Finding Equilibrium Points II

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I. Pure Exchange: Walras

agent's problem: Agents: $i \in \mathcal{I} \quad |\mathcal{I}|$ *finite* "large"

$$\bar{x}_i \in \arg\max u_i(x_i) \text{ so that } \langle p, x_i \rangle \leq \langle p, e_i \rangle, \ \ x_i \in C_i$$

e_i: endowment of agent i, $e_i \in \text{ int } C_i$

u_i: utility of agent i, concave, usc

$$u_i : C_i \to \mathbb{R}, \quad C_i \subset \mathbb{R}^n \text{ (survival set) convex}$$

market clearing: $s(p) = \sum_{i \in \mathcal{I}} (e_i - \bar{x}_i)$ excess supply

equilibrium price: $\bar{p} \in \Delta$ such that $s(\bar{p}) \geq 0$

Δ unit simplex

The Walrasian

$$W(p,q)=\langle q,s(p)\rangle,\ W:\Delta\times\Delta\to\mathbb{R}$$

$\bar{p}$ equilibirum price (Ky Fan Inequality)

$$\Leftrightarrow \bar{p}\in \arg\max_p (\inf_q W(p,q))\ \&\ s(\bar{p})\geq 0$$

Properties of W :

continuous in p ($e_i\in \operatorname{int} C_i$, '$i$-inf-compact') usc

linear in q, Δ compact convex

$W(p,p)\geq 0, \forall p\in\Delta$

i.e., W is a Ky Fan function

II. Numerical Approaches

- Augmented Walrasian:

$\bar{p} \in$ argmax-inf W

$\cong$ saddle point $(\bar{p}, \bar{q})$ of $\tilde{W}_r$

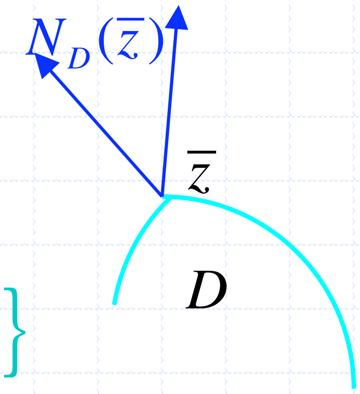
$$\tilde{W}_r(p,q) = \inf_z \left\{ W(p,z) \,\middle|\, \|z-q\| \le r \right\},$$

$\|\bullet\|$ an appropriate norm ($|\bullet|_\infty$ e.g.)

Variational Inequality

$$\max_x u_i(x_i) \text{ so that } \langle p, x_i \rangle \le \langle p, e_i \rangle,\ x_i \in C_i$$

$$\sum_i (e_i - c_i) = s(p) \ge 0.$$



$\Updownarrow$

$$N_D(\bar{z}) = \left\{ v \,\middle|\, \langle v, z - \bar{z} \rangle \le 0, \forall z \in D \right\}$$

$$G(p,(x_i),(\lambda_i)) = \left[\sum_i (e_i - x_i); \left(\lambda_i p - \nabla u_i(x_i) \right); \langle p, e_i - x_i \rangle \right]$$

$$D = \Delta \times \left(\prod_i C_i \right) \times \left(\prod_i \mathbb{R}_+ \right)$$

$$-G(\bar{p},(\bar{x}_i),(\bar{\lambda}_i)) \in N_D(\bar{p},(\bar{x}_i),(\bar{\lambda}_i))$$

D (unfortunately) is unbounded

III. Stochastic Equilibrium Model

Pure Exchange model

Agent-*i* problem-stochastic

$$\max_{x_i^0, y_i \in \mathbb{R}^n, x_{i,\bullet}^1 \in \mathcal{M}} u_i^0(x_i^0) + E_i\left\{u_i^1(\xi, x_{i,\xi}^1)\right\}$$

$$\text{so that } \left\langle p^0, x_i^0 + T_i^0 y_i \right\rangle \le \left\langle p^0, e_i^0 \right\rangle$$

$$\left\langle p_\xi^1, x_{i,\xi}^1 \right\rangle \le \left\langle p_\xi^1, e_{i,\xi}^1 + T_{i,\xi}^1 y_i \right\rangle, \forall \xi \in \Xi$$

$$x_i^0 \in C_i^0,\ \ x_{i,\xi}^1 \in C_{i,\xi}^1,\ \ \forall \xi \in \Xi$$

- $E_i\{.\}$ rational expectation w.r.t. *i*-beliefs

Stochastic program with recourse: 2-stage
Well-developed solution procedures
Well-developed “Approximation Theory”

Simplest-classical assumptions

Ξ finite (support)

$u_i^0 : C_i^0 \to \mathbb{R},\ \forall \xi \in \Xi,\ u_i^1(\xi,\bullet) : C_{i,\xi}^1 \to \mathbb{R}$ concave

continuous. Numerics: differentiable

$T_{\mathrm{i}}^0, T_{i,\xi}^1$: input-ouput matrices

(production, investment, etc.)

