Algebraic Cobordism

A. Algebraic cobordism of schemes

B. Cobordism motives

Motives and Periods
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Outline: Part A

- Oriented Borel-Moore homology
- Universality and Riemann-Roch
- Fundamental classes

Oriented Borel-Moore homology

Regular embeddings and l.c.i. morphisms Recall:

A *regular embedding of codimension d* is a closed immersion $i : Z \to X$ such that $\mathcal{I}_Z$ is locally generated by a regular sequence of length d.

Example A regular embedding of codimension 1 is a Cartier divisor

Definition A morphism $f : Y \to X$ in $\mathbf{Sch}_k$ is an *l.c.i. morphism* if f can be factored as $p \circ i$, with $i : Y \to P$ a regular embedding and $p : P \to X$ smooth and quasi-projective.

$X \in \mathbf{Sch}_k$ is an *l.c.i. scheme* if $X \to \operatorname{Spec} k$ is an l.c.i. morphism.

$\mathbf{Lci}_k \subset \mathbf{Sch}_k$ is the full subcategory of l.c.i. schemes. Let $\mathbf{Sch}'_k :=$ the subcategory of projective morphisms in $\mathbf{Sch}_k$.

Oriented homology

An *oriented Borel-Moore homology theory* A_* on $\mathbf{Sch}_k$ consists of the following data:

(D1) An additive functor $A_* : \mathbf{Sch}'_k \to \mathbf{Ab}_*$, $X \mapsto A_*(X)$.

(D2) For $f : Y \to X$ an l.c.i. morphism in $\mathbf{Sch}_k$, a homomorphism of graded groups $f^* : A_*(X) \to A_{*-d}(Y)$,
$d :=$ the codimension of f.

(D3) For each pair (X, Y) in $\mathbf{Sch}_k$, a (commutative, associative) bilinear graded pairing $A_*(X) \otimes A_*(Y) \to A_*(X \times_k Y)$

$$u \otimes v \mapsto u \times v,$$

and a unit element $1 \in A_0(\mathrm{Spec}(k))$.

These satisfy six conditions:

(BM1) $\mathrm{id}_X^* = \mathrm{id}_{A_*(X)}$. For composable l.c.i. morphism f and g, $(f \circ g)^* = g^* \circ f^*$.

(BM2) Given a Tor-independent cartesian square in $\mathbf{Sch}_k$:
$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array},$$
with f projective, g l.c.i. . Then $g^* f_* = f'_* g'^*$.

(BM3) For f and g morphisms in $\mathbf{Sch}_k$: If f and g are projective, then $(f \times g)_*(u \times v) = f_*(u) \times g_*(v)$.
If f and g are l.c.i. , then $(f \times g)^*(u \times v) = f^*(u) \times g^*(u')$.

(PB) For a line bundle L on $Y \in \mathbf{Sch}_k$ with zero section $s : Y \to L$ define $\tilde{c}_1(L) : A_*(Y) \to A_{*-1}(Y)$ by $\tilde{c}_1(L)(\eta) = s^*(s_*(\eta))$.

Let $E \to X$ be a rank $n+1$ vector bundle, with associated projective space bundle $q : \mathbb{P}(E) \to X$. Then

$$\oplus_{i=0}^{n} A_{*+i-n}(X) \xrightarrow{\Sigma_{i=0}^{n-1} \tilde{c}_1(O(1)_E)^i \circ q^*} A_*(\mathbb{P}(E))$$

is an isomorphism.

(EH) Let $p : V \to X$ be an affine space bundle. Then

$$p^* : A_*(X) \to A_{*+r}(V)$$

is an isomorphism.

(CD) ***.

Examples (1) The Chow group functor

$$X \mapsto \mathrm{CH}_*(X)$$

with projective push-forward and l.c.i. pull-back given by Fulton.

(2) The Grothendieck group of coherent sheaves

$$X \mapsto G_0(X)[\beta, \beta^{-1}].$$

($\deg \beta = 1$). L.c.i. pull-back exists because an l.c.i. morphism has finite Tor-dimension.

(3) Algebraic cobordism ($\mathrm{char} k = 0$) $X \mapsto \Omega_*(X)$.
L.c.i. pull-backs are similar to Fulton's, but require a bit more work.

Note. There are "refined intersections" for Ω_*, similar to Fulton's refined intersection theory for CH_*.

