Matching Rules

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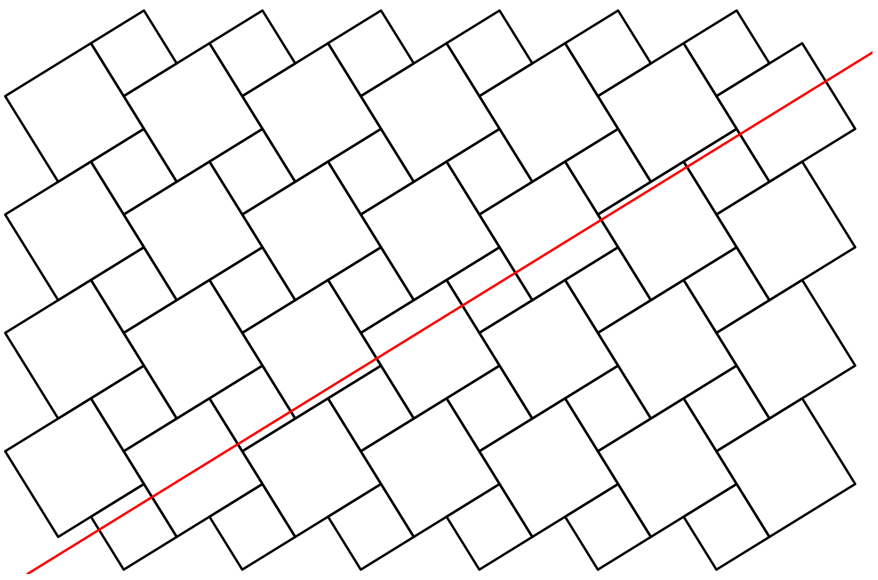
Quasiperiodic Projection Tilings



Irrational sections through a periodic klotz tiling.

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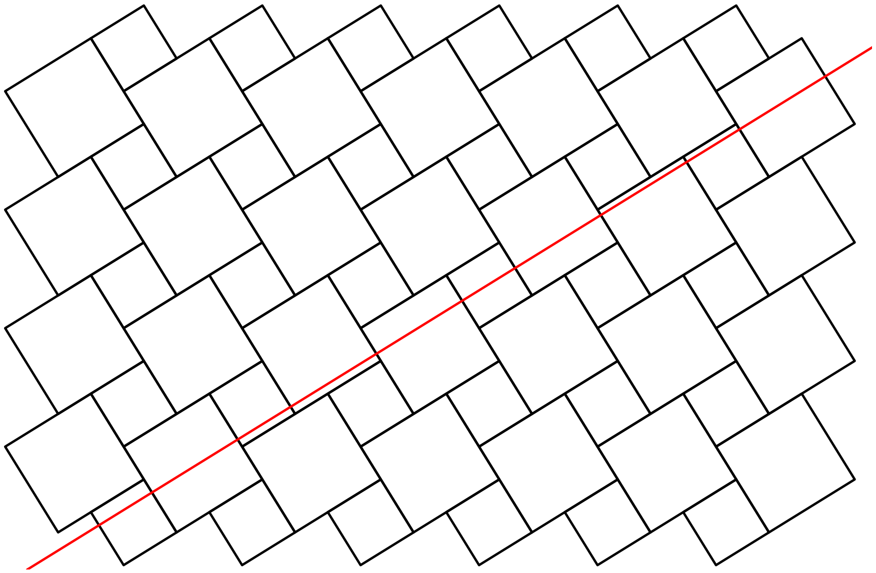

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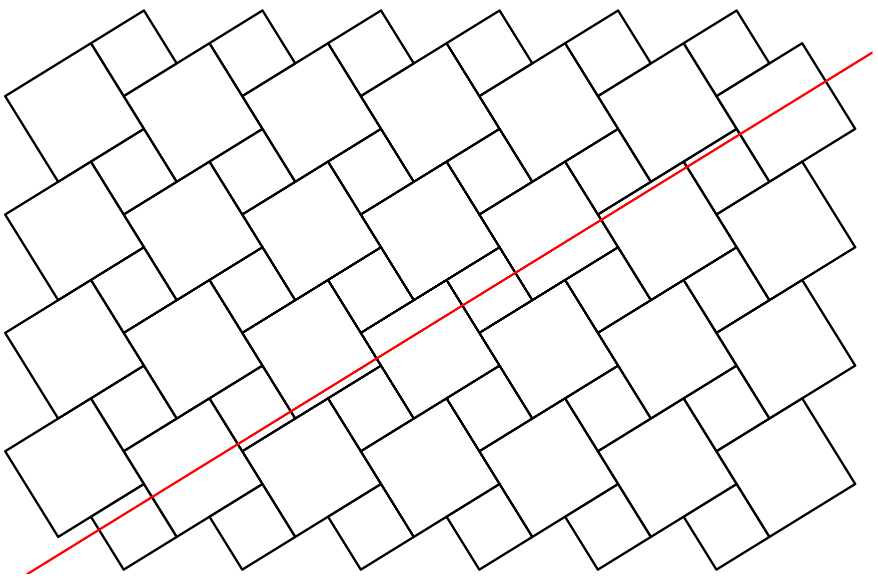

