LECTURE 3: LIQUIDITY MANAGEMENT

. Previous models have assumed that shareholders were not cash constrained
$\Rightarrow$ no need for liquidity

. Polar assumption: no access to external finance [Jeanblanc-Picqué – Shiryaev 1995, also Radner – Shepp 1995, Shreve – Lehoczky – Gaver, SIAM J. Control and Optimization, 1984, 22(1)]

1°) The model

Assets A_t Reserves M_t	Equity E_t

net operating income dx_t	Profits / Losses $dL_t + dM_t$

Two simplifications: (no debt) (no interest on reserves)

dL_t: dividends

Assumptions: . $\tau = \inf\{t \,/\, M_t < 0\} \rightarrow$ liquidation (0 value)
[Here default is involuntary]

. $dx_t = \mu dt + \sigma dW_t$

[NB: $\mu > 0 \Rightarrow$ solvent but liquidity risk: unpredictable cashflow; contrast with Leland, Merton $dx_t = \beta A_t dt$ ↳ solvency risk]

. $dL_t \geq 0$ (shareholders cannot inject cash)

2) Optimal dividend policy

$$E(m_0) = \underset{dL_t \geq 0}{\mathrm{Max}}\, \mathcal{E}_0\left[\int_0^\tau e^{-rt} dL_t\right]$$

a) simpler problem A: L_0 arbitrary (initial dividend)

then $dL_t = u(m_t)dt$ $\qquad 0 \leq u(m) \leq K$ ↳ given

Hamilton Jacobi equation

$$rE(m) = \max_{0\le u\le K}\left\{ u + (\mu-u)E'(m) + \frac{\sigma^2}{2}E''(m)\right\}$$

corresponds to the control problem

$$\begin{cases} E(m) = \max\limits_{0\le u_t\le K} \mathcal{E}\left[\int_0^{\tau} e^{-rt}u_t\,dt\right] \\ dm_t = (\mu-u_t)dt + \sigma dW_t \end{cases}$$

<u>solution</u>: for K large enough

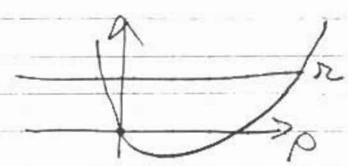
$$\begin{cases} u^*(m) = 0 & m \le m^*(K) \\ = K & m > m^*(K) \end{cases}$$

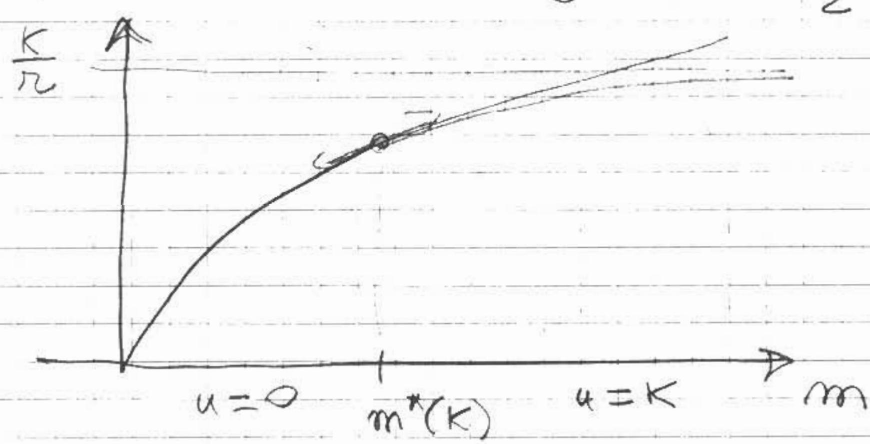
$$E(m) = \frac{e^{\rho_1 m} - e^{\rho_2 m}}{\rho_1 e^{\rho_1 m^*(K)} - \rho_2 e^{\rho_2 m^*(K)}} \qquad m \le m^*(K)$$

$$E(m) = \frac{K}{r} + \frac{e^{\rho_3 m}}{\rho_3 e^{\rho_3 m^*(K)}} \qquad m > m^*(K)$$

$\rho_1 < 0 < \rho_2 \rightarrow \quad r = \mu\rho + \frac{\sigma^2}{2}\rho^2$

$\rho_3 < 0 \longrightarrow \quad r = (\mu-K)\rho + \frac{\sigma^2}{2}\rho^2$





If initial dividend unconstrained

$$SV(m) = \max_{L_0\ge 0}\left[L_0 + E(m-L_0)\right]$$

$L_0 > 0 \Rightarrow E'(m-L_0) = 1 \Rightarrow L_0 = m - m^*(K)$

$$\begin{cases} SV(m) = E(m) & m \le m^*(K) \\ = m - m^*(K) + E(m^*(K)) & m > m^*(K) \end{cases}$$