Homology and cohomology

Every morphism in $\mathbf{Sm}/k$ is l.c.i., hence:

Proposition *Let A_* be an O.B.M.H.T. on $\mathbf{Sch}_k$. Then the restriction of A to $\mathbf{Sm}/k$, with*

$$A^n(X) := A_{\dim X - n}(X),$$

defines an O.C.T. A^* on $\mathbf{Sm}/k$:

- *The product $\cup$ on $A^*(X)$ is $x \cup y = \delta_X^*(x \times y)$.*
- $1_X = p_X^*(1)$ *for $p_X : X \to \operatorname{Spec} k$ in $\mathbf{Sm}/k$.*
- $c_1(L) = \tilde{c}_1(L)(1_X)$ *for $L \to X$ a line bundle.*

Consequence:

Let A_* be an O.B.M.H.T. on $\mathbf{Sch}_k$. There is a unique formal group law $F_A \in A_*(k)[[u, v]]$ with

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(f_*(1_Y)) = \tilde{c}_1(L \otimes M)(f_*(1_Y))$$

for all $X \in \mathbf{Sch}_k$, all $(f : Y \to X) \in \mathcal{M}(X)$.

Examples

(1) CH_* has the additive formal group law: $F_{\mathrm{CH}}(u, v) = u + v$

(2) $G_0[\beta, \beta^{-1}]$ has the multiplicative formal group law: $F_{G_0}(u, v) = u + v - \beta uv$.

(3) Ω_* has the universal formal group law: $(F_\Omega, \Omega_*(k)) = (F_{\mathbb{L}}, \mathbb{L}_*)$

Universality and Riemann-Roch

Universality

Theorem *Algebraic cobordism* Ω_* *is the universal O.B.M.H.T. on* $\mathbf{Sch}_k$*.*

Also:

Theorem *The canonical morphism* $\vartheta_{\mathrm{CH}} : \Omega_* \otimes_{\mathbb{L}} \mathbb{Z} \to \mathrm{CH}_*$ *is an isomorphism, so* CH_* *is the universal additive theory on* $\mathbf{Sch}_k$*.*

A new result (due to S. Dai) is

Theorem *The canonical morphism*

$$\vartheta_{G_0} : \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to G_0[\beta, \beta^{-1}]$$

is an isomorphism, so $G_0[\beta, \beta^{-1}]$ *is the universal multiplicative theory on* $\mathbf{Sch}_k$*.*

Twisting

The τ-twisting construction is modified: one leaves f_* alone and twists f^* by $\widetilde{\mathsf{Td}_\tau^{-1}}(N_f)$:

$$f^*_{(\tau)} = \widetilde{\mathsf{Td}_\tau^{-1}}(N_f) \circ f^*.$$

Here $f : Y \to X$ is an l.c.i. morphism and $N_f \in K_0(Y)$ is the virtual normal bundle: If we factor f as $p \circ i$, p smooth, i a regular embedding, then:

p has a relative tangent bundle T_p
i has a normal bundle N_i and

$$N_f := [N_i] - [i^* T_p].$$

$\widetilde{\mathsf{Td}_\tau^{-1}}(N_f)$ is the inverse Todd class operator, defined as we did Td_τ^{-1}, using the operators $\tilde{c}_1$ instead of the classes c_1.

Riemann-Roch for singular varieties

Twisting $\mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ to give it the multiplicative group law and using Dai's theorem, we recover the Fulton-MacPherson Riemann-Roch transformation $\tau : G_0 \to \mathrm{CH}_{*,\mathbb{Q}}$:

Using the universal property of $G_0[\beta, \beta^{-1}]$ gives

$$\tau_\beta : G_0[\beta, \beta^{-1}] \to \mathrm{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]^{(\times)};$$

τ is the restriction of τ_β to degree 0.

Fundamental classes

Fundamental classes for l.c.i. schemes

Definition Let $p_X : X \to \operatorname{Spec} k$ be an l.c.i. scheme. For an O.B.M.H.T. A on $\mathbf{Sch}_k$, set

$$1_X^A := p_X^*(1)$$

If X has pure dimension d over k, then 1_X^A is in $A_d(X)$. 1_X is the *fundamental class* of X.

Properties:

- For $X = X_1 \amalg X_2 \in \mathbf{Lci}_k$,
  $1_X = i_{1*}(1_{X_1}) + i_{2*}(1_{X_2})$.

- For $f : Y \to X$ an l.c.i. morphism in $\mathbf{Lci}_k$, $f^*(1_X) = 1_Y$.

Fundamental classes of non-l.c.i. schemes

Some theories have more extensive pull-back morphisms, and thus admit fundamental classes for more schemes.

Example Both CH_* and $G_0[\beta, \beta^{-1}]$ admit pull-back for arbitrary *flat* maps, still satisfying all the axioms. Thus, functorial fundamental classes in CH_* and $G_0[\beta, \beta^{-1}]$ exist for *all* $X \in \mathbf{Sch}_k$.

