Algebraic Cobordism

Riemann-Roch and applications

Motives and Periods

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Outline

- Twisting a theory
- Panin's Riemann-Roch theorem
- Operations in cobordism
- Degree formulas
- Applications

Twisting a theory

Definition Let $L \to X$ be a line bundle. The *inverse Todd class* of L is

$$\mathsf{Td}_{\tau}^{-1}(L) := \sum_{i=0}^{\infty} \tau_i c_1(L)^i.$$

Note. $c_1(L)^{\dim X+1} = 0.$

Todd classes

Given: A^*: an O.C.T. on $\mathbf{Sm}/k$

$\tau_i \in A^{-i}(k)$, $i = 0, 1, \ldots$; $\tau_0 = 1$.

Let $\sigma_i(t) :=$ the ith elementary symmetric function in $t_1, t_2, \ldots$.

Let $f_\tau(t) = \sum_{i=0}^{\infty} \tau_i t^i$ and

$$F_\tau(t_1, t_2, \ldots) := \prod_{i=1}^{\infty} f_\tau(t_i).$$

Then

$$F_\tau(t_1, t_2, \ldots) = \mathrm{td}_\tau^{-1}(\sigma_1(t), \sigma_2(t), \ldots)$$

for a unique $\mathrm{td}_\tau^{-1} \in A^*(k)[\sigma_1, \sigma_2, \ldots]$.

Definition Let $E \to X$ be a vector bundle. Set

$$\mathrm{Td}_\tau^{-1}(E) := \mathrm{td}_\tau^{-1}(c_1(E), c_2(E), \ldots)$$

Note. This also works if we only assume $\tau_0 \in A^0(k)$ is a unit.

Properties:

- For $L \to X$ a line bundle: $\mathsf{Td}^{-1}(L) = \sum_{i=0}^{\infty} \tau_i c_1(L)^i$.

- $\mathsf{Td}_\tau^{-1}(-)$ is functorial: $f^* \mathsf{Td}_\tau^{-1}(E) = \mathsf{Td}_\tau^{-1}(f^*E)$.

- $\mathsf{Td}_\tau^{-1}(-)$ is multiplicative: $\mathsf{Td}_\tau^{-1}(E) = \mathsf{Td}_\tau^{-1}(E')\,\mathsf{Td}_\tau^{-1}(E'')$ for each exact sequence

$$0 \to E' \to E \to E'' \to 0$$

- $E \mapsto \mathsf{Td}_\tau^{-1}(E)$ descends to a group homomorphism

$$\mathsf{Td}_\tau^{-1} : K_0(X) \to A^0(X)^\times$$

Twisting a theory

For $f : Y \to X$ in $\mathbf{Sm}/k$, set

$$N_f := [f^*T_X] - [T_Y] \in K_0(Y).$$

Define:

$A^*_\tau(X) := A^*(X)$
$f^*_\tau := f^*$

For $f : Y \to X$ projective, $d = \mathrm{codim} f$, define
$f^\tau_* : A^*(Y) \to A^{*+d}(X)$ by

$$f^\tau_*(y) := f_*(y \cdot \mathrm{Td}_\tau^{-1}(N_f)).$$

Proposition *(1)* $X \mapsto A^*_\tau(X)$ *defines an O.C.T. on* $\mathbf{Sm}/k$.

(2) Let $\lambda_\tau(t) = \sum_{i=0}^\infty \tau_i t^{i+1}$. *For* $p : L \to X$ *a line bundle,*

$$c_1^\tau(L) = \lambda_\tau(c_1(L)).$$

(3) A^*_τ *has formal group law*

$$F_A^\tau(u, v) = \lambda_\tau(F_A(\lambda_\tau^{-1}(u), \lambda_\tau^{-1}(v))).$$

Proof: The functoriality of f_* follows from the identity

$$N_{fg} = g^* N_f + N_g$$

in K_0, and the multiplicativity of Td_τ^{-1}.

The formula for $c_1^\tau(L)$ follows from the definition.

(PB) for A_τ^* follows from (PB) for A^* and the fact that $\mathrm{Td}_\tau^{-1}(L)$ is a unit.