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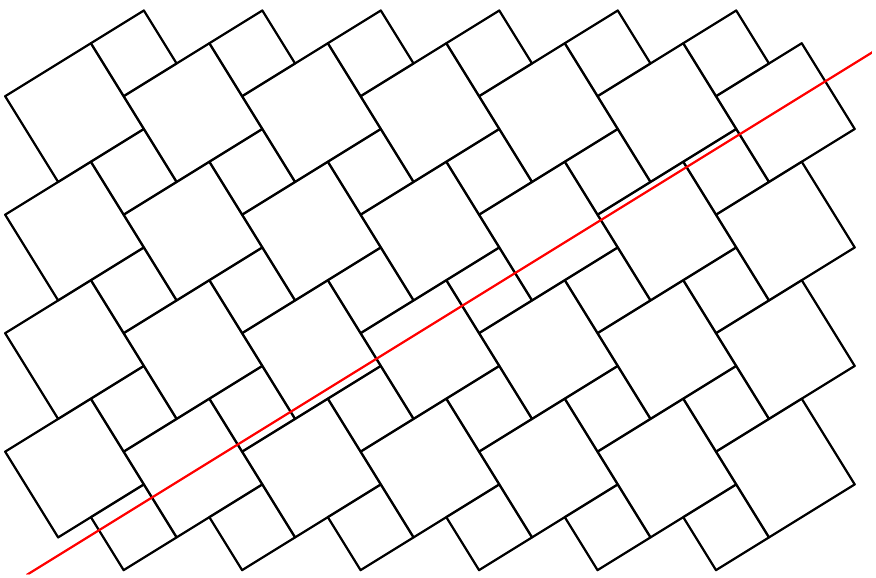

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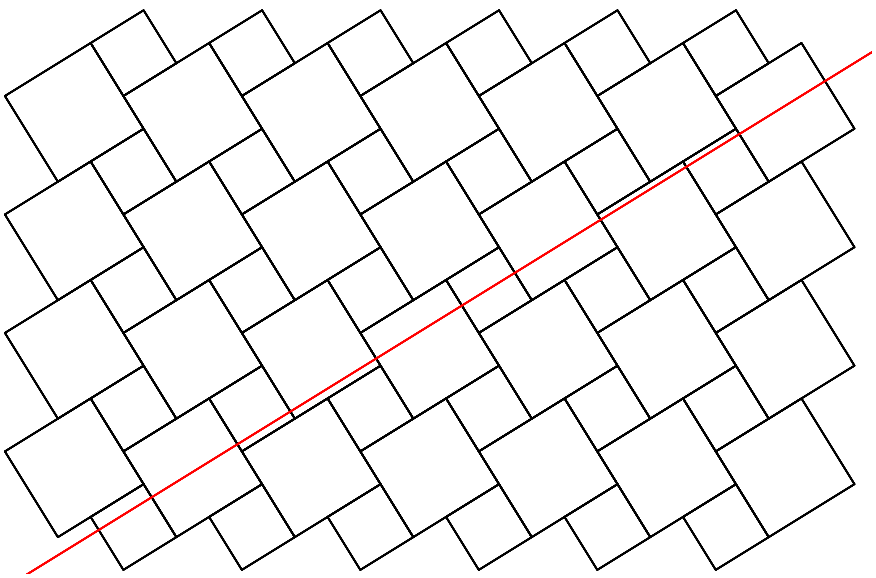

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Translation module is projection of higher-dimensional lattice on physical space.

Cut positions touching boundaries of acceptance domains are called singular.

Local Isomorphism

Two tilings are locally isomorphic (LI), if every finite patch occuring in one also occurs in the other.

Localy isomorphic tilings are indistinguishable by any local means.

All correlation functions are the same, so they are physically completely equivalent!