This is NOT the case for all theories.

We give an example for Ω_*.

Let $S_1 \subset \mathbb{P}^5$ be $\mathbb{P}^2$ embedded by $\mathcal{O}(2)$.

Let $S_2 \subset \mathbb{P}^5$ be $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(2, 1)$.

Let $R_1 \subset S_1$, $R_2 \subset S_2$ be smooth hyperplane sections. Note:

1. $\deg c_1(\mathcal{O}(2))^2 = \deg c_1(\mathcal{O}(2, 1))^2 = 4$

2. R_1 and R_2 are both $\mathbb{P}^1$'s

So: R_1 and R_2 are both rational normal curves of degree 4 in $\mathbb{P}^4$.

We may assume $R_1 = R_2 = R$.

Let $C(S_1)$, $C(S_2)$ and $C(R)$ be the projective cones in $\mathbb{P}^6$.

Proposition *Let A be a O.B.M.H.T. on $\mathbf{Sch}_k$. If we can extend fundamental classes in A for $\mathbf{Sm}/k$ to $C(S_1)$, $C(S_2)$ and $C(R)$, functorial for l.c.i. morphisms, then*

$$[\mathbb{P}^2] = [\mathbb{P}^1 \times \mathbb{P}^1] \text{ in } A_2(k)$$

This is NOT the case for $A = \Omega$, since

$$c_2(\mathbb{P}^2) = 3, \; c_2(\mathbb{P}^1 \times \mathbb{P}^1) = 4.$$

Consequence for Gromov-Witten theory

The formalism of Gromov-Witten theory can be extended to a cobordism valued version, at least if the relevant moduli stack is an **Lci** stack.

One needs a theory of algebraic cobordism for (Deligne-Mumford) stacks: one can make a cheap version with $\mathbb{Q}$-coefficients by the universal twisting of $\mathrm{CH}_{*\mathbb{Q}}$.

BUT: there are problems if the intrinsic normal cone of the moduli stack is not **Lci**.

Part B: Cobordism motives

Outline: Part B

- Motives over an O.C.T.
- Cobordism motives
- Motivic computations
- Algebraic cobordism of Pfister quadrics

Motives over an O.C.T.

We follow the discussion of Nenashev-Zainoulline.

A-correspondences

Definition A^* an O.C.T. on $\mathbf{Sm}/k$. X, Y smooth projective k-varieties. Set

$$\mathrm{Cor}^0_A(X,Y) := A^{\dim Y}(X \times Y).$$

Cor^0_A is the category with
objects: smooth projective k-varieties $\mathbf{SmProj}/k$,
morphisms:

$$\mathrm{Hom}_{\mathrm{Cor}^0_A}(X,Y) := \mathrm{Cor}^0_A(X,Y)$$

and composition law:

$$\gamma_{Y,Z} \circ \gamma_{X,Y} := p_{X,Z*}(p^*_{X,Y}(\gamma_{X,Y}) \cdot p^*_{Y,Z}(\gamma_{Y,Z}))$$

- Cor^0_A is a tensor category: $X \oplus Y = X \amalg Y$ and $X \otimes Y := X \times Y$.

- Sending $f : X \to Y$ to the "graph"

$$\Gamma_f := (\mathrm{id}_X, f)_*(1_X) \in A^{\dim Y}(X \times Y)$$

gives the functor

$$m_A : \mathbf{SmProj}/k \to \mathrm{Cor}^0_A.$$

Definition $\mathcal{M}^{\mathrm{eff}}_A$ is the pseudo-abelian hull of Cor^0_A:

Obects are pairs (X, α), $\alpha \in \mathrm{End}_{\mathrm{Cor}^0_A}(X)$, $\alpha^2 = \alpha$.

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{eff}}_A}((X, \alpha), (Y, \beta)) := \beta \mathrm{Hom}_{\mathrm{Cor}^0_A}(X, Y)\alpha$$

with the evident composition.

Definition Let $\mathrm{Cor}^*_A(X,Y) := A^{\dim Y + *}(X \times Y)$.

$\widetilde{\mathrm{Cor}}_A$ is the category with objects pairs (X,n), X a smooth projective k-variety $n \in \mathbb{Z}$, morphisms

$$\mathrm{Hom}_{\widetilde{\mathrm{Cor}}_A}((X,n),(Y,m)) := \mathrm{Cor}^{m-n}_A(X,Y)$$

Cor_A is the additive category generated by $\widetilde{\mathrm{Cor}}_A$ and $\mathcal{M}_A$ is the pseudo-abelian hull of Cor_A.

For $M = (X,\alpha) \in \mathcal{M}^{\mathrm{eff}}_A$, write $M(m) := ((X,\alpha),m)$.

