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Optimization of the flow of dividends

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Contents

§1. Introduction

1. In the recent papers [1], [2] Radner and Shepp considered a model of the evolution of the capital $X = (X_t)_{t\geqslant 0}$ of a company assuming that

$$dX_t = \mu\,dt + \sigma\,dW_t - dZ_t, \tag{1.1}$$

where $W = (W_t)_{t\geqslant 0}$ is a standard Wiener process, and the coefficients (μ, σ) can be chosen in a predictable way as functions of the data observed, with values in an a priori admissible set A. The non-negative non-decreasing non-anticipating process $Z = (Z_t)_{t\geqslant 0}$ appearing in (1.1) characterizes a *strategy of payment of dividends* by the company.

We assume that the initial capital is non-negative, $X_0 = x \geqslant 0$, and after X hits zero we have bankruptcy and $dX_t = dZ_t = 0$ for $t \geqslant \tau$, where τ is the moment of bankruptcy.

As a criterion for optimal functioning of the company Radner and Shepp consider the quantity

$$V(x) = \sup \mathsf{E}_x \int_0^\infty e^{-\lambda t}\,dZ_t, \qquad \lambda > 0, \tag{1.2}$$

where E_x is the mathematical expectation corresponding to $X_0 = x$,

$$\int_0^\infty e^{-\lambda t}\,dZ_t = Z_0 + \int_{(0,\infty)} e^{-\lambda t}\,dZ_t, \tag{1.3}$$

and sup is taken over all admissible strategies from the set A and admissible dividend processes $Z = (Z_t)_{t\geqslant 0}$.

2. In the present paper we consider the Radner–Shepp model (1.1) assuming that the set A is *one-element*, $A = \{(\mu, \sigma)\}$ with $\mu > 0$, $\sigma > 0$. We want to find optimal dividend processes $Z = (Z_t)_{t\geqslant 0}$ under the following assumptions on their structure.

A. Processes $Z = (Z_t)_{t\geqslant 0}$ are such that

$$dZ_t = u(X_t)\,dt, \qquad Z_0 = Z_0(x), \tag{1.4}$$

where $u = u(x)$, $Z_0 = Z_0(x)$ are arbitrary measurable functions satisfying $0 \leqslant u(x) \leqslant K < \infty$, $0 \leqslant Z_0(x) \leqslant x$.

B. Processes $Z = (Z_t)_{t\geqslant 0}$ are such that

$$Z_t = \sum_{i\geqslant 0} e^{-\lambda T_i}\mathfrak{z}_i I(T_i \leqslant t), \tag{1.5}$$

where $0 = T_0 < T_1 < T_2 < \dots$ are (random) moments of payments of dividends, and $\mathfrak{z}_0, \mathfrak{z}_1, \dots$ are non-negative amounts of dividends paid. In addition, we assume that there is a fee (transaction cost) $\gamma > 0$ for each payment and the cost function has the form

$$V(x) = \sup \mathsf{E}_x \sum_{i\geqslant 0} e^{-\lambda T_i}(\mathfrak{z}_i - \gamma), \tag{1.6}$$

where sup is taken over all multivariant point processes $(T_i, \mathfrak{z}_i)_{i\geqslant 0}$ (see [3]).

C. The process $Z = (Z_t)_{t\geqslant 0}$ is an arbitrary non-negative non-decreasing non-anticipating process, right-continuous for $t > 0$.

The solution of the problem of finding the structure of the optimal payment process given in [1], [2] in the general case **C** will be established below, making use of the ideas concerned with local time and diffusion with reflection. What we do is the same as in [1], [2], but the stochastic analysis technique is somewhat different.

The relevance of cases **A** and **B**, in addition to their natural importance, is that they suggest the structure of optimal solution in the general case **C** (by the limit passage with $K \to \infty$ and $\gamma \to 0$ in **A** and **B**, respectively).

The results corresponding to the three cases **A**, **B**, **C** are presented in §2, §3, §4 respectively.

§2. The case A

1. Let $W = (W_t)_{t\geqslant 0}$ be a standard Wiener process given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathsf{P})$. We assume that the evolution of the company's capital is described by the equation

$$dX_t = (\mu - u(X_t))\,dt + \sigma\,dW_t, \tag{2.1}$$

where $u = u(x)$ is an arbitrary measurable function with $0 \leqslant u(x) \leqslant K < \infty$ (K is a given constant). We note that by Zvonkin's result [6], the stochastic differential equation (2.1) has a unique strong solution $X = (X_t)_{t\geqslant 0}$ such that X_t are $\mathcal{F}_t^W \equiv \sigma(W_s; s \leqslant t)$-measurable, $t > 0$. In this way (2.1) defines the controlled process $X = X^u$ by means of the control $u = u(x)$.

Drawing on the meaning of the model described in §1 we assume that $x \geqslant 0$, and if $\tau = \inf\{t : X_t = 0\}$, then $X_t(\omega) = 0$ for all $t \geqslant \tau(\omega)$ (formally, $X = 0$ on $[\tau, \infty)$, see [3]). Here the equation (2.1) for X 'exists' up to the first instant of X hitting zero.

Writing (1.4) in the integral form

$$Z_t = Z_0(x) + \int_0^t u(X_s)\,ds, \tag{2.2}$$

we find that

$$X_t = (x - Z_0(x)) + \int_0^t (\mu - u(X_s))\,ds + \sigma W_t \tag{2.3}$$

(for all $0 \leqslant t \leqslant \tau$).

Let

$$V(x;u;Z_0) = Z_0(x) + \mathsf{E}_{x-Z_0(x)} \int_0^\tau e^{-\lambda t} u(X_t)\,dt, \tag{2.4}$$

$$V(x) = \sup V(x;u;Z_0), \tag{2.5}$$

where sup is taken over all admissible $u(x)$ and $Z_0(x)$ ($0 \leqslant u(x) \leqslant K$, $0 \leqslant Z_0(x) \leqslant x$).

It is clear that

$$V(x) = \max_{0 \leqslant Z_0(x) \leqslant x} \{Z_0(x) + V_0(x - Z_0(x))\}, \tag{2.6}$$

where

$$V_0(x) = \sup V(x;u), \tag{2.7}$$

and sup is taken over all admissible u; $Z_0(x) \equiv 0$, $V(x;u) = V(x;u,0)$.

Clearly, the most difficult task is to find the function $V_0(x)$ and the corresponding optimal control $u_0 = u_0(x)$. This is why we now assume that $Z_0 \equiv 0$ and we return to the general case in **6** below. To simplify the notation we denote $V_0(x)$ by $V(x)$, omitting the index 0.

2. So, suppose that

$$V(x) = \sup \mathsf{E}_x \int_0^\tau e^{-\lambda t} u(X_t)\,dt,$$

where sup is taken over all controls $u = u(x)$ with $0 \leqslant u(x) \leqslant K$.

Using standard techniques of stochastic control theory (see, for example, [5], [7]), to get the required function $V = V(x)$ and the corresponding optimal control $\widetilde{u} = \widetilde{u}(x)$ with $V(x;\widetilde{u}) = V(x)$, it is sufficient to establish the following *testing properties*:

(A_1) there is a function $\widetilde{V} = \widetilde{V}(x)$ such that for any admissible control $u = u(x)$

$$V(x;u) \not\geqslant \widetilde{V}(x), \qquad x \geqslant 0$$

$\leqslant$

(and so $V(x) \leqslant \widetilde{V}(x)$), and

(A_2) there is a control $\widetilde{u} = \widetilde{u}(x)$ such that

$$V(x;\widetilde{u}) = \widetilde{V}(x), \qquad x \geqslant 0.$$

Clearly, if such a function $\widetilde{V}(x)$ exists, then $\widetilde{V}(x) = V(x)$ and as optimal we can take the control $\widetilde{u} = \widetilde{u}(x)$.

The function $\widetilde{V} = V(x)$ and control $\widetilde{u} = \widetilde{u}(x)$ can be found by means of Bellman's equation. The verification of the properties $(A_1), (A_2)$ is usually done by means of Itô's formula using the martingale properties of stochastic integrals.