The formal group law follows from the formula for $c_1^\tau(L)$:

$$F_A^\tau(c_1^\tau(L), c_1^\tau(M)) = c_1^\tau(L \otimes M) \Longrightarrow$$

$$\begin{aligned} F_A^\tau(\lambda_\tau(c_1(L)), \lambda_\tau(c_1(M))) &= \lambda_\tau(c_1(L \otimes M)) \\ &= \lambda_\tau(F_A(c_1(L), c_1(M))). \end{aligned}$$

Panin's Riemann-Roch theorem

A^*, B^*: O.C.T. on $\mathbf{Sm}/k$
$\phi : A^* \to B^*$ a natural transformation of underlying cohomology theories:

$$\phi(x \cdot_A y) = \phi(x) \cdot_B \phi(y)$$
$$\phi(f_A^*(x)) = f_B^*(\phi(x)).$$

By (PB) there is a unique power series $\mathrm{td}_\phi^{-1}(t) = \sum_{i=0}^\infty \tau_i t^i$ such that

$$\phi(c_1^A(L)) = \mathrm{td}_\phi^{-1}(c_1^B(L)) \cdot c_1^B(L).$$

Theorem (Panin) *Suppose that τ_0 is a unit. Then ϕ defines a natural transformation of O.C.T.*

$$\phi : A^* \to B^*_\tau.$$

Explicit R-R

In concrete terms: Let $\mathsf{td}_\tau(t) = 1/\,\mathsf{td}_\tau^{-1}(t)$. Define $\mathsf{Td}_\tau(E)$ using $\mathsf{td}_\tau(t)$ instead of $\mathsf{td}_\tau^{-1}(t)$.

Let $f : Y \to X$ be a projective morphism. Then

$$\begin{aligned}\mathsf{Td}_\tau^{-1}(N_f) &= \mathsf{Td}_\tau^{-1}([f^*T_X] - [T_Y]) \\ &= \frac{\mathsf{Td}_\tau(T_Y)}{f^*(\mathsf{Td}_\tau(T_X))}.\end{aligned}$$

Thus

$$\phi(f_*^A(x)) = f_*^{B^\tau}(\phi(x)) = f_*^B(\phi(x) \cdot \mathsf{Td}^{-1}(N_f))$$

so we recover the "classical" R-R theorem:

$$\phi(f_*^A(x)) \cdot \mathsf{Td}_\tau(T_X) = f_*^B(\phi(x) \cdot \mathsf{Td}_\tau(T_Y)).$$

Grothendieck-R-R

We take the original example: Let $ch : K_0(X) \to \mathrm{CH}^*(X)_{\mathbb{Q}}$ be the Chern character.
ch is characterized (by the splitting principle) as the unique additve homomorphism with

$$ch([L]) = e^{c_1^{\mathrm{CH}}(L)}.$$

CH has the additive group law $\Longrightarrow$ ch is a ring homomorphism.

Modify ch to the natural transformation of cohomology theories

$$ch_\beta : K_0[\beta, \beta^{-1}] \to \mathrm{CH}^*_{\mathbb{Q}}[\beta, \beta^{-1}]$$

by $ch_\beta([L]\beta^n) = e^{\beta c_1^{\mathrm{CH}}(L)}\beta^n$.

What is $\mathrm{td}_{ch}^{-1}(t)$?

$c_1^K(L) = (1 - L^{-1})\beta^{-1}$, so

$$\begin{aligned} ch_\beta(c_1^K(L)) &= \beta^{-1}[ch_\beta(1) - ch_\beta(L^{-1})] \\ &= \beta^{-1}[1 - e^{-\beta c_1^{\mathrm{CH}}(L)}]. \end{aligned}$$

Thus

$$\mathrm{td}_{ch}^{-1}(t) = \frac{1 - e^{-\beta t}}{\beta t}.$$

Restricting to degree 0 and sending β to 1, we recover the usual Chern character, Todd class and the Grothendieck-Riemann-Roch theorem.