$\alpha \in \mathrm{Cor}^n_A(X,Y)$ acts as a homomorphism

$$\alpha_* : A^*(X) \to A^{*+n}(Y).$$

We have ${}^t\alpha \in \mathrm{Cor}^{n+\dim X - \dim Y}(Y,X)$; set

$$\alpha^* := {}^t\alpha_* : A^*(Y) \to A^{*+\dim X - \dim Y + n}(X).$$

• Cor_A is a tensor category, $1 = m_A(\mathsf{Spec}\, k)$ and

$$\mathsf{Hom}_{\mathsf{Cor}_A}(1(n), m_A(X)) = A_n(X).$$

• Sending X to $(X, 0)$ defines tensor functors

$$\mathsf{Cor}^0_A \to \mathsf{Cor}_A$$
$$\mathcal{M}^{\mathsf{eff}}_A \to \mathcal{M}_A$$

• A natural transformation of O.C.T.'s on $\mathbf{Sm}/k$, $\vartheta : A \to B$, induces tensor functors

$$\vartheta_* : \mathsf{Cor}^0_A \to \mathsf{Cor}^0_B$$
$$\vartheta_* : \mathcal{M}^{\mathsf{eff}}_A \to \mathcal{M}^{\mathsf{eff}}_B$$

etc.

We add the ground field k to the notation when necessary: $\mathsf{Cor}^0_A(k)$, $\mathcal{M}_A(k)$, etc.

If R is a commutative ring, set

$$\mathsf{Cor}^0_{A,R} := \mathsf{Cor}^0_A \otimes R$$
$$\mathsf{Cor}_{A,R} := \mathsf{Cor}_A \otimes R$$

$\mathcal{M}^{\mathsf{eff}}_{A,R}$ and $\mathcal{M}_{A,R}$ are the respective pseudo-abelian hulls.

Examples

(1) For $A^* = \mathrm{CH}^*$, we have the well-known categories:

$\mathrm{Cor}^0_{\mathrm{CH}}(k)$ is the category of correspondences mod rational equivalence, $\mathcal{M}^{\mathrm{eff}}_{\mathrm{CH}}(k)$ is the category of effective Chow motives, $\mathcal{M}_{\mathrm{CH}}(k)$ is the category of Chow motives (all over k).

(2) For $A^* = \Omega^*$, we call $\mathrm{Cor}^0_{\Omega}(k)$ the category of *cobordism correspondences*, $\mathcal{M}^{\mathrm{eff}}_{\Omega}(k)$ the category of *effective cobordism motives*, $\mathcal{M}_{\Omega}(k)$ the category of *cobordism motives* (over k).

(3) We can also take e.g. $A^* = K_0[\beta, \beta^{-1}]$; we write $\mathrm{Cor}^0_{K_0}$, $\mathcal{M}^{\mathrm{eff}}_{K_0}$, etc.

Cobordism motives

Vishik-Yagita have considered the category $\mathcal{M}_{\Omega}^{\mathrm{eff}}(k)$ and discussed its relation with Chow motives.

Remarks

(1) Since Ω^* is universal, there are canonical functors

$$\vartheta_*^A : \mathsf{Cor}^0_\Omega(k) \to \mathsf{Cor}^0_A(k)$$
$$\vartheta_*^A : \mathcal{M}^{\mathsf{eff}}_\Omega(k) \to \mathcal{M}^{\mathsf{eff}}_A(k)$$

etc. Thus, identities in $\mathcal{M}^{\mathsf{eff}}_\Omega(k)$ or $\mathcal{M}_\Omega(k)$ yield identities in $\mathcal{M}^{\mathsf{eff}}_A(k)$ or $\mathcal{M}_A(k)$ for all O.C.T.'s A on $\mathbf{Sm}/k$.

(2) $\Omega^* \otimes \mathbb{Q}$ is isomorphic to the "universal twist" of $\mathsf{CH}^* \otimes \mathbb{L} \otimes \mathbb{Q}$, so one can hope to understand $\mathcal{M}_{\Omega,\mathbb{Q}}$ by modifying $\mathcal{M}_{\mathsf{CH},\mathbb{L}\otimes\mathbb{Q}}$ by a twisting construction, i.e., a deformation of the composition law. We will see that $\mathcal{M}_{\Omega,\mathbb{Q}}$ is NOT equvalent to $\mathcal{M}_{\mathsf{CH},\mathbb{L}\otimes\mathbb{Q}}$.