Let us realize our plan of finding $\widetilde{V}$ and $\widetilde{u}$. We introduce some operators $L(u)$, $0 \leqslant u \leqslant K$, acting on $\Psi = \Psi(x)$, $x \geqslant 0$, of class $C^2(0, \infty)$, given by

$$L(u)\Psi = (\mu - u)\frac{d\Psi}{dx} + \frac{\sigma^2}{2}\frac{d^2\Psi}{dx^2} - \lambda\Psi. \tag{2.8}$$

Suppose we have found a bounded function $\widetilde{V} \in C^2(0, \infty)$ with $\widetilde{V}(0) = 0$ such that *Bellman's inequality* is satisfied:

$$\sup_{0 \leqslant u \leqslant K} \left[L(u)\widetilde{V}(x) + u\right] \leqslant 0. \tag{2.9}$$

We show that then property (A_1) is satisfied, that is, for any admissible control $u = u(x)$ we have $V(x; u) \leqslant \widetilde{V}(x)$, $x \geqslant 0$.

To this end we apply Itô's formula to $(e^{-\lambda t}\widetilde{V}(X_t))_{t \geqslant 0}$:

$$e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) = \widetilde{V}(X_0) + \int_0^{t\wedge\tau} e^{-\lambda s}L(u)\widetilde{V}(X_s)\,ds + \int_0^{t\wedge\tau} e^{-\lambda s}\sigma\widetilde{V}'(X_s)\,dW_s.$$

Hence, taking E_x and taking into account $X_0 = x$ and (2.9), we find that

$$\begin{aligned}
\widetilde{V}(x) &= \mathsf{E}_x e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) \\
&\quad - \mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s}L(u)\widetilde{V}(X_s)\,ds - \mathsf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\widetilde{V}'(X_s)\,dW_s \\
&\geqslant \mathsf{E}_x e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) \\
&\quad + \mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s}u(X_s)\,ds - \mathsf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\widetilde{V}'(X_s)\,dW_s.
\end{aligned} \tag{2.10}$$

Hence we see that if $\widetilde{V} = \widetilde{V}(x)$ is such that $|\widetilde{V}'(x)| \leqslant C$ for some constant $C \geqslant 0$, then the stochastic integral in (2.10) is a martingale and its mathematical expectation is zero.

Letting $t \to \infty$ in (2.10) we find that

$$\mathsf{E}_x e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) \to 0$$

(if $\tau < \infty$, then $\widetilde{V}(X_\tau) = 0$, and if $\tau = \infty$, then $\widetilde{V}(X_t)$ is bounded and $e^{-\lambda(t\wedge\tau)} \to 0$ as $t \to \infty$).

Thus from (2.10) we find that

$$\widetilde{V}(x) \geqslant V(x; u),$$

that is, the testing condition (A_2) is satisfied.

3. We note that in the computations performed in (2.10) *inequality* appeared as a result of the assumption that $\widetilde{V} = \widetilde{V}(x)$ satisfies *Bellman's inequality* (2.9).

Suppose now that in (2.9) equality takes place, that is, *Bellman's equation* is satisfied:

$$\sup_{0\leqslant u\leqslant K}\left[L(u)\widetilde{V}(x) + u\right] = 0. \tag{2.11}$$

By (2.8) this equation is equivalent to

$$\left(\mu\frac{d\widetilde{V}}{dx} + \frac{\sigma^2}{2}\frac{d^2\widetilde{V}}{dx^2} - \lambda\widetilde{V}\right) + \sup_{0\leqslant u\leqslant K}\left[u\left(1 - \frac{d\widetilde{V}}{dx}\right)\right] = 0. \tag{2.12}$$

Hence we can see that if the function $\widetilde{V} = \widetilde{V}(x)$ is known, then the control $\widetilde{u} = \widetilde{u}(x)$ on which (2.12) is satisfied should be the following:

$$\widetilde{u}(x) = \begin{cases} K, & 1 - \dfrac{d\widetilde{V}}{dx} \geqslant 0, \\ 0, & 1 - \dfrac{d\widetilde{V}}{dx} < 0. \end{cases} \tag{2.13}$$

Using this we find from (2.12) that for x such that $\widetilde{u}(x) = 0$ the following equation should hold:

$$\mu\widetilde{V}'(x) + \frac{\sigma^2}{2}\widetilde{V}''(x) - \lambda\widetilde{V}(x) = 0, \tag{2.14}$$

and if $\widetilde{u}(x) = K$, then we have

$$\mu\widetilde{V}'(x) + \frac{\sigma^2}{2}\widetilde{V}''(x) - \lambda\widetilde{V}(x) + K(1 - \widetilde{V}'(x)) = 0. \tag{2.15}$$

From the intuitive considerations about the structure of the optimal control $\widetilde{u} = \widetilde{u}(x)$ we can assume that there is an $\widetilde{x}$ such that for $x \geqslant \widetilde{x}$ we have to use $\widetilde{u}(x) = K$ (that is, to pay dividends with maximal possible speed if capital is 'large') and for $x < \widetilde{x}$ we put $\widetilde{u}(x) = 0$ (that is, we do not pay a dividend if the capital is 'small').

Thus, we seek the function $\widetilde{V} = \widetilde{V}(x)$ and the threshold $\widetilde{x}$ of switching the equations as a solution to the following Stefan problem with free boundary $\widetilde{x}$:

$$\mu\widetilde{V}'(x) + \frac{\sigma^2}{2}\widetilde{V}''(x) - \lambda\widetilde{V}(x) = 0, \qquad x < \widetilde{x}, \tag{2.16}$$

$$(\mu - K)\widetilde{V}'(x) + \frac{\sigma^2}{2}\widetilde{V}''(x) - \lambda\widetilde{V}(x) + K = 0, \qquad x > \widetilde{x}. \tag{2.17}$$

4. For $x > 0$ we consider the equation

$$(\mu - K)U'(x) + \frac{\sigma^2}{2}U''(x) - \lambda U(x) + K = 0,$$

equivalent to

$$\frac{\mu - K}{\sigma^2/2} U'(x) + U''(x) - \frac{\lambda}{\sigma^2/2} U(x) + \frac{K}{\sigma^2/2} = 0. \tag{2.18}$$

Clearly, in the investigation of the properties of $U(x)$ we can put $\sigma^2/2 = 1$ from the very beginning, which results in replacing λ, μ, K by $\lambda/(\sigma^2/2)$, $\mu/(\sigma^2/2)$, $K/(\sigma^2/2)$ in the final result. The general solution of (2.18) (with $\sigma^2/2 = 1$) has the form

$$U(x) = C_1 e^{\rho_1 x} + C_2 e^{\rho_2 x} + \frac{K}{\lambda}, \tag{2.19}$$

where

$$\rho_1 = \frac{K - \mu}{2} + \sqrt{\left(\frac{K-\mu}{2}\right)^2 + \lambda},$$

$$\rho_2 = \frac{K - \mu}{2} - \sqrt{\left(\frac{K-\mu}{2}\right)^2 + \lambda}$$

are the roots of the quadratic equation

$$\rho^2 + (\mu - K)\rho - \lambda = 0$$

(here $\rho_1 > 0$, $\rho_2 < 0$ since $\lambda > 0$).

From (2.7) it follows that $\widetilde{V}(x) \leqslant K/\lambda$ (since $u(x) \leqslant K$). Hence among the solutions in (2.19) we should choose bounded ones, which gives $C_1 = 0$. By the interpretation of the problem the function $\widetilde{V}(x)$ should be non-decreasing in x. Therefore, the required solution $\widetilde{V}(x)$ (for $x > \widetilde{x}$) should have the form

$$U(x) = \frac{K}{\lambda} - Be^{\rho_2 x}, \tag{2.19a}$$

where the constant (so far unknown) $B \geqslant 0$.