Operations

Landweber-Novikov classes

These are the coefficients of the universal inverse Todd class:

Take variables $t_1, t_2, \ldots$ with deg $t_i := -i$ and extend Ω^* to $\Omega^*[t_1, t_2, \ldots] := \Omega^*[\mathbf{t}]$.

Let $f_{\mathbf{t}}(t) := \sum_i t_i t^i$ $(t_0 = 1)$ be the universal inverse Todd genus.

For $E \to X$ a vector bundle, write

$$\mathrm{Td}_{\mathbf{t}}^{-1}(E) = \sum_J c_J(E) t^J; \quad c_J \in \Omega^{|J|}(X).$$

Since $\mathrm{Td}_{\mathbf{t}}^{-1}$ is multiplicative, sending E to $c_J(E)$ descends to a natural map

$$c_J : K_0(X) \to \Omega^{|J|}(X),$$

the Jth *Landweber-Novikov class*.

Examples

(1) $c_n(E) = c_{n,0,0,\ldots}(E)$.

(2) The Newton class $S_n(E) := c_{0,\ldots,0,1}(E)$ ($n-1$ 0's). For L a line bundle

$$S_n(L) = c_1(L)^n.$$

S_n is additive: $S_n(E \oplus E') = S_n(E) + S_n(E')$.

Landweber-Novikov operations

We using the twisting construction to promote the classes c_J to operations on Ω^*.

Let $\Omega^*[\mathbf{t}]^{(\mathbf{t})}$ be the twist of $\Omega^*[\mathbf{t}]$ by the universal Todd genus.

The universality of Ω^* gives a unique transformation

$$\nu_{LN} : \Omega^* \to \Omega^*[\mathbf{t}]^{(\mathbf{t})}.$$

For $x \in \Omega^n(X)$, write

$$\nu_{LN}(x) = \sum_J S_J^{LN}(x) t^J; \quad S_J^{LN}(x) \in \Omega^{n+|J|}(X).$$

The transformation

$$S_J^{LN} : \Omega^* \to \Omega^{*+|J|}$$

is the Jth *Landweber-Novikov operation*.

The definition of pushforward in the twisted theory gives the formula for s_J^{LN}:

For $f : Y \to X \in \mathcal{M}(X)$,

$$S_J^{LN}(f) = f_*(c_J(N_f)).$$

Proposition *Sending* $f : Y \to X \in \mathcal{M}^*(X)$ *to* $f_*(c_J(N_f)) \in \Omega^{*+|J|}(X)$ *descends to a natural homomorphism*

$$S_J^{LN} : \Omega^*(X) \to \Omega^{*+|J|}(X).$$

Note. Let $c_J^{CF}(E) := \vartheta_{\mathsf{CH}}(c_J(E)) \in \mathsf{CH}^{|J|}(X)$. The classes $c_J^{CF}(E)$ are the *Conner-Floyd Chern classes* of E.

Ex.: $c_{(n)}(E) = c_n(E)$, the usual nth Chern class.

Brosnan/Voevodsky Steenrod operations

Fix a prime p. Let $b_n := t_{p^n-1}$ ($\deg b_n = p^n - 1$).

Extend CH^*/p to $\mathrm{CH}^*/p[\mathbf{b}] := \mathrm{CH}/p[b_1, b_2, \ldots]$.

Form the universal mod p genus

$$f_{\mathbf{b}}^{(p)}(t) := \sum_n b_n t^{p^n-1} \in \mathrm{CH}^*/p(k)[\mathbf{b}][t] = \mathbb{F}_p[\mathbf{b}][t].$$

Let $\mathrm{CH}^*/p[\mathbf{b}]^{(\mathbf{b})}$ be the twisted theory and

$$\nu^{(p)} : \Omega^* \to \mathrm{CH}^*/p[\mathbf{b}]^{(\mathbf{b})}$$

the canonical map.