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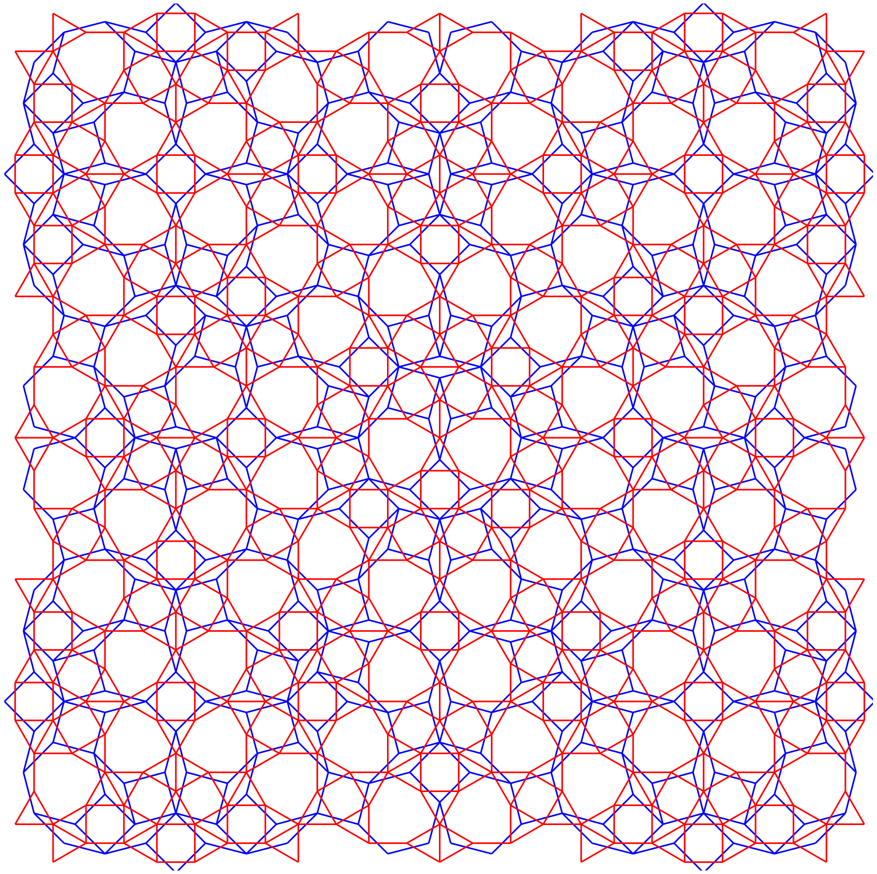
If projection of lattice to $E^{\perp}$ is dense, then all parallel sections yield locally isomorphic tilings.

Symmetry of an aperiodic tiling: define it as symmetry of the LI class.

Mutual Local Derivability



One tiling must be locally constructible from the other, and vice versa.

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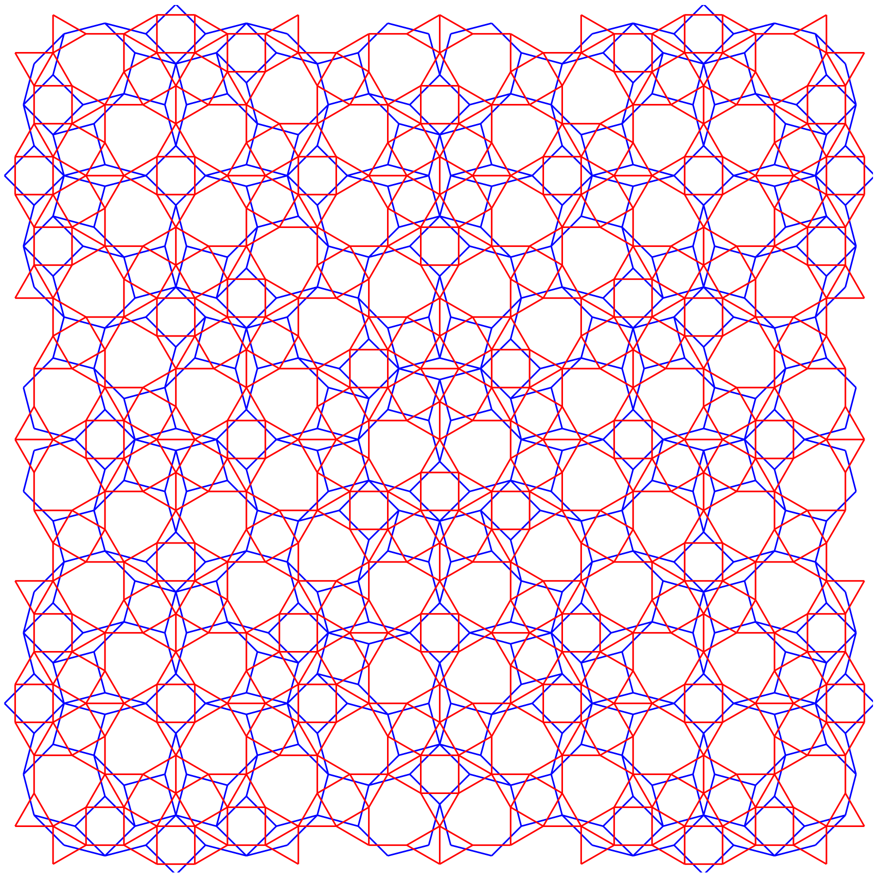