In the domain $0 < x < \widetilde{x}$ the required function $\widetilde{V} = \widetilde{V}(x)$ satisfies (2.16) and so it is of the form

$$U(x) = A_1 e^{r_1 x} + A_2 e^{r_2 x}, \tag{2.19b}$$

where

$$r_1 = -\frac{\mu}{2} + \sqrt{\left(\frac{\mu}{2}\right)^2 + \lambda}, \qquad r_2 = -\frac{\mu}{2} - \sqrt{\left(\frac{\mu}{2}\right)^2 + \lambda}.$$

Since we should have $U(0) = 0$, from (2.19b) we find that $A_1 + A_2 = 0$ and so the required solution belongs to the family

$$U(x) = A_1 e^{-\frac{\mu}{2}x}(e^{\Delta x} - e^{-\Delta x}),$$

where A_1 is a constant, $\Delta = \sqrt{(\mu^2/2)^2 + \lambda}$.

Putting $A = 2A_1$ we see that

$$U(x) = Ae^{-\frac{\mu}{2}x} \sinh(\Delta x). \tag{2.20}$$

Thus we have three unknown constants A, B, and $\widetilde{x}$. In addition, in the domain $x < \widetilde{X}$ the representation (2.20) 'acts' and in $x > \widetilde{x}$ we have (2.19a).

We shall seek the unknown constants $A, B, \widetilde{x}$ from the following supplementary conditions at $\widetilde{x}$:

$$U(\widetilde{x}-) = U(\widetilde{x}+), \qquad U'(\widetilde{x}-) = U'(\widetilde{x}+), \qquad U''(\widetilde{x}-) = U''(\widetilde{x}+). \tag{2.21}$$

The first condition is simply the continuity of the function at the border $\widetilde{x}$ and is quite natural. The second condition is the so-called (heuristic) 'smooth gluing condition' (in this respect see, for example, [7]). Finally, the condition that the second derivative be continuous at $\widetilde{x}$ is motivated by the requirement $U \in C^2(0, \infty)$ for applying Itô's formula.

We note that from (2.14) and (2.15) it is clear that the system of conditions (2.21) is equivalent to the system

$$U(\widetilde{x}-) = U(\widetilde{x}+), \qquad U'(\widetilde{x}-) = 1, \qquad U'(\widetilde{x}+) = 1. \tag{2.22}$$

In the domain $x < \widetilde{x}$ from (2.20) we find that

$$U'(x) = Ae^{-\frac{\mu}{2}x}\left\{\Delta\cosh(\Delta x) - \frac{\mu}{2}\sinh(\Delta x)\right\}, \tag{2.23}$$

$$U''(x) = Ae^{-\frac{\mu}{2}x}\left\{\left(\left(\frac{\mu}{2}\right)^2 + \Delta^2\right)\sinh(\Delta x) - \mu\Delta\cosh(\Delta x)\right\}. \tag{2.24}$$

By (2.19a) in $x > \widetilde{x}$ we have

$$U'(x) = -B\rho_2 e^{\rho_2 x}, \qquad U''(x) = -B\rho_2^2 e^{\rho_2 x}. \tag{2.25}$$

Hence (2.22) takes the form

$$\begin{aligned} &Ae^{-\frac{\mu}{2}\widetilde{x}}\sinh(\Delta\widetilde{x}) = \frac{K}{\lambda} - Be^{\rho_2\widetilde{x}}, \\ &Ae^{-\frac{\mu}{2}\widetilde{x}}\left\{\Delta\cosh(\Delta\widetilde{x}) - \frac{\mu}{2}\sinh(\Delta\widetilde{x})\right\} = 1, \\ &-B\rho_2 e^{\rho_2\widetilde{x}} = 1. \end{aligned} \tag{2.26}$$

By the last equation $Be^{\rho_2\tilde{x}} = -1/\rho_2$. Therefore, dividing the second equation in (2.26) by the first one we find that $\widetilde{x}$ is a solution of

$$\tanh(\Delta\widetilde{x}) = \frac{2(\lambda + K\rho_2)\Delta}{\mu(\lambda + K\rho_2) + 2\lambda\rho_2}. \tag{2.27}$$

This equation was obtained under the assumption that $0 < \widetilde{x} < \infty$. We discuss the question of when the solution of (2.29) really satisfies this condition.

To this end we fix the values of $\mu > 0$, $\lambda > 0$ and let $K \to \infty$. Then

$$\rho_2 \sim -\frac{\lambda}{K - \mu},$$

and the right-hand side of (2.27) converges to

$$\frac{\mu\sqrt{\mu^2+4\lambda}}{\mu^2+2\lambda},$$

which is greater than zero (since $\mu > 0$) and less than one (since $\lambda > 0$).

Thus, for large K equation (2.27) has a solution $\widetilde{x} = \widetilde{x}(K)$, which is in fact unique, and

$$\widetilde{x}(K) \to \widetilde{x}(\infty),$$

where $\widetilde{x}(\infty)$ is a solution of

$$\tanh(\Delta x) = \frac{\mu\Delta}{2\Delta^2-\lambda}, \tag{2.28}$$

with $\Delta = \sqrt{(\mu/2)^2+\lambda}$.

On the other hand, if $K \to \infty$, then

$$\rho_2 \to r_2 = -\frac{\mu}{2} - \sqrt{\lambda + \left(\frac{\mu}{2}\right)^2},$$

and the right-hand side of (2.27) converges to -1. Consequently, by the properties of $\tanh x$, equation (2.27) does not have a positive solution for small K.

Thus it becomes clear that for the existence of a solution $0 < \widetilde{x} < \infty$ the parameters λ, μ, K should satisfy

$$0 < \frac{2(\lambda + K\rho_2)\Delta}{\mu(\lambda+K\rho_2)+2\lambda\rho_2} < 1. \tag{2.29}$$

For our purpose it is sufficient (with fixed $\lambda > 0$ and $\mu > 0$) to find out for which K_* the root of the equation (2.27) becomes zero. We can see directly from (2.27) that the condition $\lambda + K\rho_2 = 2$ should hold, that is, K_* should be a root of the equation

$$\frac{\lambda}{K_*} = |\rho_2(K_*)|, \tag{2.30}$$

where

$$\rho_2(K) = \frac{K-\mu}{2} - \sqrt{\left(\frac{K-\mu}{2}\right)^2 + \lambda}.$$

(Such a number K_* exists and is unique, which follows from the properties of the functions λ/K and $|\rho_2(K)|$.)

Thus if $K = K_*$, then $\widetilde{x}(K_*) = 0$ and the function we seek is

$$U(x) = \frac{K_*}{\lambda} - Be^{-|\rho_2(K_*)|x}.$$

Since we should have $U(0) = 0$, it follows that $B = K_*/\lambda$, and so for $K = K_*$

$$U(x) = \frac{K_*}{\lambda}\left(1 - e^{-|\rho_2(K_*)|x}\right). \tag{2.31}$$

Next we show that (for $K = K_*$) the function $U(x)$ in fact coincides with the function $V_0(x)$ introduced in (2.7). Therefore (see (2.6)),

$$V(x) = \max_{0 \leqslant Z_0(x) \leqslant x} \{Z_0(x) + U(x - Z_0(x))\}. \tag{2.32}$$

From (2.31) we deduce that in (2.32) the maximum is reached when $Z_0(x) = x$. Thus if $K = K_*$, then it is appropriate to pay a dividend of the size of the available capital x. So in this case $\tau = 0$, $Z_0(x) = x$, and $X_t = 0$ for $t > 0$.

5. And so we assume that $K > K_*$. In this case, as was shown above, the Stefan problem (2.16)–(2.17) with boundary conditions (at $\widetilde{x}$) (2.22) (or, equivalently, (2.21)) has a solution where $\widetilde{x}$ is defined by (2.27) and

$$U(x) = \begin{cases} Ae^{-\frac{a}{2}x}\sinh(\Delta x), & x < \widetilde{x}, \\ \dfrac{K}{\lambda} - Be^{\rho_2 x}, & x \geqslant \widetilde{x}. \end{cases} \tag{2.33}$$

where

$$B = \frac{1}{|\rho_2|} e^{|\rho_2|\widetilde{x}}$$

(from the third equation in (2.26)), and the constant A is obtained from either of the first two equations in (2.26). To emphasize the dependence of $U(x)$ on K we shall also write $U(x; K)$.