$C_{\mathrm{i}}^0, C_{i,\xi}^1$: closed, convex, non-empty interior

$e_{\mathrm{i}}^0 \in \operatorname{int} C_{\mathrm{i}}^0,\ e_{i,\xi}^1 \in \operatorname{int} C_{i,\xi}^1$ for all ξ

Market Clearing

Agents: $i \in \mathcal{I},\ |\mathcal{I}|$ *finite* "large"

$$\left(\overline{x}_i^0, \overline{y}_i, \left\{\overline{x}_{i,\xi}^1\right\}_{\xi\in\Xi}\right) \in \arg\max\{\text{agent-}i \text{ problem}\}$$

excess supply:

$$\sum_{i\in\mathcal{I}}\left(e_i^0 - (\overline{x}_i^0 + T_i^0\overline{y}_i)\right) = s^0(p^0, \{p_\xi^1\}_{\{\xi\in\Xi\}}) \geq 0$$

$\forall \xi \in \Xi$:

$$\sum_{i\in\mathcal{I}}\left(e_{i,\xi}^1 + T_{i,\xi}^1\overline{y}_i - \overline{x}_{i,\xi}^1\right) = s_\xi^1(p^0, \{p_\xi^1\}_{\{\xi\in\Xi\}}) \geq 0$$

Here-&-Now vs. Wait-&-See

- Basic Process: decision --> observation --> decision

$$(x_i^0, y_i) \;\rightarrow\; \xi \;\rightarrow\; x_{i,\xi}^1$$

- Here-&-now problem!

  not all contingencies available at time 0

  (x_i^0, y_i) can't depend on ξ!

- Wait-&-see problem

  implicitly all contingencies available at time 0

  choose $\left(x_{i,\xi}^0, y_{i,\xi}^0, x_{i,\xi}^1\right)$ after observing x

- incomplete ⇆ complete market ?

Fundamental Theorem of Stochastic Optimization

A here-and-now problem can be "reduced" to a wait-and-see problem by introducing the

appropriate 'contingency' costs (price of nonanticipativity)

Contingencies prices (nonanticipativity)

Here-&-now

$$\max E\left\{f(\xi, z^0, z^1_\xi)\right\}$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0),\ \forall \xi.$$

Explicit nonanti. constraints

$$\max E\left\{f(\xi, z^0_\xi, z^1_\xi)\right\}$$

$$z^0_\xi \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0),\ \forall \xi.$$

$$z^0_\xi = E\{z^0_\xi\}\ \forall \xi$$

$$w_\xi \perp \text{c}^{\text{ste}}\ \text{fcns}$$

$$\Rightarrow E\{w_\xi\} = 0$$

Progressive Hedging

- **Step 0.** $w^0(\bullet)$ so that $E\left\{w^0(\xi)\right\} = 0,\ \nu = 0$
- **Step 1. for all ξ:**

$$\left(z_\xi^{0,\nu}, z_\xi^{1,\nu}\right) \in \arg\max f(\xi;\, z^0, z^1) - \left\langle w_\xi^\nu, z^0 \right\rangle$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_0},\ z^1 \in C^1(\xi, x^0) \subset \mathbb{R}^{n_1}$$

- **Step 2.** $w_\xi^{\nu+1} = w_\xi^\nu + \rho\left[z_\xi^{0,\nu} - E\left\{z_\xi^{0,\nu}\right\}\right],\quad \rho > 0$
  - and return to Step 1, $\nu = \nu + 1$
- **Convergence:** add proximal term

$-\frac{\rho}{2}\left\| z_\xi^{0,\nu} - E\left\{z_\xi^{0,\nu}\right\}\right\|^2$, linear rate in $\left(z^\nu, w^\nu\right)$