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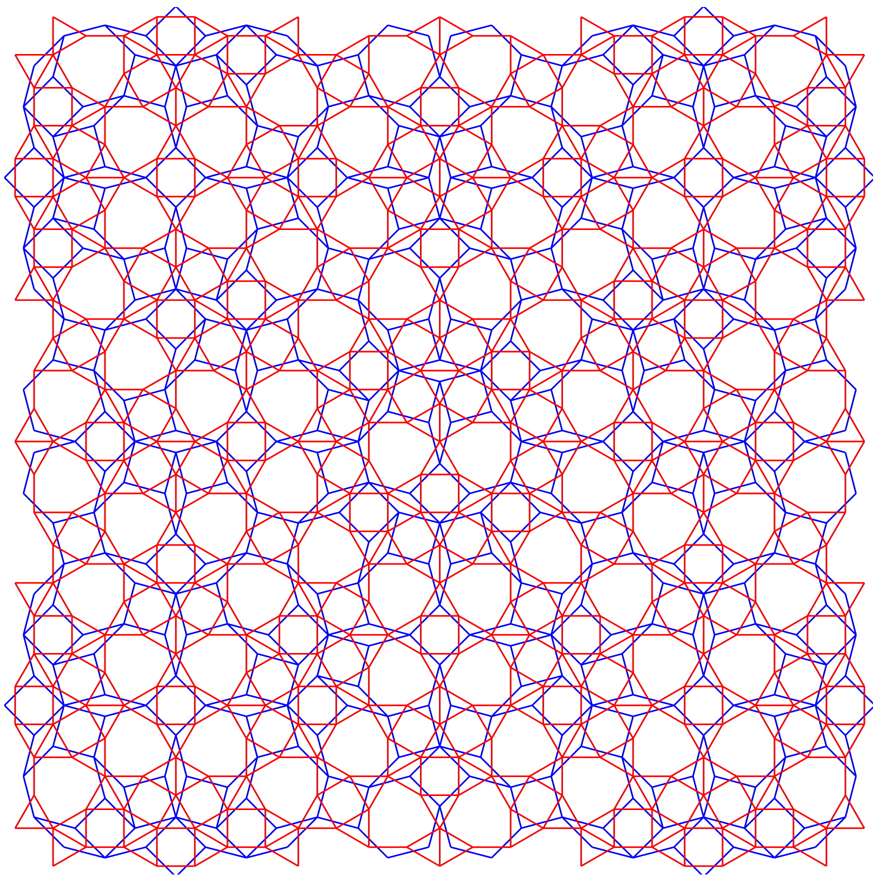

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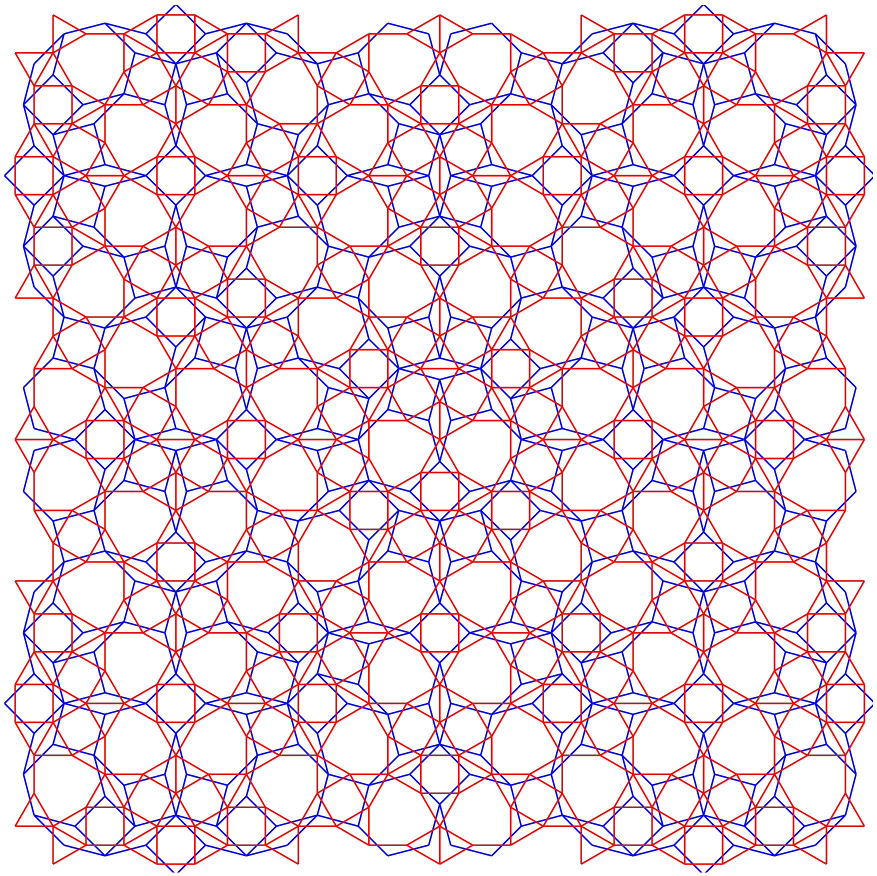