(3) The work of Vishik-Yagita allows one to lift identies in $\mathcal{M}^{\mathsf{eff}}_{\mathsf{CH}}(k)$ or $\mathcal{M}_{\mathsf{CH}}(k)$ to $\mathcal{M}^{\mathsf{eff}}_\Omega(k)$ or $\mathcal{M}_\Omega(k)$

Example [The Lefschetz motive in $\mathcal{M}_{\Omega}^{\text{eff}}$] Let's compare $\text{End}_{\text{Cor}^0_{\Omega}}(\mathbb{P}^1)$ with $\text{End}_{\text{Cor}^0_{\text{CH}}}(\mathbb{P}^1)$

$$\Omega^1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[0 \times \mathbb{P}^1] \oplus \mathbb{Z}[\mathbb{P}^1 \times 0] \oplus \mathbb{Z}[\mathbb{P}^1] \times [(0,0)]$$

Set: $\alpha = [0 \times \mathbb{P}^1]$; $\beta = [\mathbb{P}^1 \times 0]; \gamma = [\mathbb{P}^1] \cdot [(0,0)]$.

$\text{Cor}^0_{\Omega} \to \text{Cor}^0_{\text{CH}}$ just sends γ to zero. We have the composition laws:

$\text{End}_{\text{Cor}^0_{\text{CH}}}(\mathbb{P}^1)$		
$\circ$	α	β
α	α	0
β	0	β

$\text{End}_{\text{Cor}^0_{\Omega}}(\mathbb{P}^1)$			
$\circ$	α	β	γ
α	α	0	0
β	γ	β	γ
γ	γ	0	0

So

$$\mathrm{End}_{\mathrm{Cor}^0_\Omega}(\mathbb{P}^1) \to \mathrm{End}_{\mathrm{Cor}^0_{\mathrm{CH}}}(\mathbb{P}^1) = \mathbb{Z} \times \mathbb{Z}$$

is a non-commutative extension with square-zero kernel (γ). Hence:

• The Lefschetz Chow motive $L := (\mathbb{P}^1, \alpha)$ lifts to "the Lefschetz Ω-motive"

$$L_\Omega(\lambda) := (\mathbb{P}^1, \alpha + \lambda\gamma)$$

for any choice of $\lambda \in \mathbb{Z}$. Since $[\Delta_{\mathbb{P}^1}] = \alpha + \beta - \gamma$:

$$(\mathbb{P}^1, \mathrm{id}) = (\mathbb{P}^1, \alpha + \lambda\gamma) \oplus (\mathbb{P}^1, \beta - (1 + \lambda)\gamma) \cong L_\Omega(\lambda) \oplus 1.$$

Also: $L_\Omega(\lambda) \cong L_\Omega(\lambda')$ for all λ, λ', but are not equal as summands of $(\mathbb{P}^1, \mathrm{id})$.

One can also compute:

• $\mathrm{End}(L_\Omega^{\otimes n}) = \mathbb{Z} \cdot \mathrm{id}$ for all $n \geq 0$, and $\mathrm{Hom}(1, L_\Omega) = 0$, but $\mathrm{Hom}(L_\Omega, 1) = \mathbb{Z} \cdot \gamma$.

• Let $t : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the switch of factors. Then t induces the identity on $L_\Omega^{\otimes 2}$.

Remark We have seen that

$$\mathrm{End}_{\mathcal{M}^{\mathrm{eff}}_{\Omega}}(m_{\Omega}(\mathbb{P}^1)) \to \mathrm{End}_{\mathcal{M}^{\mathrm{eff}}_{\mathrm{CH}}}(m_{\mathrm{CH}}(\mathbb{P}^1))$$

is surjective with kernel a square zero ideal. A similar computation shows that

$$\mathrm{End}_{\mathcal{M}^{\mathrm{eff}}_{\Omega}}(m_{\Omega}(\mathbb{P}^n)) \to \mathrm{End}_{\mathcal{M}^{\mathrm{eff}}_{\mathrm{CH}}}(m_{\mathrm{CH}}(\mathbb{P}^n)) = \prod_{i=1}^{n} \mathbb{Z}$$

is surjective for all n with $ker^{n+1} = 0$, but $ker^n \neq 0$.