Disintegration: agent's problem

with $p_{\bullet} = \left(p^0, \left\{ p^1_\xi \right\}_{\xi \in \Xi} \right)$

$$\left(\bar{x}^0_{i,\xi},\ \bar{y}^0_{i,\xi},\ \bar{x}^1_{i,\xi} \right) \in$$

'*i*-contingency' costs

$$\underset{x_i^0, y_i, x_i^1}{\arg\max} \left\{ u_i^0(x_i^0) - \left\langle \bar{w}_{i,\xi}, (x_i^0, y_i) \right\rangle + u_i^1(\xi, x_i^1) \right\}$$

$$\left\langle p^0, x_i^0 \right\rangle \le \left\langle p^0, e_i^0 - T_i^0 y_i \right\rangle$$

$$\left\langle p^1_\xi, x_i^1 \right\rangle \le \left\langle p^1_\xi, e^1_{i,\xi} + T^1_{i,\xi} y_i \right\rangle,$$

$$x_i^0 \in C_i^0,\ \ x_i^1 \in C^1_{i,\xi}.$$

solved for each ξ **<u>separately</u>**

Incomplete to 'Complete' Market

$\forall \xi \in \Xi$ (separately),

agent's problem:

$$\left(\bar{x}_i^0, \bar{y}_i^0, \bar{x}_{i,\xi}^1\right) \in \arg\max \left\{ u_i^{w_{i,\xi}}\left(x_i^0, y_i^0, x_i^1\right) \text{on } \widehat{C}_{i,\xi}(p^0, p_\xi^1) \right\}$$

$$\text{for } \{w_{i,\xi}\}_{\xi \in \Xi} \text{ associated with } (p^0, p_\xi^1)$$

clear market:

$$s^0(p^0, p_\xi^1) \geq 0, \quad s_\xi^1(p^0, p_\xi^1) \geq 0$$

Arrow-Debreu 'stochastic' equilibrium problem

THE WALRASIAN

$$W(p_{\bullet}, q_{\bullet}) = \langle q_{\bullet}, s(p_{\bullet}) \rangle$$

$$= \left\langle (q^0, \{q_\xi^1\}_{\xi\in\Xi}), \left(s^0(p^0, \{p_\xi^1\}_{\xi\in\Xi}), \left\{ s_\xi^1(p^0, \{p_\xi^1\}_{\xi\in\Xi}) \right\}_{\xi\in\Xi} \right) \right\rangle$$

$$W : \prod_{1+|\Xi|} \Delta \times \prod_{1+|\Xi|} \Delta \to \mathbb{R}$$

linear w.r.t. $q_{\bullet}$, continuous w.r.t. $p_{\bullet}$

$$W(p_{\bullet}, p_{\bullet}) \geq 0.$$

provided $s(\bullet)$ continuous w.r.t. $p_{\bullet}$

↑ another lecture, § V

Ky Fan functions & inequality

- $K : B \times B \to \mathbb{R}$ is a Ky Fan function if

  (a) $\forall y : x \mapsto K(x, y)$ usc

  (b) $\forall x : y \mapsto K(x, y)$ convex

- Theorem. K Ky Fan fcn, dom $K = B \times B$, B compact

  $$\Rightarrow \text{ argmax-inf } K \neq \emptyset$$

  if $K(x, x) \geq 0$ on dom K, $\bar{x} \in$ argmax-inf K

  $$\Rightarrow \inf_y K(\bar{x}, y) \geq 0.$$

- The Walrasian is a Ky Fan function

  yields existence of equilibrium price.

IV. Experimentation

with PATH Solver (experimental)

- Economy: (5 goods)
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks
- 2-stages,   solved under # of scenarios
- utilities: CSE-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - only used for transfer in stage 1
- so far: mostly calibration

  numerically: `blink' (5000 iterations).

with PATH Solver (stochastic)

- objectives: $u_i^0(x_i^0) + u_i^1(x_i^1) \quad \Rightarrow$

$$u_i^0(x_i^0) - \left\langle w_{i,\xi}^{\nu}, (x_i^0, y_i) \right\rangle - \frac{\rho_i}{2} \left| (x_i^0, y_i) - (\hat{x}_i^{0,\nu}, \hat{y}_i^{\nu}) \right|^2 + u_i^1(x_i^1)$$

- updating: $(\hat{x}_i^{0,\nu}, \hat{y}_i^{\nu}) = E_i \left\{ (x_{i,\xi}^{0,\nu}, y_{i,\xi}^{\nu}) \right\}$

$$w_{i,\xi}^{\nu+1} = w_{i,\xi}^{\nu} + \rho_i((x_{i,\xi}^{0,\nu}, y_{i,\xi}^{\nu}) - (\hat{x}_i^{0,\nu}, \hat{y}_i^{\nu}))$$