Lemma *The formal group law of* $\mathsf{CH}^*/p[\mathbf{b}]^{(\mathbf{b})}$ *is the additive group.*

Proof.

$$\begin{aligned} c_1^{(\mathbf{b})}(L) &= c_1^{\mathsf{CH}/p}(L) \cdot f^{(p)}(c_1^{\mathsf{CH}/p}(L)) \\ &= \sum_n c_1^{\mathsf{CH}/p}(L)^{p^n} b_n. \end{aligned}$$

So

$$\begin{aligned} c_1^{(\mathbf{b})}(L \otimes M) &= \sum_n c_1^{\mathsf{CH}/p}(L \otimes M)^{p^n} b_n \\ &= \sum_n (c_1^{\mathsf{CH}/p}(L) + c_1^{\mathsf{CH}/p}(M))^{p^n} b_n \\ &= \sum_n (c_1^{\mathsf{CH}/p}(L)^{p^n} + c_1^{\mathsf{CH}/p}(M)^{p^n}) b_n \\ &= c_1^{(\mathbf{b})}(L) + c_1^{(\mathbf{b})}(M). \end{aligned}$$

Since $\mathrm{CH}^* = \Omega^*_+$, $\nu^{(p)} : \Omega^* \to \mathrm{CH}^*/p[\mathbf{b}]^{(\mathbf{b})}$ descends to

$$S^{(p)} : \mathrm{CH}^*/p \to \mathrm{CH}^*/p[\mathbf{b}]^{(\mathbf{b})}.$$

Write

$$S^{(p)} := \sum_J S_J^{(p)} b^J.$$

Definition The homomorphism

$$S_J^{(p)} : \mathrm{CH}^*/p \to \mathrm{CH}^{*+|J|_p}/p$$

is the Jth *mod p Steenrod operation*
$(|(j_1, \ldots, j_r)|_p := \sum_i j_i(p^i - 1))$.

As for the Landweber-Novikov operations:

$$S_J^{(p)}([f : Y \to X]) = f_*(c_{J(p)}^{CF}(N_f)).$$

($J \mapsto J^{(p)}$ places the ith entry of J in position $p^i - 1$ and fills in with 0's).

This shows these Steenrod operations agree with those of Brosnan/Voevodsky.

Divisibility results We make the $\mathbb{Z}$-version of our construction:

$$\tilde{f}^{(p)}_{\mathbf{b}}(t) := \sum_n b_n t^{p^n - 1} \in \mathrm{CH}^*(k)[\mathbf{b}][t] = \mathbb{Z}[\mathbf{b}][t].$$

Twist $\mathrm{CH}^*[\mathbf{b}]$ to $\mathrm{CH}^*[\mathbf{b}]^{(\mathbf{b})}$.

The universal property gives $\tilde{S}^{(p)} : \Omega^* \to \mathrm{CH}^*[\mathbf{b}]^{(\mathbf{b})}$.

For each index J, this gives the commutative diagram

$$\begin{array}{ccc} \Omega^* & \xrightarrow{\nu_{\mathrm{CH}}} & \mathrm{CH}^* \\ {\scriptstyle \tilde{S}^{(p)}_J}\downarrow & & \downarrow{\scriptstyle S^{(p)}_J} \\ \mathrm{CH}^{*+|J|_p} & \longrightarrow & \mathrm{CH}^{*+|J|_p}/p \end{array}$$

So for $x \in \Omega^*(X)$:

If $\nu_{\mathrm{CH}}(x) = 0$, then p divides $\tilde{S}^{(p)}_J$ in $\mathrm{CH}^{*+|J|_p}(X)$ for all J.

Taking $X = \operatorname{Spec} k$ and noting $\mathrm{CH}^*(k) = \mathrm{CH}^0(k) = \mathbb{Z}$ gives

Proposition *Let Y be a smooth projective variety over k of dimension $d > 0$. Then for all J with $|J|_p = d$,*

$$p \mid \tilde{S}_J^{(p)}([Y]) \in \mathrm{CH}^0(k) = \mathbb{Z}.$$

Example For $J = (0, \ldots, 0, 1)$ with the 1 in the nth spot, we have $\tilde{S}_J^{(p)} = S_{p^n-1}$, the $p^n - 1$st Newton class. Thus: For all smooth projective varieties Y of dimension $d = p^n - 1$

$$\deg(S_{p^n-1}(T_Y)) \in p\mathbb{Z}.$$

Jndecomposability

Definition $p : X \to \operatorname{Spec} k$ a smooth projective variety over k.