Definition Let A be an O.C.T. on $\mathbf{Sm}/k$, $\vartheta_A : \mathcal{M}_\Omega^{\mathrm{eff}}(k) \to \mathcal{M}_A^{\mathrm{eff}}(k)$ the canonical functor. Define the *Lefschetz A-motive*

$$L_A := \vartheta_A(L_\Omega).$$

Proposition *For $M = (X, \alpha)$, $N = (Y, \beta)$ in $\mathcal{M}_A^{\mathrm{eff}}(k)$,*

$$\mathrm{Hom}_{\mathcal{M}_A^{\mathrm{eff}}}(M \otimes L_A^{\otimes m}, N \otimes L_A^{\otimes n})$$
$$= (\mathrm{id} \times \alpha)^*(\beta \times \mathrm{id})_* \Omega^{\dim Y - m + n}(X \times Y).$$

Hence

Theorem *The inclusion functor $\mathcal{M}_A^{\mathrm{eff}}(k) \to \mathcal{M}_A(k)$ identifies $\mathcal{M}_A(k)$ with the localization of $\mathcal{M}_A^{\mathrm{eff}}(k)$ with respect to $- \otimes L_A$. Also*

$$(X, \alpha)(m) := (X, \alpha, m) \cong (X, \alpha) \otimes L_A^{\otimes m}.$$

The nilpotence theorem

Theorem *Take* $X, Y \in \mathbf{SmProj}/k$. *Then*

$$\vartheta_{\mathsf{CH}} : \mathsf{Cor}^0_{\Omega}(X,Y) \to \mathsf{Cor}^0_{\mathsf{CH}}(X,Y)$$

is surjective. If $X = Y$, *then the kernel* $ker(X)$ *of* ϑ_{CH} *is nilpotent.*

Proof. Surjectivity: Since $\mathsf{CH}^* = \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}$, $\Omega^*(T) \to \mathsf{CH}^*(T)$ is surjective for all $T \in \mathbf{Sm}/k$.

Nilpotence of the kernel: $\mathrm{CH}^* = \Omega^* \otimes_{\mathbb{L}} \mathbb{Z} \Longrightarrow$

$$ker(X) = \sum_{n>0} \mathbb{L}^{-n} \Omega^{\dim X + n}(X \times X).$$

Composition is $\mathbb{L}$-linear, hence operates as:

$$\mathbb{L}^{-n}\Omega^{\dim X+n}(X \times X) \otimes \mathbb{L}^{-m}\Omega^{\dim X+m}(X \times X) \xrightarrow{\circ} \mathbb{L}^{-n-m}\Omega^{\dim X+n+m}(X \times X).$$

Also: $\Omega^d(T) = 0$ for $d > \dim T$. Thus

$$ker(X)^{\circ \dim X + 1} = 0.$$

Proposition (Vishik-Yagita)
(1) For $X \in \mathbf{SmProj}/k$, each idempotent in $\mathrm{Cor}^0_{\mathrm{CH}}(X,X)$ *lifts to an idempotent in* $\mathrm{Cor}^0_{\Omega}(X,X)$*.*

(2) For M, N in $\mathcal{M}^{\mathrm{eff}}_{\Omega}(k)$, each isomorphism $f : \vartheta_{\mathrm{CH}}(M) \to \vartheta_{\mathrm{CH}}(N)$ lifts to an isomorphism $\tilde{f} : M \to N$.

Theorem (Isomorphism) *$\vartheta_{\mathrm{CH}} : \mathcal{M}^{\mathrm{eff}}_{\Omega}(k) \to \mathcal{M}^{\mathrm{eff}}_{\mathrm{CH}}(k)$ and $\vartheta_{\mathrm{CH}} : \mathcal{M}_{\Omega}(k) \to \mathcal{M}_{\mathrm{CH}}(k)$ both induce bijections on the set of isomorphism classes of objects.*

Proof. For $\mathcal{M}^{\mathrm{eff}}$, this follows from the proposition. For $\mathcal{M}$, this follows by localization.

Note. These result are also valid for motives with R-coefficients, R a commutative ring.

Motivic computations

Elementary computations

• $m_A(\mathbb{P}^n) \cong \oplus_{i=0}^n L_A^{\otimes i} \cong \oplus_{i=0}^n 1_A(i)$.
• Let $E \to B$ be a vector bundle of rank $n+1$, $\mathbb{P}(E) \to B$ the projective-space bundle. Then

$$m_A(\mathbb{P}(E)) \cong \oplus_{i=0}^n m_A(B)(i).$$

• Let $\mu : X_F \to X$ be the blow-up of X along a codimension d closed subscheme F. Then

$$m_A(X_F) \cong m_A(X) \oplus \oplus_{i=1}^{d-1} m_A(F)(i).$$

So:

$$A^*(X_F) \cong A^*(X) \oplus \oplus_{i=1}^{d-1} A^{*-d+i}(F).$$

Indeed, we have all these isomorphisms in $\mathcal{M}_{\mathrm{CH}}$, hence in $\mathcal{M}_\Omega$ by the isomorphism theorem, and thus in $\mathcal{M}_A$ by applying ϑ_A.