$m^*(K)$ is obtained by using continuity of E:

$$\frac{e^{\rho_1 m^*(K)} - e^{\rho_2 m^*(K)}}{\rho_1 e^{\rho_1 m^*(K)} - \rho_2 e^{\rho_2 m^*(K)}} = \frac{K}{r} + \frac{e^{\rho_3 m^*(K)}}{\rho_3 e^{\rho_3 m^*(K)}} \quad (1)$$

ρ_3 = negative solution of $\frac{\sigma^2}{2}\rho^2 + (\mu - K)\rho - r = 0$

$$\rho_3(K) = \frac{K - \mu - \sqrt{(\mu - K)^2 + 2r\sigma^2}}{\sigma^2}$$

right hand side of (1):

$$\frac{K}{r} + \frac{1}{\rho_3(K)} = \frac{K}{r} + \frac{\sigma^2}{K - \mu - \sqrt{(K-\mu)^2 + 2r\sigma^2}}$$

$$= \frac{K}{r} + \frac{K - \mu + \sqrt{(K-\mu)^2 + 2r}}{-2r}$$

$$= \frac{1}{2r}\left[\mu + K - \sqrt{(K-\mu)^2 + 2r}\right]$$

$$= \frac{\mu}{r} + \frac{1}{2r}\left[K - \mu - \sqrt{(K-\mu)^2 + 2r}\right]$$

$$= \frac{\mu}{r} - \frac{1}{K - \mu + \sqrt{(K-\mu)^2 + 2r}}$$

When $K \to \infty$ $\quad m^*(K) \to m^*$ defined by

$$E(m^*) = \frac{e^{\rho_1 m^*} - e^{\rho_2 m^*}}{\rho_1 e^{\rho_1 m^*} - \rho_2 e^{\rho_2 m^*}} = \frac{\mu}{r}$$

b) <u>Back to original problem</u>

$K \to \infty$ $\quad E$ characterized by

$$\max\left[\mathcal{L}E(m),\ 1 - E'(m)\right] = 0$$

$$\left(\text{or } rE(m) = \max_{u \geq 0}\left\{u + (\mu - u)E'(m) + \frac{\sigma^2}{2}E''(m)\right\}\right.$$

with $\mathcal{L}E \equiv \mu E' + \frac{\sigma^2}{2}E'' - rE$

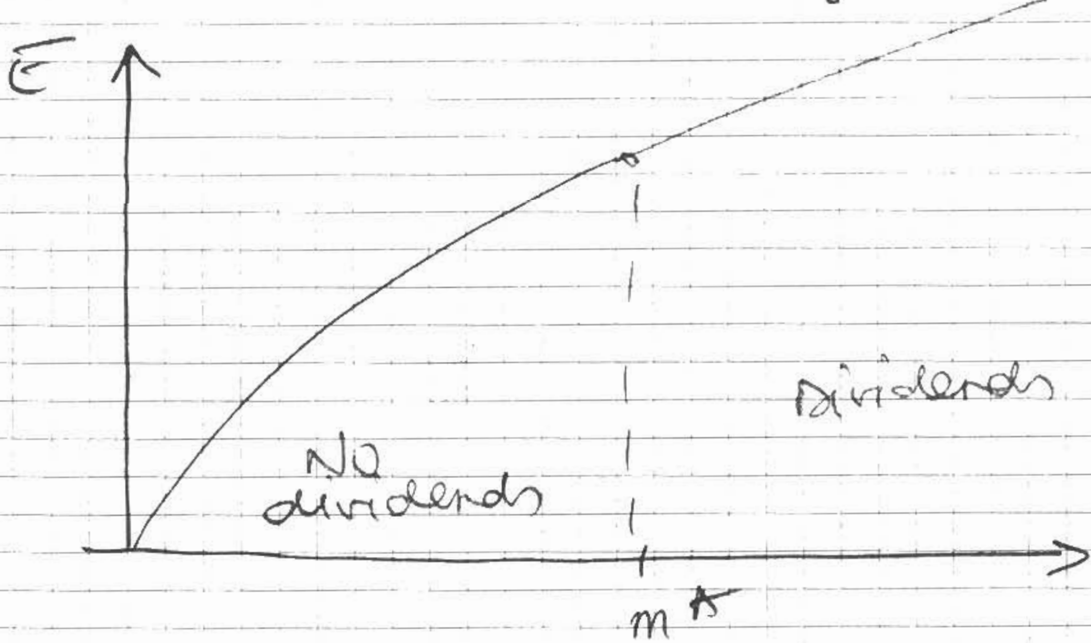
$$\begin{cases} 0 \leq m \leq m^* & E(m) = \dfrac{e^{\rho_1 m} - e^{\rho_2 m}}{\rho_1 e^{\rho_1 m^*} - \rho_2 e^{\rho_2 m^*}} \\ m > m^* & E(m) = m - m^* + E(m^*) \end{cases}$$