$I(X) \subset \mathbb{Z}$ is the ideal generated by $\{\deg_k k(x)\}$, x a closed point of X. Equivalently: $I(X) \subset \mathrm{CH}_0(k) = \mathbb{Z}$ is the image of $p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(k)$.

Proposition *Y, Z smooth projective varieties over k with $\dim Z > 0$, $\dim Y > 0$. Let $X = Y \times Z$, $d = \dim X$. Then for all J with $|J|_p = d$, we have*

$$\tilde{S}_J^{(p)}(X) \in p \cdot I(Z) \cap (p^2).$$

Note. $\tilde{S}_J^{(p)}(X) = \deg c_{J(p)}(-T_X)$

$$\implies \tilde{S}_J^{(p)}(X) \in I(X).$$

Proof of the proposition.

$\tilde{S}^{(p)} : \Omega^* \to \mathrm{CH}^*[\mathbf{b}]^{(\mathbf{b})}$ is a natural transformation of O.C.T.s, hence respects products. Thus

$$\tilde{S}^{(p)}(X) = \tilde{S}^{(p)}(Y) \cdot \tilde{S}^{(p)}(Z).$$

For fixed index J:

$$\tilde{S}^{(p)}_J(X) = \sum_{\substack{J', J'' \\ J'+J''=J}} \tilde{S}^{(p)}_{J'}(Y) \cdot \tilde{S}^{(p)}_{J''}(Z)$$

But $p | \tilde{S}^{(p)}_{J'}(Y)$ and $\tilde{S}^{(p)}_{J''}(Z) \in I(Z)$.

Consequences

Definition J an index and X a smooth projective variety of dimension $d = |J|_p$. Set

$$s_J^{(p)}(X) := \frac{1}{p} \cdot \tilde{S}_J^{(p)}([X])$$

Proposition
(1) $s_J^{(p)}(X)$ *is an integer,* $ps_J^{(p)}(X) \in I(X)$.

(2) $s_J^{(p)}(Y \times Z) \cong 0 \mod I(Z) \cap (p)$ *if* $\dim Z > 0$, $\dim Y > 0$.

(3) $X \mapsto s_J^{(p)}(X)$ *descends to a homomorphism*

$$s_J^{(p)} : \Omega^{-|J|_p}(k) \to \mathbb{Z}.$$

Degree formulas

The degree homomorphism

Recall that the classifying map $\phi_{\Omega,k} : \mathbb{L}_* \to \Omega_*(k)$ is an isomorphism for any field k (of characteristic zero).

Let X be an irreducible finite type k-scheme. Restriction to the generic point $\eta \in X$ defines

$$i_\eta^* : \Omega^*(X) \to \Omega^*(k(\eta)).$$

Definition The *degree map* deg : $\Omega^*(X) \to \Omega^*(k)$ is defined by

$$\deg := \phi_{\Omega,k} \circ \phi_{\Omega,k(\eta)}^{-1} \circ i_\eta^*.$$

For a general X, we have one degree map for each irreducible component (use $\Omega_*(X)$ instead of $\Omega^*(X)$).

The generalized degree formula

For simplicity we give the statement for X irreducible. Let $\tilde{X} \to X$ be a resolution of singularites.

Theorem *Take* $x \in \Omega_*(X)$*. Then there are elements* $\alpha_i \in \Omega_*(k)$ *and* $f_i : Z_i \to X$ *in* $\mathcal{M}(X)$ *such that*

1. $Z_i \to f_i(Z_i)$ *is birational*

2. *No* $f_i(Z_i)$ *contains a generic point of* X

3. $x - \deg(x) \cdot [\tilde{X} \to X] = \sum_{i=1}^{r} \alpha_i \cdot [f_i : Z_i \to X]$.

The proof is quite easy:

Essentially by definition

$$i_\eta^*(x - \text{deg}(x) \cdot [\tilde{X} \to X]) = 0.$$

Thus there is an open $j : U \to X$ such that $j^*(x - \text{deg}(x) \cdot [\tilde{X} \to X]) = 0$.