Cellular varieties

Definition $X \in \mathbf{SmProj}/k$ is called *cellular* if there is a filtration by closed subsets

$$X = X^0 \supset X^1 \supset \ldots \supset X^d \supset X^{d+1} = \emptyset;\ d = \dim X,$$

such that $\mathrm{codim}_X X^i \geq i$ and either $X^i \setminus X^{i+1} \cong \coprod_{i=1}^{n_i} \mathbb{A}^{d-i}$ or $X^i = X^{i+1}$. If $X_{\bar{k}}$ is cellular, call X *geometrically cellular*.

• For X cellular as above, we have

$$m_A(X) \cong \oplus_{i=0}^{d} 1_A(i)^{n_i}$$

because we have this isomorphism for $A = \mathrm{CH}$.

Examples Projective spaces and Grassmannians are cellular. A smooth quadric over k is geometrically cellular.

Quadratic forms

First some elementary facts about quadratic forms:

• Each quadratic form over k can be diagonalized. If $q = \sum_{i=1}^{n} a_i x_i^2$, let $Q_q \subset \mathbb{P}^{n-1}$ be the quadric $q = 0$. The *dimension* of q is n.

• For $q_1 = \sum_{i=1}^{n} a_i x_i^2$, $q_2 = \sum_{j=1}^{m} b_j y_j^2$, we have the orthogonal sum

$$q_1 \perp q_2 := \sum_{i=1}^{n} a_i x_i^2 + \sum j = 1^m b_j y_j^2$$

and tensor product

$$q_1 \otimes q_2 := \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j z_{ij}^2.$$

Pfister forms and Pfister quadrics

• For $a \in k^\times$ we have $\langle\langle a\rangle\rangle := x^2 - ay^2$ and for $a_1, \ldots, a_n \in k^\times$ the *n-fold Pfister form*

$$\alpha := \langle\langle a_1, \ldots, a_n\rangle\rangle := \langle\langle a_1\rangle\rangle \otimes \ldots \otimes \langle\langle a_n\rangle\rangle$$

The quadric $Q_\alpha \subset \mathbb{P}^{2^n-1}$ is the associated *Pfister quadric.*

• The isomorphism class of $\alpha = \langle\langle a_1, \ldots, a_n\rangle\rangle$ depends only on the symbol

$$\{a_1, \ldots, a_n\} \in k_n(k) := K_n^M(k)/2.$$

$\langle\langle a_1, \ldots, a_n\rangle\rangle)$ is isomorphic to a hyperbolic form if and only if Q_α is isotropic, i.e. $\langle\langle a_1, \ldots, a_n\rangle\rangle = 0$ has a non-trivial solution in k.

The Rost motive

Proposition (Rost) *(1) Let* $\alpha = \langle\langle a_1, \ldots, a_n\rangle\rangle$ *and let* Q_α *be the associated Pfister quadric. Then there is a motive* $M_\alpha \in \mathcal{M}^{\mathrm{eff}}_{\mathrm{CH}}(k)$ *with*

$$m_{\mathrm{CH}}(Q_\alpha) \cong M_\alpha \otimes m_{\mathrm{CH}}(\mathbb{P}^{2^{n-1}-1}).$$

(2) Let $\bar{k}$ *be the algebraic closure. There are maps*

$$L^{\otimes 2^{n-1}-1} \to M_\alpha \to 1$$

which induce

$$M_{\alpha\bar{k}} \cong 1 \oplus L^{\otimes 2^{n-1}-1} \textit{ in } \mathcal{M}^{\mathrm{eff}}_{\mathrm{CH}}(\bar{k}).$$

M_α is the *Rost motive*.

The Rost cobordism-motive

Applying the Vishik-Yagita bijection, there is a unique (up to isomorphism) cobordism motive

$$M_\alpha^\Omega \in \mathcal{M}_\Omega^{\mathrm{eff}}(k)$$

with $\vartheta_{\mathrm{CH}}(M_\alpha^\Omega) \cong M_\alpha$. In addition:

1. $m_\Omega(Q_\alpha) \cong M_\alpha^\Omega \otimes m_\Omega(\mathbb{P}^{2^{n-1}-1})$.

2. There are maps $L_\Omega^{\otimes 2^{n-1}-1} \to M_\alpha^\Omega \to 1$ which induce

$$M_{\alpha\bar{k}}^\Omega \cong 1 \oplus L_\Omega^{\otimes 2^{n-1}-1} \text{ in } \mathcal{M}_\Omega^{\mathrm{eff}}(\bar{k}).$$

Algebraic cobordism of Pfister quadrics

Vishik-Yagita use the Rost cobordism motive to compute $\Omega^*(Q_\alpha)$. The computation is in two parts:

1. Compute the image of base-change $\Omega^*(Q_\alpha) \to \Omega^*(Q_{\alpha\bar{k}})$. $\Omega^*(Q_{\alpha\bar{k}})$ is easy because $Q_{\alpha\bar{k}}$ is cellular.