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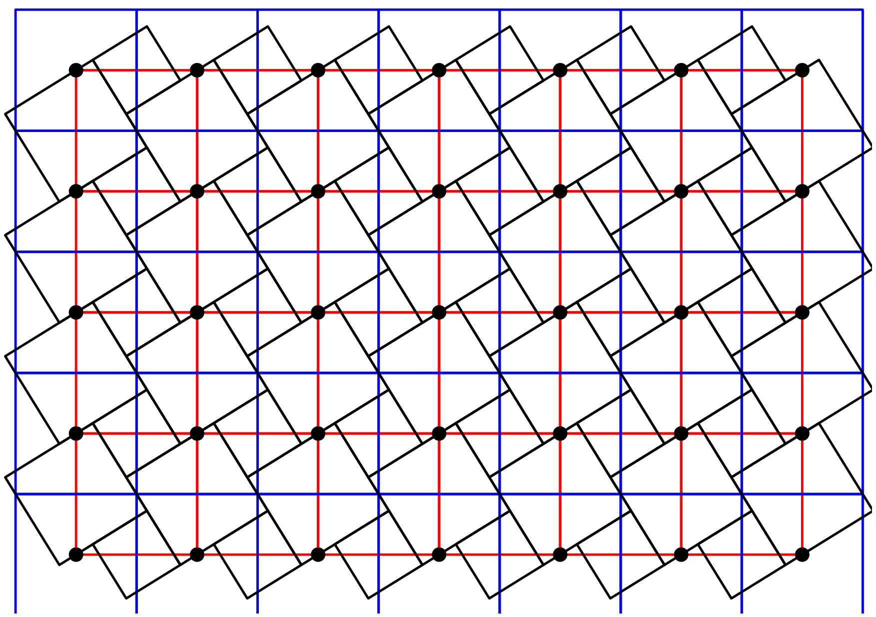
MLD induces a bijection between LI classes.

Dualisation

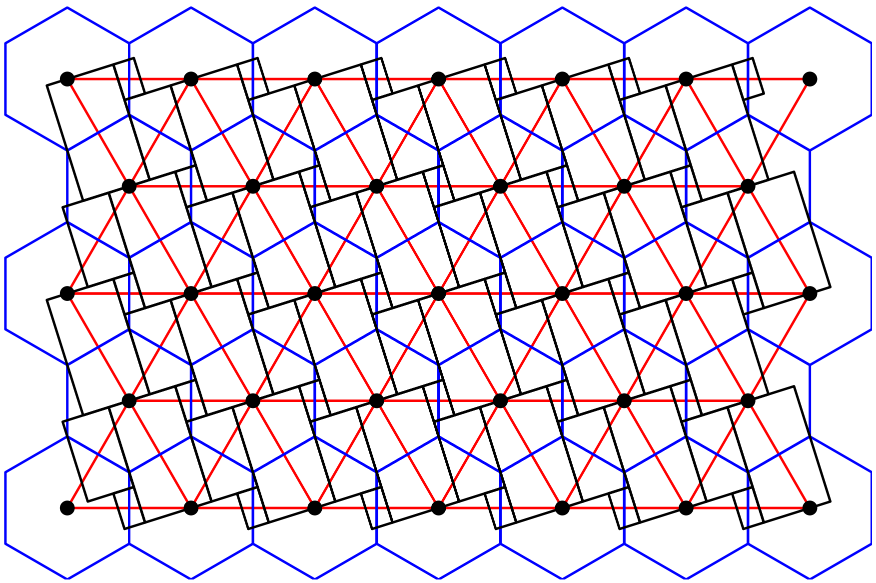


Construct klotz tiling from a pair of dual Voronoi and Delaunay cell complexes

Voronoi cells projected on the tiling space serve as tiles, projections of their dual Delaunay cells on internal space as acceptance domains.

The roles of Voronoi and Delaunay cells can also be exchanged.