Let $W = X \setminus U$ with $i : W \to X$. The exact localization sequence

$$\Omega_*(W) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

gives us an element $w \in \Omega_*(W)$ with

$$i_*(w) = x - \text{deg}(x) \cdot [\tilde{X} \to X].$$

Then use noetherian induction.

Corollary *Let X be in $\mathbf{Sm}/k$. Then*

$$\Omega^*(X) = \oplus_{n=0}^{\dim X} \mathbb{L}\Omega^n(X).$$

Indeed, $[\mathrm{id}_X]$ is in $\Omega^0(X)$ and $[Z_i \to X]$ is in $\Omega^n(X)$ for some n, $1 \leq n \leq \dim X$.

Degree formulas of Rost and Merkurjev

Theorem (Degree formula) *$f: Y \to X$ a morphism of smooth projective k-varieties of dimension d, p a prime. Then*

$$s_J^{(p)}(Y) \equiv \deg f \cdot s_J^{(p)}(X) \mod I(X).$$

Proof. The generalized degree formula yields (in $\Omega^*(X)$)

$$[f: Y \to X] = \deg f \cdot [\mathrm{id}: X \to X] + \sum_i \alpha_i [f_i : Z_i \to X];$$

$\dim Z_i < \dim X$, $k(Z_i) = k(f_i(Z_i))$, $\alpha_i \in \Omega^*(k)$.

Push forward to $\operatorname{Spec} k$: $[Y] = \deg f \cdot [X] + \sum_{ij} n_{ij} [Y_{ij} \times Z_i] \in \Omega^*(k)$.

($\alpha_i = \sum_j n_{ij} [Y_{ij}]$) $\dim Z_i < \dim X \Longrightarrow \dim Y_{ij} > 0$.

Apply $s_J^{(p)}$ and use the indecomposibility of $s_J^{(p)}$ $(+\ I(Z_i) \subset I(X))$:

$$s_J^{(p)}(Y) \equiv \text{deg}\, f \cdot s_J^{(p)}(X) + \sum{}' n_{ij} s_J^{(p)}(Y_{ij} \times Z_i) \mod I(X)$$

where $\sum'$ is over the i with $\dim Z_i = 0$.

But such Z_i are closed points of X, so

$$n_{ij} s_J^{(p)}(Y_{ij} \times Z_i) = n_{ij} s_J^{(p)}(Y_{ij}) \cdot \text{deg}(Z_i) \equiv 0 \mod I(X).$$

Examples (1) Let X be a conic over k: $X_{\bar{k}} \cong \mathbb{P}^1$ but $I(X) = (2)$. Let Y be a smooth irreducible projective curve over k, and $f : Y \to X$ a morphism. Then $\deg f$ and $g(Y)$ have opposite parity:

Take $p = 2$, $J = (1)$. Then $s_J^{(2)}(Y) = -(1/2)c_1(T_Y) = g(Y) - 1$ and the degree formula yields

$$g(Y) - 1 \equiv \deg f \cdot (g(X) - 1) = -\deg f \mod 2.$$

(2) Take $J = (0, \ldots, 0, 1)$ ($n-1$ zeros). Then $s_J^{(p)} = (1/p)\tilde{S}_{p^n-1}$; write s_{p^n-1} for $s_J^{(p)}$. The degree formula reads:

$$s_{p^n-1}(Y) = \deg f \cdot s_{p^n-1}(X) \mod I(X).$$

This is Rost's original degree formula.

Applications

Correspondences and rational maps

Theorem *Let X and Y be smooth projective varieties over k, $d = \dim X$. Suppose there is an index J with $|J|_p = d$ such that $s_J^{(p)}(X) \not\equiv 0 \mod I(X)$.*

Let $\gamma \in \mathrm{CH}_d(X \times Y)$ be an irreducible correspondence. Suppose that
a) $\deg_X \gamma$ is prime to p
b) $\nu_p(I(Y)) \geq \nu_p(I(X))$ (ν_p the p-adic valuation $\nu_p(p^n) = n$)

Then

1) $\dim Y \geq \dim X$
2) If $\dim Y = \dim X$ then $s_J^{(p)}(Y) \not\equiv 0 \mod I(Y)$,
$\nu_p(I(Y)) = \nu_p(I(X))$ and $\deg_Y \gamma$ is prime to p.