Now we give a proof of the fact that for $K > K_*$ the function $U(x)$ from (2.33) indeed coincides with the function $V(x)$ defined in (2.5) and, moreover, the optimal strategy of payment of dividends is this: $Z_0(x) = 0$ for all $x \geqslant 0$ and

$$\widetilde{u}(x) = \begin{cases} 0, & x < \widetilde{x}, \\ K, & x \geqslant \widetilde{x}. \end{cases}$$

Applying Itô's formula to $U(x)$ from (2.33) we find that (cf. (2.10)) for the control $\widetilde{u} = \widetilde{u}(x)$ we have

$$\begin{aligned} U(x) &= \mathsf{E}_x e^{-\lambda(t\wedge\tau)} U(X_{t\wedge\tau}) \\ &\qquad - \mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} L(\widetilde{u}) U(X_s)\,ds - \mathsf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s} U'(X_s)\,dW_s \\ &= \mathsf{E}_x e^{-\lambda(t\wedge\tau)} U(X_{t\wedge\tau}) \\ &\qquad + \mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} \widetilde{u}(X_s)\,ds - \mathsf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s} U'(X_s)\,dW_s. \end{aligned} \tag{2.34}$$

Since $0 \leqslant U'(x) \leqslant C$, the mathematical expectation of the last term in (2.34) is zero, and going to the limit as $t \to \infty$ we find that

$$U(x) = \mathsf{E}_x \int_0^\tau e^{-\lambda s} \widetilde{u}(X_s)\,ds.$$

Thus, given that $Z_0(x) = 0$, we get $U(x) = V(x)$ and the control $\widetilde{u} = \widetilde{u}(x)$ is optimal.

6. Now we consider the possibility of instantaneous payment of dividend ($Z_0(x) \neq 0$). Then by (2.6)

$$V(x) = \max_{0 \leqslant Z_0(x) \leqslant x} \{Z_0(x) + U(x - Z_0(x))\}. \tag{2.35}$$

From the properties of $U(x)$ it follows that if $x < \widetilde{x}$, then $U'(x) \geqslant 1$ and $U'(\widetilde{x}) = 1$. Therefore, for $x < \widetilde{x}$ the maximum in (2.35) is reached when $Z_0(x) = 0$ and, consequently, instantaneous payment of dividend should not be made. However, if $x > \widetilde{x}$, then $U'(x) < 1$ and $U(\widetilde{x}) > \widetilde{x}$. The maximum in (2.35) is reached when $Z_0(x) = x - \widetilde{x}$ and

$$V(x) = (x - \widetilde{x}) + U(\widetilde{x}). \tag{2.36}$$

In other words, in the case $x > \widetilde{x}$ the dividend of $x - \widetilde{x}$ should be paid at once and the process X should then start from the state $X_0 = \widetilde{x}$.

Collecting all the results proved, we formulate them in the following assertion.

Theorem A. *In the model A (see (1.4)) the optimal process of payment of dividend and the function $V = V(x)$ (see (1.2)) are described in the following way. Let $\lambda > 0$, $\mu > 0$.*

1) If $K \leqslant K_*$, where K_* is the root of equation (2.30), then having the initial capital $X_0 = x$ we pay the dividend $Z_0(x) = x$ at once and $X_t = 0$ for all $t > 0$. The function V is given by $V(x) = x$.

2) If $K > K_*$, then

$$Z_0(x) = \begin{cases} 0, & x < \widetilde{x}(K), \\ x - \widetilde{x}(K), & x \geqslant \widetilde{x}(K), \end{cases}$$

where $\widetilde{x}(K)$ is the root of (2.27). The optimal function is

$$\widetilde{u}(x) = \begin{cases} 0, & x < \widetilde{x}(K), \\ K, & x \geqslant \widetilde{x}(K). \end{cases}$$

In addition

$$V(x) = \begin{cases} U(x, K), & x < \widetilde{x}(K), \\ (x - \widetilde{x}(K)) + U(\widetilde{x}(K), K), & x \geqslant \widetilde{x}(K), \end{cases}$$

where

$$U(x, K) = \begin{cases} Ae^{-\frac{\mu}{2}x} \sinh(\Delta x), & x < \widetilde{x}(K), \\ \frac{K}{\lambda} - Be^{-|\rho_2| x}, & x \geqslant \widetilde{x}(K), \end{cases}$$

and the constants A, B are defined by (2.26). At $\widetilde{x}(K)$ we have

$$V(\widetilde{x}(K)) = \frac{\mu}{\lambda}. \tag{2.37}$$

Remark 1. Assertion (2.37) follows by continuity from (2.16) taking account of $V'(\widetilde{x}) = 1$ and $V''(\widetilde{x}, K) = 0$.

Remark 2. If $K \to \infty$, then

$$\widetilde{x}(K) \to \widetilde{x}(\infty),$$

where $\widetilde{x}(\infty)$ is the root of (2.28) and $U(x; K) \to \overline{V}(x)$, where $\overline{V}(x)$ is defined below in (4.5).

§3. The case B

1. In this case the dividend payment moments $0 = T_0 < T_1 < \dots$ and the amounts paid $\mathfrak{z}_0, \mathfrak{z}_1, \dots$ form a multivariant point process $(T_i, \mathfrak{z}_i)_{i \geqslant 0}$. Here

$$Z_t = \sum_{i \geqslant 0} e^{-\lambda T_i} \mathfrak{z}_i I(T_i \leqslant t). \tag{3.1}$$

We assume that (3.1) 'acts' for $t < \tau = \inf\{s : X_s = 0\}$ and

$$V(x) = \sup \mathsf{E}_x \sum_{i \geqslant 0} e^{-\lambda T_i} (\mathfrak{z}_i - \gamma) I(T_i \leqslant t), \tag{3.2}$$

where $\gamma > 0$ is interpreted as the transaction cost for each payment of dividend, which corresponds to frequent 'switching' of the evolution process of the capital $X = (X_t)_{t \geqslant 0}$ having stochastic differential

$$dX_t = \mu\, dt + \sigma\, dW_t - d Z_t. \tag{3.3}$$

If $\mathcal{F}_t^X = \sigma(X_s, s \leqslant t)$, then we assume that the moments T_i are Markov moments (stopping times) with respect to $(\mathcal{F}_t^X)_{t \geqslant 0}$ and the random variables $\mathfrak{z}_i$ are $\mathcal{F}_{T_i}^X$-measurable.

2. We shall find the function $V(x)$ and the optimal strategy $\widetilde{\pi} = (\widetilde{T}_i, \widetilde{\mathfrak{z}}_i)_{i \geqslant 0}$ following the same ideas as in §2 based on 'testing properties'.