Dualisation II

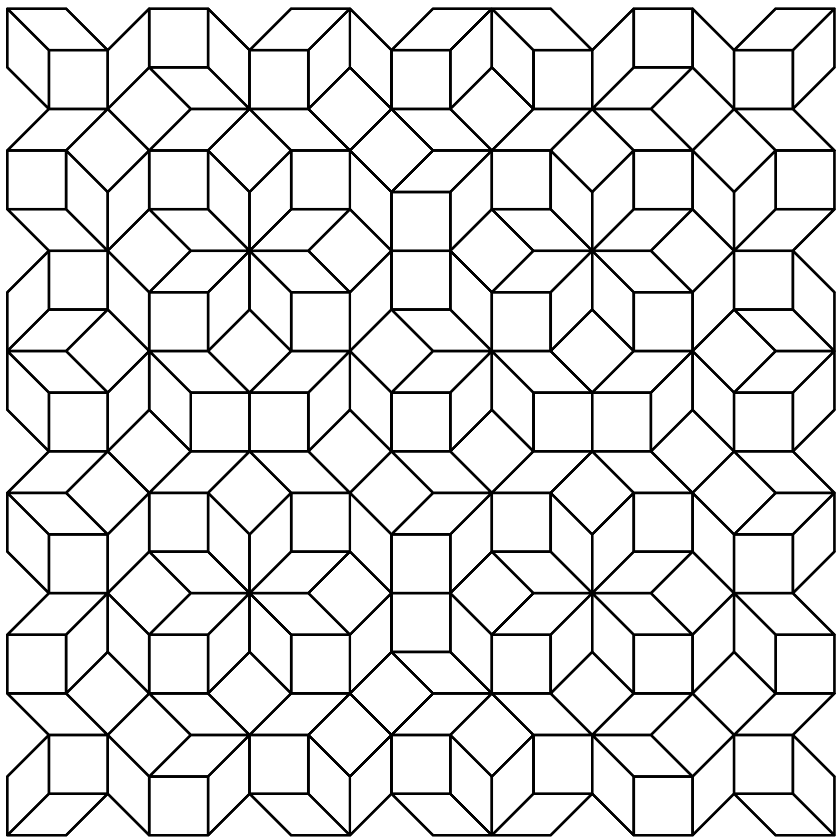
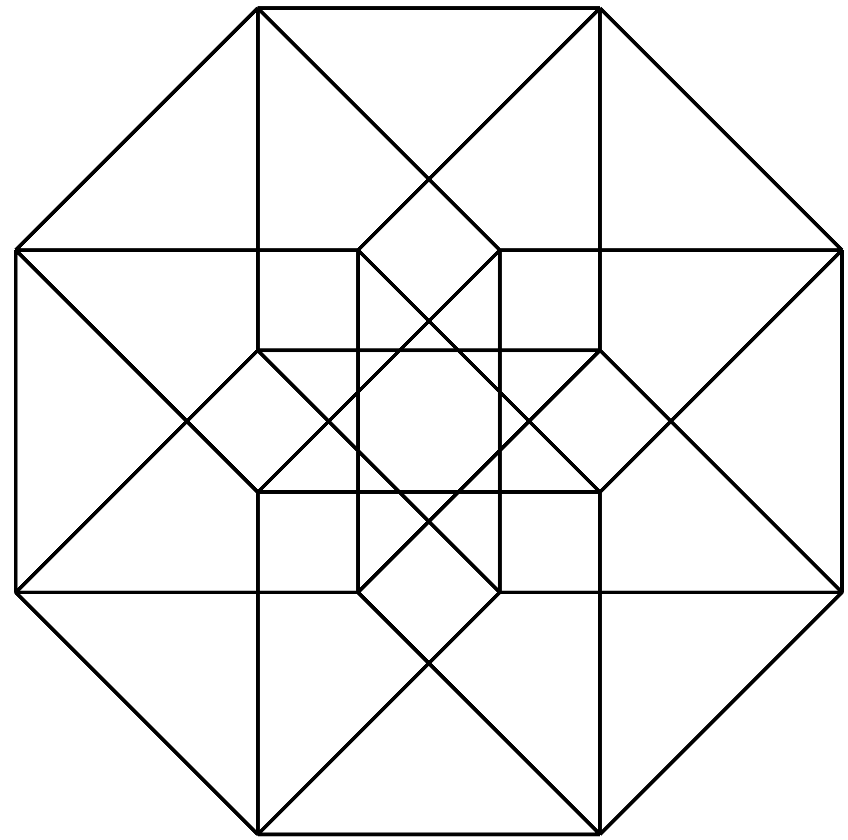


Voronoi and Delaunay cell complexes are different in general.

The same construction can be carried out with arbitrary point sets, not just lattices.

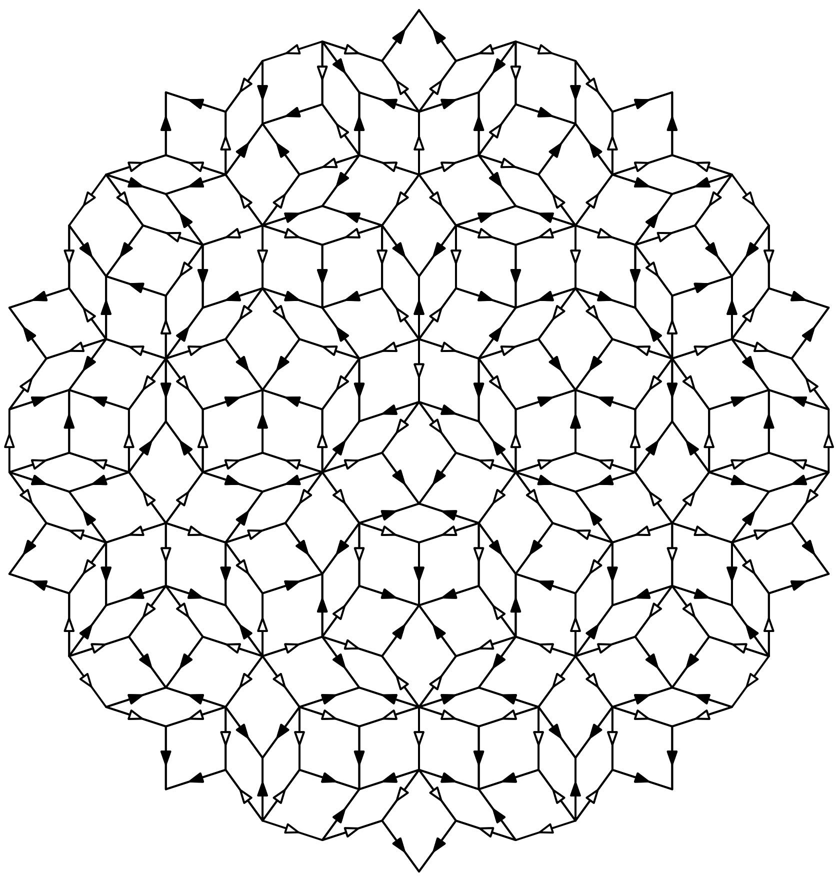
The construction can be generalized to Laguerre cell complexes, where different weights are assigned to the points.

