_1^*)$$

for some positive constant C, where the second and third inequality follow from the fact that the process M^* never leaves the interval $[0, m_1^*]$. Taking expectations and letting T go to ∞ yields the result. The proof for the case in which (26) holds is similar, and therefore omitted. ■

Proof of Corollary 1. To establish this result, we show that V^* is a decreasing function of p and f, and that $V^{*\prime}$ is an increasing function of p and f. To prove the first claim, start without loss of generality from a situation in which p and f are such that condition (28) holds, and consider the impact of a decrease in p or f, $p' \leq p$ and $f' \leq f$ with at least one strict inequality. Then the firm can keep the same dividend policy L^*, while adjusting its issuance policy so as to maintain the same dynamics for cash reserves (31) as when the issuance costs are p and f. Indeed, to do so, it needs only to issue amounts $i' = p'(m_0^* + f')$ worth of equity instead of $i^* = p(m_0^* + f)$, at the same dates $(\tau_n^*)_{n \geq 1}$. That is, the new issuance and dividend policy of the firm is $((\tau_n^*)_{n \geq 1}, (i_n')_{n \geq 1}, L^*)$ with $i_n' = i' < i^*$ for all $n \geq 1$. Since the dividend policy and the dynamics of cash reserves are the same as in the initial situation, while the amounts of equity issued are strictly lower, this policy yields a strictly higher value for the firm than in the initial situation. Thus V^* is a decreasing function of p and f, as claimed. Now, using the notation of the proof of Lemma 3, one has $V^* = W(\cdot, m_1^*)$ over $\mathbb{R}_+$. Since $(\partial W / \partial m_1)(\cdot, m_1) < 0$ over $[0, m_1]$, the above argument implies that an increase in either p or f leads to an increase in m_1^*. Since $(\partial^2 W / \partial m \partial m_1)(\cdot, m_1) > 0$ over $[0, m_1)$, it follows that $V^{*\prime}$ is an increasing function of p and f. Hence the result. ■

Proof of Lemma 5. Proposition 2 along with (36) implies that for each $n \geq 1$,

$$S_{\tau_n^*}^* N_{\tau_n^*}^* - S_{\tau_n^*-}^* N_{\tau_n^*-}^* = V^*(M_{\tau_n^*}^*) - V^*(M_{\tau_n^*-}^*) = V^*(m_0^*) - V^*(0) = p(m_0^* + f) = i^*. \tag{52}$$

Next, it follows from (37) that the issuance proceeds at date τ_n^* are given by:

$$i^* = I_{\tau_n^*}^* - I_{\tau_n^*-}^* = S_{\tau_n^*}^* (N_{\tau_n^*}^* - N_{\tau_n^*-}^*). \tag{53}$$

It then follows immediately from (52)–(53) that $S_{\tau_n^*}^* = S_{\tau_n^*-}^*$, as claimed. ■

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