$\rho_i > 0$ yields convergence

V. Continuity, Stability Issues

equilibrium points

solutions of V.I., …

Variational Convergence

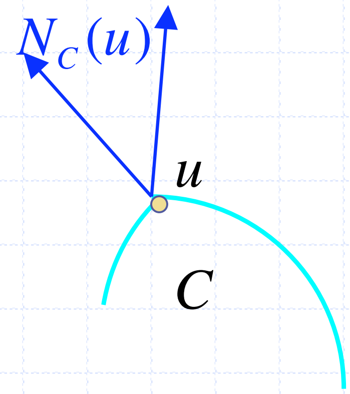
- solutions of optimization problems
  - $\arg\min f^{\nu} \to \arg\min f$: epi-convergence
  - $\arg\max f^{\nu} \to \arg\max f$: hypo-convergence
- stability of saddle points
  - saddle pts $K^{\nu} \to$ saddle pts K : epi/hypo-convergence
- stability of maxinf points
  - $\max\inf K^{\nu} \to \max\inf K$: lopsided convergence (tightly)

Walras Equilibrium points

- $\forall i \in \mathcal{I}$: $\bar{x}_i(p) \in \arg\max_{C_i} \left\{ u_i(x_i) \,\middle|\, \langle p, x_i \rangle \le \langle p, e_i \rangle \right\}$
- $s(p) = \sum_i (e_i - \bar{x}_i(p))$ excess supply
- find $\bar{p} \in \Delta$ (unit simplex) so that $s(\bar{p}) \ge 0$
- Walrasian: $W(p,q) = \langle q, s(p) \rangle$ Ky Fan fcn
- $\bar{p} \in \arg\max\text{–}\inf W \Leftrightarrow s(\bar{p}) \ge 0$
- conditions: $e_i \in \operatorname{int} C_i$, "globally compact"
- Convergence: $u_i^v \to_{hypo} u_i$, $e_i^v \to e_i \Rightarrow$

  W^v converge lopsided tightly to W

Variational Inequalities



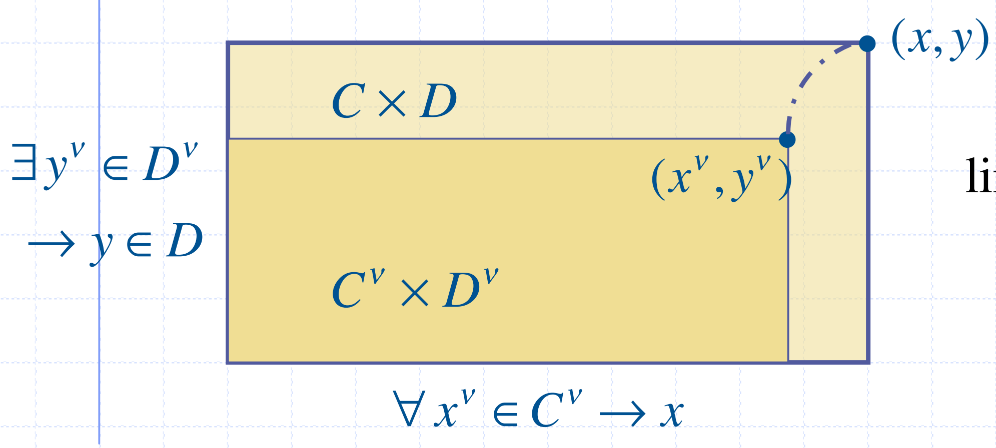
- $C \subset \mathbb{R}^n$ non-empty, convex
- $G : C \to \mathbb{R}^n$ continuous
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

  where $v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \leq 0, \forall u \in C$
- with $K(u,v) = \langle G(u), v - u \rangle$ on dom $K = C \times C$