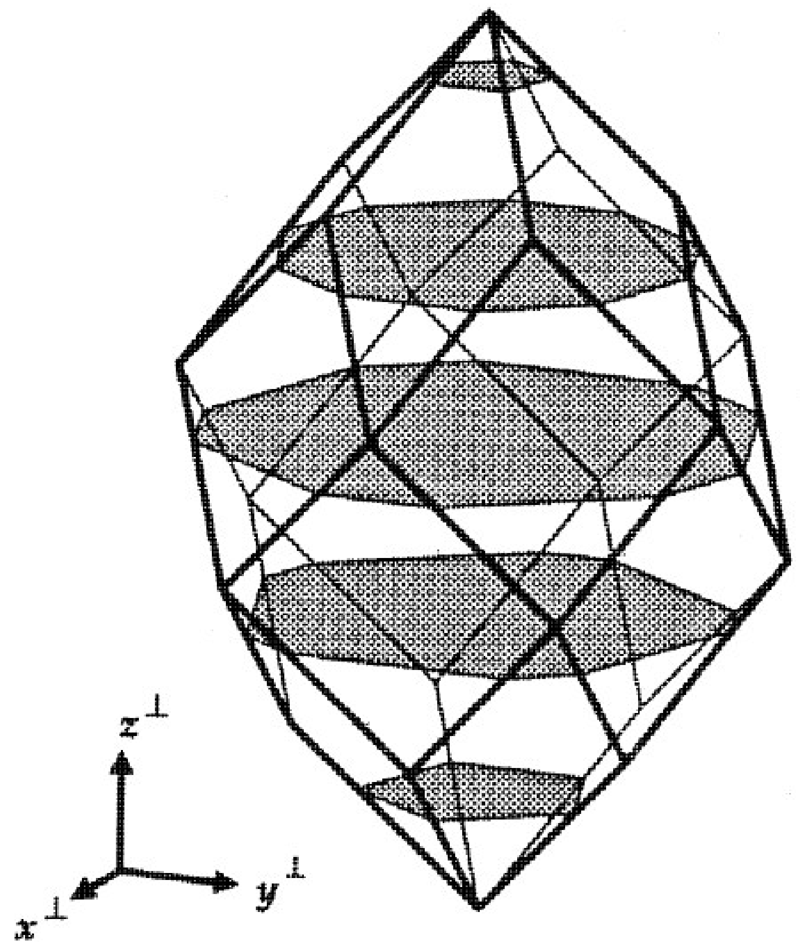
The Octagonal Tiling

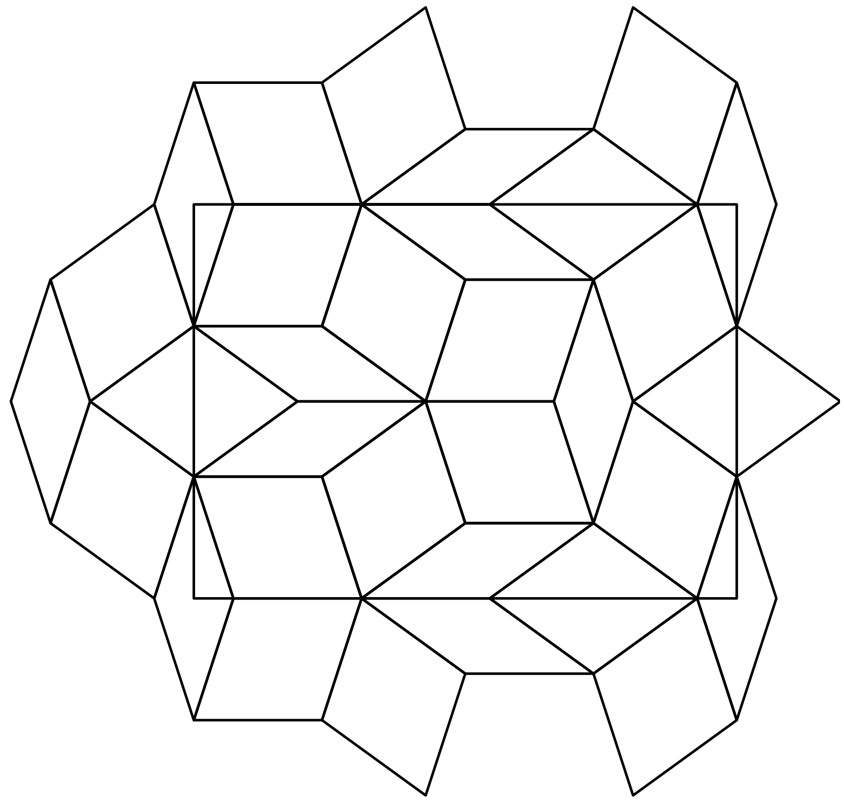
Octagonal tiling as projection from $\mathbb{Z}^4$, with acceptance domain.

The Penrose Rhombus Tiling





Penrose rhombus tiling from $\mathbb{Z}^5$; acceptance domain of generalized Penrose tilings.