To this end we assume that a certain function $\widetilde{V} = \widetilde{V}(x)$ of class $C^2(0, \infty)$ is considered as a 'candidate' for $V(x)$. Then for a strategy π (with $\mathfrak{z}_0 = 0$) and the corresponding process X we have ($\tau = \inf\{s : X_s = 0\}$)

$$\begin{aligned}
e^{-\lambda(t \wedge \tau)} \widetilde{V}(X_{t \wedge \tau}) &= \widetilde{V}(X_0) + \int_0^{t \wedge \tau} (-e^{-\lambda s} \lambda \widetilde{V}(X_s))\, ds \\
&\quad + \int_0^{t \wedge \tau} e^{-\lambda s} V'(X_{s-})\, dX_s + \frac{1}{2} \int_0^{t \wedge \tau} e^{-\lambda s} \widetilde{V}''(X_s) \sigma^2\, ds \\
&\quad + \sum_{0 < s \leqslant t \wedge \tau} e^{-\lambda s} \{\widetilde{V}(X_s) - \widetilde{V}(X_{s-}) - \Delta X_s \widetilde{V}'(X_{s-})\} \\
&= \widetilde{V}(X_0) + \int_0^{t \wedge \tau} e^{-\lambda s} L\widetilde{V}(X_s)\, ds + \int_0^{t \wedge \tau} \sigma e^{-\lambda s} \widetilde{V}'(X_s)\, dW_s \\
&\quad + \sum_{i \geqslant 1} e^{-\lambda T_i} [\widetilde{V}(X_{T_i}) - \widetilde{V}(X_{T_i -})] I(T_i \leqslant t \wedge \tau) \\
&= \widetilde{V}(X_0) + \int_0^{t \wedge \tau} e^{-\lambda s} L\widetilde{V}(X_s)\, ds \\
&\quad + \int_0^{t \wedge \tau} \sigma e^{-\lambda s} \widetilde{V}'(X_s)\, dW_s - \sum_{i \geqslant 1} e^{-\lambda (T_i \wedge \tau)} (\mathfrak{z}_i - \gamma) \\
&\quad + \sum_{i \geqslant 1} e^{-\lambda T_i} [\widetilde{V}(X_{T_i}) - \widetilde{V}(X_{T_i -}) - (\Delta X_{T_i} + \gamma)] I(T_i \leqslant \tau \wedge t).
\end{aligned} \tag{3.4}$$

where

$$L\widetilde{V}(x) \equiv \mu \widetilde{V}'(x) + \frac{\sigma^2}{2}\widetilde{V}''(x) - \lambda \widetilde{V}(x),$$

and we have used the fact that $\Delta X_{T_i} = -\mathfrak{z}_i$.

From (3.4) we can see that

$$\begin{aligned}\widetilde{V}(x) = \mathsf{E}_x e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) - \mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} L\widetilde{V}(X_s)\,ds \qquad (3.5)\\ - \mathsf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\widetilde{V}'(X_s)\,dW_s + \mathsf{E}_x \sum_{i\geqslant 1} e^{-\lambda T_i}(\mathfrak{z}_i - \gamma) I(T_i \leqslant t\wedge\tau)\\ - \mathsf{E}_x \sum_{i\geqslant 1} e^{-\lambda T_i}\big[\widetilde{V}(X_{T_i}) - \widetilde{V}(X_{T_i-}) - \Delta X_{T_i} - \gamma\big] I(T_i \leqslant t\wedge\tau).\end{aligned}$$

If we assume that the function $\widetilde{V} = \widetilde{V}(x)$ satisfies (cf. (2.9))

$$L\widetilde{V}(x) \leqslant 0, \qquad x > 0, \tag{3.6}$$

and

$$\widetilde{V}(x) - \widetilde{V}(y) \geqslant (x - y) - \gamma, \qquad x \geqslant y, \tag{3.7}$$

then from (3.5) we find that

$$\begin{aligned}\widetilde{V}(x) \geqslant \mathsf{E}_x \sum_{i\geqslant 1} e^{-\lambda(T_i\wedge\tau)}(\mathfrak{z}_i - \gamma)\\ + \mathsf{E}_x e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) - \mathsf{E}_x \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\widetilde{V}'(X_s)\,dW_s. \qquad (3.8)\end{aligned}$$

Assuming additionally that $\widetilde{V}'(x)$ and $\widetilde{V}(x)$ are bounded, we let $t \to \infty$ in (3.8) to obtain

$$\widetilde{V}(x) \geqslant \mathsf{E}_x \sum_{i\geqslant 1} e^{-\lambda T_i}(\mathfrak{z}_i - \gamma) I(T_i \leqslant \tau). \tag{3.9}$$

To make conditions (3.6), (3.7) more precise (they are necessary to find $\widetilde{V}(x)$ coinciding with $V(x)$) we refer to the following heuristic argument suggesting the structure of the optimal strategy $\pi = (T_i, \mathfrak{z}_i)_{i\geqslant 0}$.

If the initial capital $X_0 = x$ is 'large', then it seems appropriate to pay a certain dividend $\mathfrak{z}_0(x)$ at once and then to begin the observation of the evolution of X with the initial state $x - \mathfrak{z}_0(x)$. Then it is clear that as in (2.6) we have

$$V(x) = \max_{0\leqslant \mathfrak{z}_0(x)\leqslant x} \{(\mathfrak{z}_0(x) - \gamma) + V_0(x - \mathfrak{z}_0(x))\},$$

where $V_0(x)$ is a function coinciding with the right-hand side of (3.2) but with $i \geqslant 1$.

At the same time, the following strategy of dividend payment seems natural: choose two thresholds $a < b$ and when X reaches b, pay the dividend $b - a$, that is, at the moments $T_i = \inf\{t > T_{i-1} : X_{t-} = b\}$ we have $X_{T_i} = a$ and $\mathfrak{z}_i = -\Delta X_{T_i} = b - a$. Then clearly $V_0(x)$ should satisfy the following condition at a and b:

$$V_0(b) = V_0(a) + (b - a) - \gamma.$$

These considerations lead us to the natural idea of finding the required function $V = V(x)$ and the thresholds a, b as a solution of the following problem:

$$\begin{aligned} LV(x) &= 0, \qquad 0 < x < b, \\ V(x) &= V(a) + (x - a) - \gamma, \qquad x \geqslant b, \\ V(0) &= 0. \end{aligned} \tag{3.10}$$

Using the last condition ($V(0) = 0$), the solution of the equation $LV(x) = 0$ has the form

$$V(x) = Ae^{-\mu x/2}\sinh(\Delta x), \qquad 0 \leqslant x \leqslant b, \tag{3.11}$$

with $\Delta = \sqrt{(\mu/2)^2 + \lambda}$ (cf. (2.20)).

Thus we have three unknown constants: A, a, and b. Applying the concept of 'smooth gluing' to the condition $V(b) = V(a) + (b - a) - \gamma$, we add two more conditions at a and b

$$V'(a) = 1, \qquad V'(b) = 1. \tag{3.12}$$

We show that a, b, and A can now be found uniquely. Since

$$V(b) - V(a) = \int_a^b V'(y)\,dy,$$

the condition $V(b) - V(a) = (b - a) - \gamma$ takes the form

$$\int_a^b (1 - V'(y))\,dy = \gamma. \tag{3.13}$$

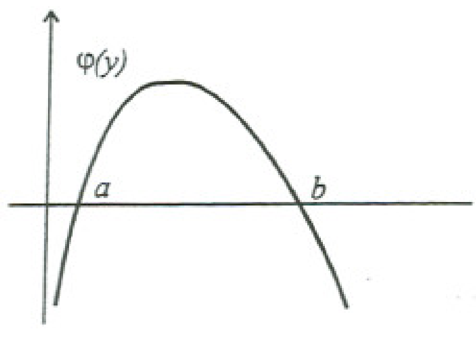
Let $\varphi(y; A) = 1 - V'(y)$. Then

$$\int_a^b \varphi(y; A)\,dy = \gamma \qquad \varphi(a; A) = 0, \qquad \varphi(b; A) = 0.$$

To show that this problem has a solution, we observe that (cf. (2.23), (2.24))

$$V'(x) = Ae^{-\frac{\mu}{2}x}\left\{\Delta\cosh(\Delta x) - \frac{\mu}{2}\sinh(\Delta x)\right\},$$

$$V''(x) = Ae^{-\frac{\mu}{2}x}\left\{\left(\left(\frac{\mu}{2}\right)^2 + \Delta^2\right)\sinh(\Delta x) - \mu\Delta\cosh(\Delta x)\right\}.$$



If $\overline{x}$ is a root of the equation

$$\tanh(\Delta x) = \frac{\mu\Delta}{(\mu/2)^2 + \Delta^2} \qquad \left(= \frac{\mu\Delta}{2\Delta^2 - \lambda} \right)$$

(cf. (2.28); $\overline{x} = \widetilde{x}(\infty)$), then $V''(\overline{x}) = 0$ and

$$V'(\overline{x}) = Ae^{-\frac{\mu}{2}\overline{x}}\Delta \cosh(\Delta\overline{x}) \frac{\lambda}{\lambda + \mu^2/2} > 0$$

if $A > 0$.