  ⇨ *K* is a Ky Fan function, K(u,u) ≥ 0.
- Find

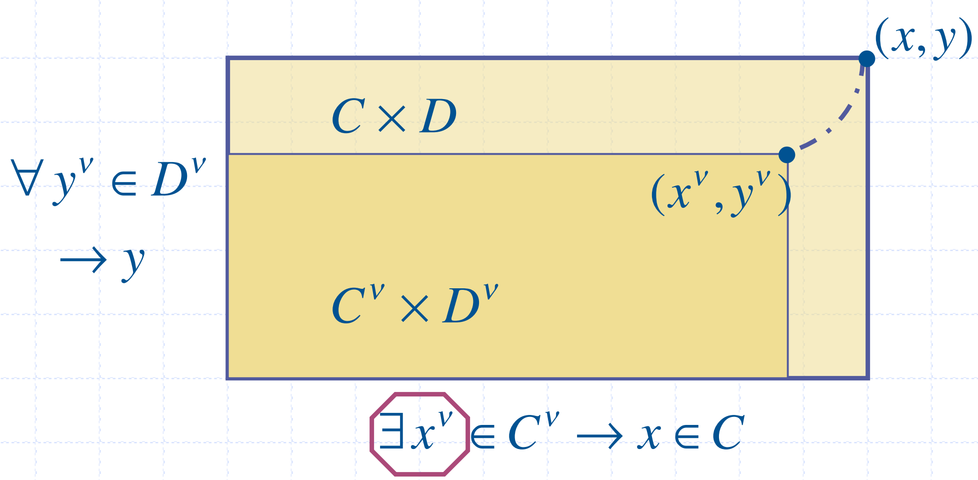
$$\bar{u} \in \arg\max\text{–}\inf K(\bullet,\bullet) \text{ so that } K(\bar{u},\bullet) \geq 0$$

Lopsided convergence: definition



$$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x, y) \text{ when } x \in C$$

$$K^\nu(x^\nu, y^\nu) \to -\infty \text{ when } x \notin C$$



$$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x, y) \text{ when } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \text{ when } x \notin D$$

Lopsided tightly

- $K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop-tightly}{\rightarrow} K_{C\times D}$ if $K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop}{\rightarrow} K_{C\times D}$ &

  (b) $\forall x \in C, \exists x^{\nu} \to x, \forall y^{\nu} \in D^{\nu}$ and $y^{\nu} \to y$:

  $$\liminf K^{\nu}(x^{\nu}, y^{\nu}) \geq K(x,y) \quad \text{if } y \in D$$

  $$K^{\nu}(x^{\nu}, y^{\nu}) \to \infty \quad \text{if } y \notin D$$

  but also $\forall \varepsilon > 0$, $\exists B_{\varepsilon}$ compact (depends on $x^{\nu} \to x$) :

  $$\inf_{B_{\varepsilon}\cap D^{\nu}} K^{\nu}(x^{\nu}, \bullet) \leq \inf_{D^{\nu}} K^{\nu}(x^{\nu}, \bullet) + \varepsilon, \ \forall \nu \geq \nu_{\varepsilon}$$

- THM. $K^{\nu}_{C^{\nu}\times D^{\nu}} \rightarrow K_{C\times D}$ lopsided tightly, $\bar{x}$ cluster point of $\{x^{\nu} \in \arg\max\text{–}\inf K^{\nu}_{C^{\nu}\times D^{\nu}}\}_{\nu\in\mathbb{N}} \Rightarrow \bar{x} \in \arg\max\text{–}\inf K_{C\times D}$

Proof ....

- $$K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop-tightly}{\longrightarrow} K_{C\times D}$$

  Let $g^{\nu} = \inf_{y\in D^{\nu}} K^{\nu}(\bullet, y), \quad g = \inf_{y\in D} K(\bullet, y).$

  $$\Rightarrow g^{\nu} \underset{hypo}{\longrightarrow} g \quad \text{when} \quad \left\{ \begin{array}{l} C^{\nu}_{g} = \left\{ x \in C^{\nu} \middle| g^{\nu}(x) > -\infty \right\} \\ C_{g} = \left\{ x \in C \middle| g(x) > -\infty \right\} \end{array} \right. \neq \emptyset$$

- then apply

  $$g^{\nu}_{C^{\nu}} \underset{hypo}{\longrightarrow} g_{C},\, x^{\nu} \in \arg\max_{C^{\nu}} g^{\nu},\, x^{\nu_k} \longrightarrow \bar{x} \in C \Rightarrow \bar{x} \in \arg\max_{C} g$$

Ky Fan functions & inequality

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  $$\Rightarrow \ \inf_y K(\bar{x}, y) \geq 0.$$

- The Walrasian is a Ky Fan function

  yields existence of equilibrium price.

Extending Ky Fan's inequality

$$K^\nu \to K \text{ lopsided}$$

$$K^\nu \text{ Ky Fan } \Rightarrow K \text{ Ky Fan}$$

$$\& \quad \text{when } \arg\max\text{–}\inf K^\nu \neq \emptyset$$

$$\text{if } \bar{x} \in \text{ cluster-pts } \{\arg\max\text{–}\inf K^\nu\}$$

$$\Rightarrow \bar{x} \in \arg\max\text{–}\inf K \;\; \& \;\; K(\bar{x}, \bullet) \geq 0$$

- Ky Fan fcns closed under tight-lopsided

  saddle fcns closed under hypo/epi-convergence

  usc fcns closed under hypo-convergence