2. Show that $\Omega^*(Q_\alpha) \to \Omega^*(Q_{\alpha\bar{k}})$ is injective.

Structure of $\mathbb{L}$

We need some information on $\mathbb{L}$ to state the main result.

Recall the Conner-Floyd Chern classes c_I and the Landweber-Novikov operations s_I. Let $\bar{s}_I(x)$ be the image of $s_I(x)$ in CH^*. For $X \in \mathbf{SmProj}/k$ of dimension $|I|$

$$\bar{s}_I([X]) = \deg c_I(-T_X) \in \mathbb{Z} = \mathrm{CH}^0(k).$$

Since the $\bar{s}_I$ are indexed by the monomials in $t_1, t_2, \ldots$, $\deg t_i = i$, we have

$$\bar{s} : \Omega^*(k) = \mathbb{L}^* \to \mathbb{Z}[\mathbf{t}]$$

with $\bar{s}([X]) = \sum_I \bar{s}_I(X) t^I = \sum_I c(-T_X) t^I$.

Theorem **(Quillen)** $\bar{s} : \Omega^*(k) = \mathbb{L}^* \to \mathbb{Z}[\mathbf{t}]$ *is an injective ring homomorphism with image of finite index in each degree.*

Definition $I(p) \subset \mathbb{L}^*$ is the prime ideal

$$I(p) := \bar{s}^{-1}(p\mathbb{Z}[\mathbf{t}]).$$

$I(p, n) \subset I(p)$ is the sub-ideal generated by elements of degree $\leq p^n - 1$.

In words: $I(p) \subset \mathbb{L}$ is the ideal generated by $[X]$, $X \in \mathbf{SmProj}/k$ all of whose Chern numbers $\deg c_I(-T_X)$ are divisible by p.

Note. The fact that $s_{2^n-1}(Q_{2^n-1}) \equiv 1 \mod 2$ for Q_{2^n-1} a quadric of dimension $2^n - 1$ implies that $I(2, r)$ is the ideal generated by the classes $[Q_{2^n-1}]$, $0 \leq 2^n - 1 \leq r$ ($[Q_0] = 2 \in \mathbb{L}^0$).

The main theorem

Fix $\alpha := \langle\langle a_1, \ldots, a_n\rangle\rangle$, $Q_\alpha \subset \mathbb{P}^{2^n-1}$ the associated Pfister quadric.

Let $h^i_\Omega \in \Omega^i(Q_{\alpha\bar{k}})$ be the class of a codimension i linear section,

Let $\ell^\Omega_i \in \Omega_i(Q_{\alpha\bar{k}})$ be the class of a linear $\mathbb{P}^i \subset Q_{\alpha\bar{k}}$.

h^i, ℓ_i: the images of h^i_Ω and ℓ^Ω_i in CH^i, CH_i.

Since $Q_{\alpha\bar{k}}$ is cellular

$$\Omega^*(Q_{\alpha\bar{k}}) = \oplus_{i=0}^{2^{n-1}-1}\mathbb{L}\cdot h^i_\Omega \oplus \mathbb{L}\cdot \ell^\Omega_i.$$

Theorem *The base-change map $p^* : \Omega^*(Q_\alpha) \to \Omega^*(Q_{\alpha\bar{k}})$ is injective and the image of p^* is*

$$\oplus_{i=0}^{2^{n-1}-1}\mathbb{L}\cdot h^i_\Omega \oplus I(2, n-2)\cdot \ell^\Omega_i$$

Idea of proof:

Use the isomorphisms

$$m_\Omega(Q_\alpha) \cong M_\alpha^\Omega \otimes m_\Omega(\mathbb{P}^{2^{n-1}-1}),\ M_{\alpha\bar{k}}^\Omega \cong 1 \oplus L_\Omega^{2^{n-1}-1}$$

to show that the image of base-change is $\oplus_{i=0}^{2^{n-1}-1}\mathbb{L}\cdot h_\Omega^i \oplus J\cdot \ell_i^\Omega$ for some ideal $J \subset \mathbb{L}$.

A result of Rost on M_α^{CH} plus Vishik-Yagita lifting shows that

$$M_\alpha^\Omega \oplus ? = m_\Omega(P_\alpha),$$

P_α: a linear section of Q_α of dimension $2^{n-1}-1$.

The "small" dimension ($\leq 2^{n-1}-1$) of P_α allows one to show that $J = I(2, n-2)$.

The injectivity is handled by the fact that P_α splits M_α.