So the function $\varphi(x; A) = 1 - V'(x)$ has the properties: $\varphi(x; A) \downarrow -\infty$ as $A \uparrow \infty$ and $\varphi(x; A) \uparrow 1$ as $A \downarrow 0$ for each $x \geqslant 0$.

Thus, beginning with large A and decreasing it, we find unique values $\widetilde{A}, \widetilde{a}, \widetilde{b}$ with

$$\int_{\widetilde{a}}^{\widetilde{b}} \varphi(y; \widetilde{A})\, dy = \gamma, \qquad \varphi(\widetilde{a}; \widetilde{A}) = 0, \quad \varphi(\widetilde{b}; \widetilde{A}) = 0. \tag{3.14}$$

Let $\widetilde{V}(x)$ be the function (3.11) with $A = \widetilde{A}$. We define the strategy $\widetilde{\pi} = (\widetilde{T}_i, \widetilde{\mathfrak{z}}_i)_{i\geqslant 0}$ in the following way:

$$\widetilde{\mathfrak{z}}_0 = \begin{cases} x - \widetilde{a} & \text{if } x \geqslant \widetilde{b}, \\ 0 & \text{if } x < \widetilde{b}, \end{cases}$$

$$\widetilde{T}_i = \inf\{t > \widetilde{T}_{i-1} : X_{t-} = \widetilde{b}\},$$

$$\widetilde{\mathfrak{z}}_i = \widetilde{b} - a.$$

We show that for the strategy $\widetilde{\pi}$ we have constructed the corresponding value

$$V(x; \widetilde{\pi}) \equiv \mathsf{E}_x \sum_{i\geqslant 0} e^{-\lambda \widetilde{T}_i}(\widetilde{\mathfrak{z}}_i - \gamma) I(\widetilde{T}_i \leqslant \tau)$$

exactly coincides with the function $\widetilde{V}(x)$ found, and moreover, $\widetilde{V}(x) = V(x)$, that is, $\widetilde{\pi}$ is optimal.

To prove $\widetilde{V}(x) = V(x; \widetilde{\pi})$ we use Itô's formula (3.5), taking the process X defined by the strategy $\widetilde{\pi}$ and assuming that $x < \widetilde{b}$.

Since $L\widetilde{V}(x) = 0$, in (3.5) we have

$$\mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} L\widetilde{V}(X_s)\, ds = 0.$$

The mathematical expectation of the stochastic integral vanishes,

$$\mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s} \widetilde{V}'(X_s)\, dW_s = 0,$$

since the derivative $\widetilde{V}'(x)$ is bounded. Next, for $T_i \leqslant \tau$

$$\widetilde{V}(X_{T_i}) - \widetilde{V}(X_{T_i-}) - \Delta X_{T_i} - \gamma = \widetilde{V}(\widetilde{a}) - \widetilde{V}(\widetilde{b}) - (\widetilde{a} - \widetilde{b}) - \gamma = 0$$

by the construction of $\widetilde{a}$ and $\widetilde{b}$.

So, for each $t > 0$

$$\widetilde{V}(x) = \mathsf{E}_x e^{-\lambda(t\wedge\tau)}\widetilde{V}(X_{t\wedge\tau}) + \mathsf{E}_x \sum_{i\geqslant 0} e^{-\lambda \widetilde{T}_i}(\widetilde{\mathfrak{z}}_i - \gamma) I(T_i \leqslant t \wedge \tau).$$

Passing to the limit as $t \to \infty$ we find the required equality

$$\widetilde{V}(x) = V(x; \widetilde{\pi}), \qquad x < \widetilde{b}.$$

Now let $x \geqslant \widetilde{b}$. Then by (3.10),

$$\widetilde{V}(x) = \widetilde{V}(\widetilde{a}) + (x - \widetilde{a}) - \gamma.$$

At the same time, by the definition of the strategy $\widetilde{\pi} = (\widetilde{T}_i, \widetilde{\mathfrak{z}}_i)_{i\geqslant 0}$, for $x \geqslant \widetilde{b}$ we have

$$V(x; \widetilde{\pi}) = \widetilde{V}(\widetilde{a}; \widetilde{\pi}) + (x - \widetilde{a}) - \gamma.$$

But $\widetilde{V}(\widetilde{a}) = \widetilde{V}(\widetilde{a}; \widetilde{\pi})$. Therefore, $\widetilde{V}(x) = V(x; \widetilde{\pi})$ for all $x \geqslant \widetilde{b}$.

It remains to show that for *any* strategy π

$$\widetilde{V}(x) \geqslant V(x; \pi).$$

We showed above that for this it is sufficient to verify that

$$L\widetilde{V}(x) \leqslant 0, \qquad x > 0, \tag{3.15}$$

and

$$\widetilde{V}(x) - \widetilde{V}(y) \geqslant (x - y) - \gamma, \qquad x \geqslant \gamma. \tag{3.16}$$

In the domain $x < \widetilde{b}$ we have $L\widetilde{V}(x) = 0$. In the domain $x \geqslant \widetilde{b}$

$$\widetilde{V}(x) = \widetilde{V}(\widetilde{a}) + (x - \widetilde{a}) - \gamma$$

and

$$L\widetilde{V}(x) = -\lambda\big[\widetilde{V}(\widetilde{a}) + (x - \widetilde{a}) - \gamma\big] + \mu \leqslant -\lambda\big[\widetilde{V}(\widetilde{a}) + (b - \widetilde{a}) - \gamma\big] + \mu = -\lambda\widetilde{V}(\widetilde{b}) + \mu = 0,$$

since by continuity from $L\widetilde{V}(x) = 0$, $x < \widetilde{b}$, we obtain $L\widetilde{V}(\widetilde{b}) = 0$.

Thus, (3.15) has been established.

Finally, (3.16) is equivalent to the inequality

$$\int_x^y (1 - \widetilde{V}'(u))\, du \leqslant \gamma,$$

which is obviously satisfied by the properties of $\varphi(u; \widetilde{A}) = 1 - \widetilde{V}'(u)$ and the way the constant $\widetilde{A}$ was defined in the process of solving (3.14).

We formulate the results obtained in the following assertion.

Theorem B. *In the model* B *(see* (1.5)*) the optimal dividend payment strategy and the function* $V = V(x)$ *(see* (3.2)*) are described in the following way.*

Let $\lambda > 0$, $\mu > 0$, $\gamma > 0$ *and suppose that the constants* $\widetilde{A}, \widetilde{a}$, *and* $\widetilde{b}$ *are defined by solving* (3.14).

1) *If* $x \geqslant \widetilde{b}$, *then we make an instantaneous payment of the dividend* $\widetilde{\mathfrak{z}}_0 = \widetilde{b} - \widetilde{a}$ *and the evolution of the capital starts with the value* $\widetilde{a}$.
2) *If* $x < \widetilde{b}$, *then* $\widetilde{T}_0 = 0$, $\widetilde{\mathfrak{z}}_0 = 0$ *and the payment of the dividend is made when the process* X *reaches the threshold* $\widetilde{b}$ *with instantaneous payment of dividend of size* $\widetilde{b} - \widetilde{a}$, *that is,*

$$\widetilde{T}_i = \inf\{t > T_{i-1} : X_{t-} = \widetilde{b}\}, \qquad \widetilde{\mathfrak{z}}_i = \widetilde{b} - \widetilde{a}.$$

Remark 1. By (3.11) the problem (3.14) takes the following form:

$$\widetilde{A}\big[e^{-\frac{\mu}{2}\widetilde{b}}\sinh(\Delta\widetilde{b}) - e^{-\frac{\mu}{2}\widetilde{a}}\sinh(\Delta\widetilde{a})\big] = (\widetilde{b} - \widetilde{a}) - \gamma,$$

$$\widetilde{A}e^{-\frac{\mu}{2}\widetilde{a}}\Big\{\Delta\cosh(\Delta\widetilde{a}) - \frac{\mu}{2}\sinh(\Delta\widetilde{a})\Big\} = 1,$$

$$\widetilde{A}e^{-\frac{\mu}{2}\widetilde{b}}\Big\{\Delta\cosh(\Delta\widetilde{b}) - \frac{\mu}{2}\sinh(\Delta\widetilde{b})\Big\} = 1,$$

where $\Delta = \sqrt{(\mu/2)^2 + \lambda}$.