cash reserve dynamics

$$\begin{cases} dm_t = \mu\, dt + \sigma\, dW_t - dL_t \\ m_t \leq m^* \quad \text{(reflected diffusion)} \end{cases}$$

where dL_t = dividend process = local time at m^*

NB: initial dividend $m_0 - m^*$



$E(m^*) = \frac{\mu}{r}$, $E'(m^*) = 1$, $rE = \mu E' + \frac{\sigma^2}{2}E''$

$\Rightarrow E''(m^*) = 0$ [Super contact]

Another way to obtain this condition is to optimize among the family $E(m^*, \cdot)$ of functions defined by the dividend threshold m^*:

$$E(m^*, m) = k(m^*)\left[e^{\rho_2 m} - e^{\rho_1 m}\right]$$

$$k(m^*) = \left[\rho_2 e^{\rho_2 m^*} - \rho_1 e^{\rho_1 m^*}\right]^{-1}$$

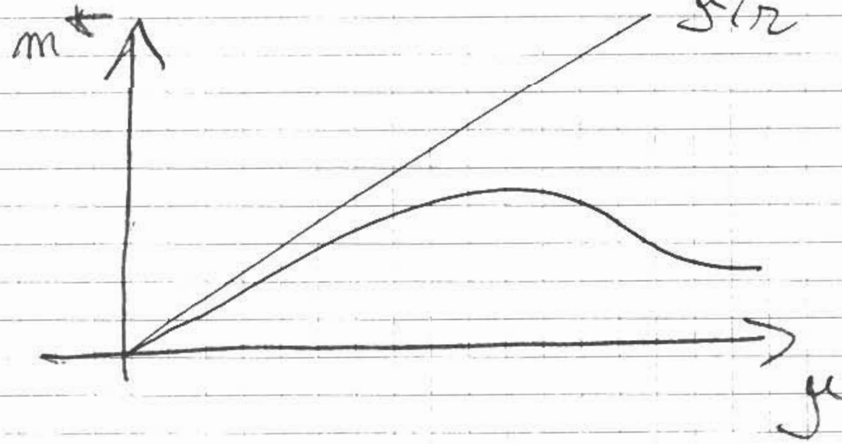
$$\min_{m^*} k(m^*) \Rightarrow \rho_2^2 e^{\rho_2 m^*} - \rho_1^2 e^{\rho_1 m^*} = 0$$

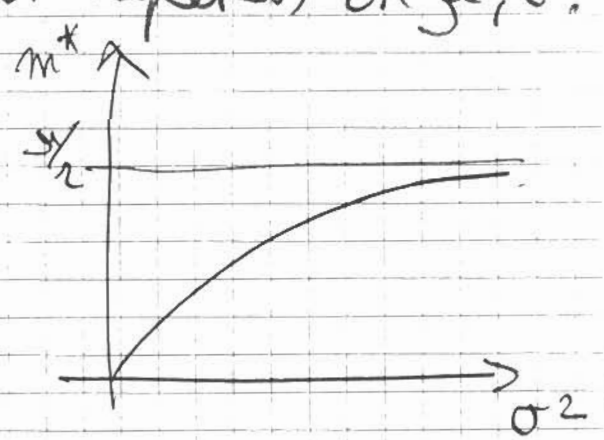
$$\Leftrightarrow \frac{\partial^2 E}{\partial m^2}(m^*, m^*) = 0$$

Notice that m^* is explicit:

$$m^* = \frac{1}{\rho_2 - \rho_1} \ln \frac{\rho_1^2}{\rho_2^2}$$

One can look at how it depends on μ, σ^2:





Interpretation