Proof. (Merkurjev)
(2): $\gamma = 1 \cdot Z$, Z irreducible. Take a resolution of singularities of Z: $Y \xleftarrow{f} \tilde{Z} \xrightarrow{g} X$, $(\deg g, p) = 1$.

The degree fomula for $g \Longrightarrow s_J^{(p)}(\tilde{Z}) \not\equiv 0 \mod I(X)$, so

$$s_J^{(p)}(\tilde{Z}) \not\equiv 0 \mod I(Y)$$

The degree formula for $f \Longrightarrow \deg f \cdot s_J^{(p)}(Y) \not\equiv 0 \mod I(Y)$.

$ps_J^{(p)}(Y) \equiv 0 \mod I(Y) \Longrightarrow (\deg f, p) = 1$ and

$$s_J^{(p)}(Y) \not\equiv 0 \mod I(Y).$$

$$(\deg f, p) = (\deg g, p) = 1 \Longrightarrow \nu_p(I(X)) = \nu_p(I(Y)).$$

(1): If $\dim Y < \dim X$, replace Y with $Y \times \mathbb{P}^n$, $n = \dim X - \dim Y$. This leaves $I(Y)$ unchanged, but now $\deg f = 0$, contrary to (2).

Corollary (Merkurjev) *Let X be a smooth projective k-variety, J an index with $s_J^{(p)}(X) \not\equiv 0 \mod I(X)$. Let Y be a smooth projective k-variety such that $\nu_p(I(Y)) \geq \nu_p(I(X))$ and $\dim Y < \dim X$. Then there is no rational map $f : X \to Y$.*

Proof. A rational map f gives $\Gamma_f \in \mathrm{CH}(X \times Y)$ of degree 1 over X, so $\dim Y \geq \dim X$ (theorem (1)).

Take $s_J^{(p)} = s_{p^n-1}$. An easy calculation gives

Lemma *Let X be a degree p hypersurface in $\mathbb{P}^{p^n}$. Then $s_{p^n-1}(X) = p^{p^n-1} - p^n - 1$. If $p|I(X)$, then $s_{p^n-1} \not\equiv 0 \mod I(X)$.*

Corollary (Hoffmann) *Let X_1, X_2 be anisotropic quadrics over k with X_2 isotropic over $k(X_1)$. Then* $\dim X_1 \geq 2^n - 1 \implies \dim X_2 \geq 2^n - 1$.

Proof. X_2 is isotropic over $k(X_1) \implies$ there is a rational map $f : X_1 \to X_2$.

May assume $\dim X_1 = 2^n - 1$ (take general hyperplane sections).

X_1, X_2 anisotropic $\implies I(X_1) = I(X_2) = (2)$ (Springer's theorem).

The lemma for $p = 2 \implies s_{2^n-1}(X_1) \not\equiv 0 \mod I(X_1)$.

Merkurjev's corollary $\implies \dim X_2 \geq 2^n - 1$.

Corollary (Izhboldin) *Let X_1, X_2 be anisotropic quadrics over k with X_2 isotropic over $k(X_1)$ and with $\dim X_1 \geq \dim X_2 = 2^n - 1$. If X_2 is isotropic over $k(X_1)$, then X_1 is isotropic over $k(X_2)$.*

Proof. May assume $\dim X_1 = \dim X_2 = 2^n - 1$.

X_2 is isotropic over $k(X_1) \Longrightarrow$ there is a rational map

$$f : X_1 \to X_2.$$

By theorem (2), there is a correspondence $\gamma' \in \mathrm{CH}(X_1 \times X_2)$ of odd degree over X_2, i.e.:

X_1 has a point over an odd degree extension of $k(X_2)$

By Springer's theorem, X_1 is isotropic over $k(X_2)$.