Remark 2. As $\gamma \to 0$,

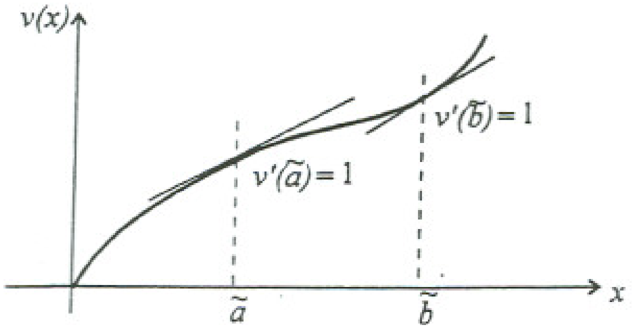
$$\widetilde{a} \to \overline{x}, \qquad \widetilde{b} \to \overline{x},$$

where $\overline{x}$ is a solution of

$$\tanh(\Delta x) = \frac{\mu\Delta}{(\mu/2)^2 + \Delta^2} \qquad \Big(= \frac{\mu\Delta}{2\Delta^2 - \lambda}\Big).$$

Remark 3. The following figure shows the behaviour of the function

$$V(x) = \begin{cases} \widetilde{A}e^{-\frac{\mu}{2}x}\sinh(\Delta x), & 0 \leqslant x \leqslant \widetilde{b}, \\ (x - \widetilde{a}) - \gamma + \widetilde{A}e^{-\frac{\mu}{2}\widetilde{a}}\sinh(\Delta\widetilde{a}), & x > b. \end{cases}$$



§4. The case C

1. We proceed to the general case where the (admissible) process of dividend payment $Z = (Z_t)_{t\geqslant 0}$ is an arbitrary non-decreasing right-continuous (for $t > 0$) process. We assume that Z_t is $\mathcal{F}_t^W = \sigma(W_s, s \leqslant t)$-measurable for each $t > 0$. Moreover, we assume that $Z_0 = Z_0(x)$ is a measurable function with $Z_0(0) = 0$ and $Z_0(x) \leqslant x$.

In this situation we take a process $X = (X_t)_{t\geqslant 0}$ satisfying

$$X_0 = x, \qquad X_{0+} = x - Z_0(x), \tag{4.1}$$

and for $t > 0$

$$X_t = (x - Z_0(x)) + \mu t + \sigma W_t - (Z_t - Z_{0+}). \tag{4.2}$$

From heuristic considerations we can expect that the optimal process $Z = (Z_t)_{t\geqslant 0}$ on which

$$V(x) = \sup_Z \left\{ Z_0(x) + \mathsf{E}_{x - Z_0(x)} \int_{(0,\infty)} e^{-\lambda t}\, dZ_t \right\} \tag{4.3}$$

is reached can be obtained by the limit passage in the cases **A** and **B** as $K \to \infty$ and $\gamma \to 0$, respectively.

To this end we note that in the case **A** the optimal threshold $\overline{x}(K)$ converges to $\overline{x}(\infty)$, which is a solution of the equation

$$\tanh(\Delta x) = \frac{\mu\Delta}{2\Delta^2 - \lambda}$$

with $\Delta = \sqrt{(\mu/2)^2 + \lambda}$.

In the case **B** the quantities $\tilde{a}$ and $\tilde{b}$ get closer as $\gamma \to 0$, converging to $\overline{x} = \overline{x}(\infty)$ (see Remark 2 in §3).

All this suggests that in the general case we can expect the structure of the optimal process $\overline{Z} = (\overline{Z}_t)_{t\geqslant 0}$ of dividend payment to be such that when $x > \overline{x}$ we need to pay at once the dividend $x - \overline{x}$ and then 'run' the process X from $\overline{x}$, where we place a reflecting barrier (since as $\gamma \to \infty$ the quantities $\tilde{a}$ and $\tilde{b}$ converge to each other). Here the accumulation of dividends will take place in local time given by the visits of X at the reflecting barrier $\overline{x}$.

2. This heuristic argument can be justified in the following way. Let $\overline{x}$ be the root of the equation

$$\tanh(\Delta \overline{x}) = \frac{\mu\Delta}{2\Delta^2 - \lambda}, \tag{4.4}$$

$$\overline{V}(x) = \begin{cases} \overline{A} e^{-\frac{\mu}{2}x} \sinh(\Delta x), & 0 \leqslant x \leqslant \overline{x}, \\ (x - \overline{x}) + \overline{V}(\overline{x}), & x > \overline{x}, \end{cases} \tag{4.5}$$

where $\overline{A}$ is such that

$$\overline{A} e^{-\frac{\mu}{2}\overline{x}} \left\{ \Delta \cosh(\Delta \overline{x}) - \frac{\mu}{2} \sinh(\Delta \overline{x}) \right\} = 1.$$

From the considerations of §2, §3 it follows that

$$\begin{gathered} -\lambda \overline{V}(x) + \mu \overline{V}'(x) + \frac{\sigma^2}{2}\overline{V}''(x) = 0, \qquad x < \overline{x}, \\ \overline{V}(0) = 0, \quad \overline{V}'(\overline{x}) = 1, \quad \overline{V}''(\overline{x}) = 0. \end{gathered} \tag{4.6}$$

Clearly, $\overline{V} \in C^2(0,\infty)$.

Let $x \leqslant \overline{x}$. Consider the solution $(\overline{X},\overline{L}) = (\overline{X}_t,\overline{L}_t)_{t\geqslant 0}$ of the stochastic differential equation with reflection (see [4], Ch. IX, §2, Exercise 2.14)

$$\overline{X}_t = x + \mu t + \sigma W_t - \overline{L}_t, \tag{4.7}$$

where $\overline{L} = (\overline{L}_t)_{t\geqslant 0}$ is a continuous non-decreasing $\mathcal{F}^W$-adapted process with $\overline{L}_0 = 0$ such that

$$\overline{L}_t = \int_0^t I(\overline{X}_s = \overline{x})\, d\overline{L}_s. \tag{4.8}$$

It is well known that a solution $(\overline{X},\overline{L})$ exists, $\overline{x}$ is the reflecting barrier for $\overline{X}$, and $\overline{L}$ is a local time determined by the process $\overline{X}$ at the boundary $\overline{x}$.

We show that for $x \leqslant \overline{x}$

$$\mathsf{E}_x \int_0^{\overline{\tau}} e^{-\lambda t}\, d\overline{L}_t = \overline{V}(x), \tag{4.9}$$

where $\overline{\tau} = \inf\{t \geqslant 0 : \overline{X}_t = 0\}$ and the function $\overline{V}(x)$ is defined by (4.5).

As in §2, §3, we apply Itô's formula to the semimartingale $\overline{X}$ (see [3]):

$$\begin{aligned} e^{-\lambda(t\wedge\overline{\tau})}\overline{V}(\overline{X}_{t\wedge\overline{\tau}}) &= \overline{V}(x) + \int_0^{t\wedge\overline{\tau}} (-\lambda e^{-\lambda s}\overline{V}(\overline{X}_s))\, ds \\ &\quad + \int_0^{t\wedge\overline{\tau}} \overline{V}'(\overline{X}_s)e^{-\lambda s}\, d\overline{X}_s + \frac{1}{2}\int_0^{t\wedge\overline{\tau}} \sigma^2 e^{-\lambda s}\overline{V}''(\overline{X}_s)\, ds. \end{aligned} \tag{4.10}$$

Taking (4.7) into account we get

$$\begin{aligned} \overline{V}(x) &= e^{-\lambda(t\wedge\overline{\tau})}\overline{V}(\overline{X}_{t\wedge\overline{\tau}}) - \int_0^{t\wedge\overline{\tau}} e^{-\lambda s} L\overline{V}(\overline{X}_s)\, ds \\ &\quad + \int_0^{t\wedge\overline{\tau}} e^{-\lambda s}\overline{V}'(\overline{X}_s)\, d\overline{L}_s - \int_0^{t\wedge\overline{\tau}} \sigma e^{-\lambda s}\overline{V}'(\overline{X}_s)\, dW_s, \end{aligned} \tag{4.11}$$

where

$$L\overline{V} = -\lambda\overline{V} + \mu\overline{V}' + \frac{\sigma^2}{2}\overline{V}''.$$

By the definition of $\overline{V} = \overline{V}(x)$ and by the properties of $(\overline{X},\overline{L})$

$$\mathsf{E}_x \int_0^{t\wedge\overline{\tau}} L\overline{V}(X_s)\, ds = 0, \qquad \mathsf{E}_x \int_0^{t\wedge\overline{\tau}} e^{-\lambda s}\overline{V}'(\overline{X}_s)\, dW_s = 0.$$

Therefore, taking account of $\overline{V}'(\overline{x}) = 1$, we obtain the required equality (4.9) by letting $t \to \infty$ in (4.11).

Now, let $x > \overline{x}$. Then we put $\overline{Z}_0(x) = x - \overline{x}$ and

$$\overline{Z}_t = \overline{Z}_0(x) I(x > \overline{x}) + \overline{L}_t. \tag{4.12}$$

Clearly, for such a process of dividend payment we have

$$\begin{aligned} \mathsf{E}_x \int_0^{\overline{\tau}} e^{-\lambda s}\, d\overline{Z}_s &= \Big(x - \overline{x} + \mathsf{E}_{\overline{x}} \int_0^{\overline{\tau}} e^{-\lambda s}\, d\overline{L}_s\Big) I(x > \overline{x}) \\ &\quad + \mathsf{E}_x \int_0^{\overline{\tau}} e^{-\lambda s}\, d\overline{L}_s \cdot I(x \leqslant \overline{x}) = \overline{V}(x), \end{aligned} \tag{4.13}$$

where $\overline{V}(x)$ is defined by(4.5).

3. Now we show that for *any* admissible dividend payment process $Z = (Z_t)_{t \geqslant 0}$

$$V(x; Z) \leqslant \overline{V}(x), \tag{4.14}$$

where $V(x; Z) = \mathsf{E}_x \int_0^{\tau} e^{-\lambda s}\, dZ_s$, $\tau = \inf\{t : X_t = 0\}$ and

$$\mathsf{E}_x \int_0^{\tau} e^{-\lambda s}\, dZ_s = Z_0(x) + \mathsf{E}_x \int_{(0,\tau)} e^{-\lambda s}\, dZ_s.$$

Let X be the process corresponding to Z. Applying Itô's formula to $e^{-\lambda t}\overline{V}(X_t)$ we get

$$\begin{aligned} e^{-\lambda(t\wedge\tau)}\overline{V}(X_{t\wedge\tau}) &= \overline{V}(x) + \int_0^{t\wedge\tau} (-\lambda e^{-\lambda s}\overline{V}(X_s))\, ds \\ &\quad + \int_0^{t\wedge\tau} e^{-\lambda s}\overline{V}'(X_{s-})\, dX_s + \frac{1}{2}\int_0^{t\wedge\tau} \sigma^2 e^{-\lambda s}\overline{V}''(X_s)\, ds \\ &\quad + \sum_{0<s\leqslant t\wedge\tau} e^{-\lambda s}\{\overline{V}(X_s) - \overline{V}(X_{s-}) - \overline{V}'(X_{s-})\Delta X_s\} \\ &= \overline{V}(x) - \int_0^{t\wedge\tau} e^{-\lambda s}\overline{V}'(X_{s-})\, dZ_s + \int_0^{t\wedge\tau} e^{-\lambda s} L\overline{V}(X_s)\, ds \\ &\quad + \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\overline{V}'(X_s)\, dW_s \\ &\quad + \sum_{0<s\leqslant t\wedge\tau} e^{-\lambda s}\{\overline{V}(X_s) - \overline{V}(X_{s-}) - \overline{V}'(X_{s-})\Delta X_s\}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \overline{V}(x) &= \int_0^{t\wedge\tau} e^{-\lambda s}\, dZ_s - \int_0^{t\wedge\tau} e^{-\lambda s}(1 - \overline{V}'(X_{s-}))\, dZ_s \\ &\quad - \int_0^{t\wedge\tau} e^{-\lambda s} L\overline{V}(X_s)\, ds - \int_0^{t\wedge\tau} \sigma e^{-\lambda s}\overline{V}'(X_s)\, dW_s \\ &\quad - \sum_{0<s\leqslant t\wedge\tau} e^{-\lambda s}\{\overline{V}(X_s) - \overline{V}(X_{s-}) - \overline{V}'(X_{s-})\Delta X_s\} + e^{-\lambda(t\wedge\tau)}\overline{V}(X_{t\wedge\tau}) \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{4.15}$$

Since $\overline{V}'(x) \geqslant 1$,

$$I_2 = -\int_0^{t\wedge\tau} e^{-\lambda s}(1 - \overline{V}'(X_{s-}))\, dZ_s \geqslant 0.$$

For $x < \overline{x}$ we have $L\overline{V}(x) = 0$, and for $x \geqslant \overline{x}$

$$-L\overline{V}(x) = \lambda\overline{V}(x) - \mu \geqslant \lambda\overline{V}(\overline{x}) - \mu = -L\overline{V}(\overline{x}) = 0.$$

Therefore,

$$I_3 = -\int_0^{t\wedge\tau} e^{-\lambda s} L\overline{V}(X_s)\, ds \geqslant 0.$$

If $\alpha < \beta$, then

$$\overline{V}(\beta) - \overline{V}(\alpha) - \overline{V}'(\beta)(\beta - \alpha) = \int_\alpha^\beta (\overline{V}'(y) - \overline{V}'(\beta))\, dy \geqslant 0,$$

since $\overline{V}'(y)$ is decreasing. Therefore,

$$I_5 = -\sum_{0<s\leqslant t\wedge\tau} e^{-\lambda s}\{\overline{V}(X_s) - \overline{V}(X_{s-}) - \overline{V}'(X_{s-})\Delta X_s\} \geqslant 0.$$

Since the derivative $\overline{V}'(x)$ is bounded,

$$\mathsf{E}_x \int_0^{t\wedge\tau} e^{-\lambda s}\overline{V}'(X_s)\, dW_s = 0.$$

Next,

$$\lim_{t\to\infty} \mathsf{E}_x e^{-\lambda(t\wedge\tau)}\overline{V}(X_{t\wedge\tau}) = 1.$$

Therefore, taking in (4.15) the mathematical expectation E_x and letting $t \to \infty$, we obtain

$$\overline{V}(x) \geqslant \mathsf{E}_x \int_0^\tau e^{-\lambda s}\, dZ_s = V(x; Z).$$

Thus the following assertion has been proved.

Theorem C. *In the model* **C** *the optimal strategy of dividend payment is described in the following way. Let $\lambda > 0$, $\mu > 0$ and let $\overline{x}$ be the root of the equation*

$$\tanh(\Delta\overline{x}) = \frac{\mu\Delta}{2\Delta^2 - \lambda}, \quad \text{where} \quad \Delta = \sqrt{\left(\frac{\mu}{2}\right)^2 + \lambda}.$$

1) *If $x > \overline{x}$, then we make an instantaneous payment of dividend equal to $x - \overline{x}$, that is, $\overline{Z}_0(x) = x - \overline{x}$.*
2) *If $x = \overline{x}$, then the dividend payment process $\overline{L} = (\overline{L}_t)_{t\geqslant 0}$ and the corresponding process $\overline{X} = (\overline{X}_t)_{t\geqslant 0}$ of capital evolution are solutions of a stochastic differential equation with reflection (4.7)–(4.8). In addition, the function $\overline{V} = \overline{V}(x)$ of optimal payment of dividends is given by (4.5).*

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