The Penrose vertex types almost enforce a Penrose tiling – but not quite.

With dotted rhombs it works – MLD to arrowed rhombs (exercise).

Perfect Matching Rules

The R-Atlas $\mathcal{A}_R$ of a tiling T consists of all its tile patches of radius R. Locally isomorphic tilings have the same R-atlases for all R.

Can one fix an LI-class by a finite R-atlas for some R?

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Admitting perfect matching rules is a property of the MLD class.

Further Types of Matching Rules

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It means, that the vertices form a Delaunay subset of a model set, and the lift to a lattice has a uniformly bounded extent in $E^{\perp}$.

Such tilings show Bragg diffraction.

The Compsition-Decomposition Method

Given a primitive substitution tiling with candidate local matching rules, expressed through some markings.

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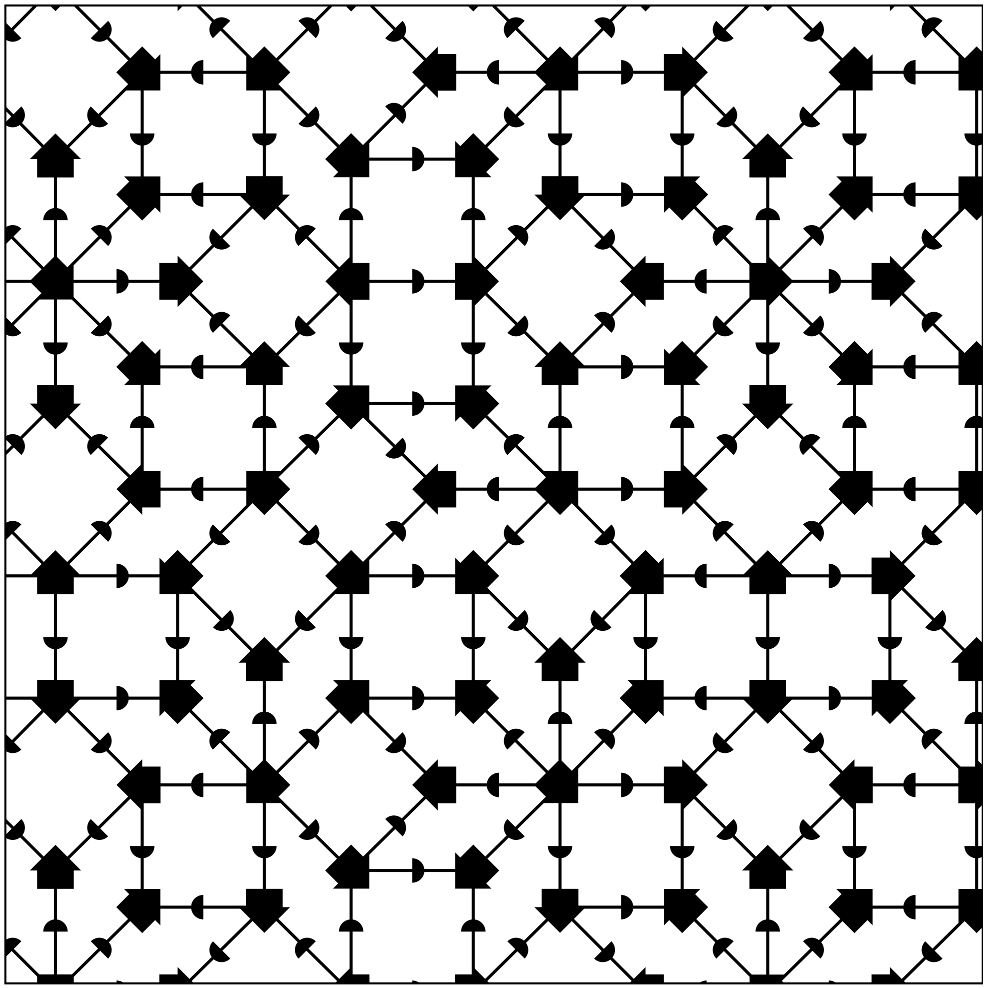
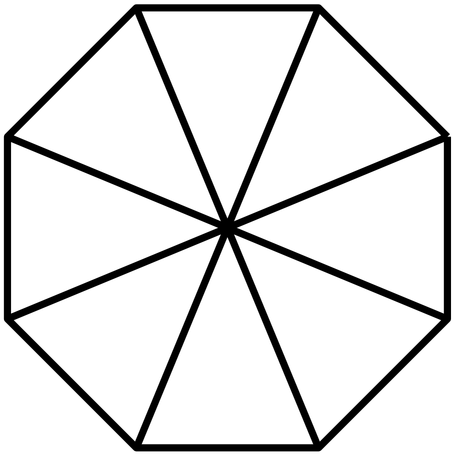
Have to show that:

- Every tiling satisfying the matching rules has a unique substitution predecessor; i.e., tiles can be composed to unique supertiles
- markings on supertiles induce equivalent matching conditions on supertile level

Method used by Robinson, Penrose, de Bruijn, Ammann,....

The Octagonal Ammann Tiling

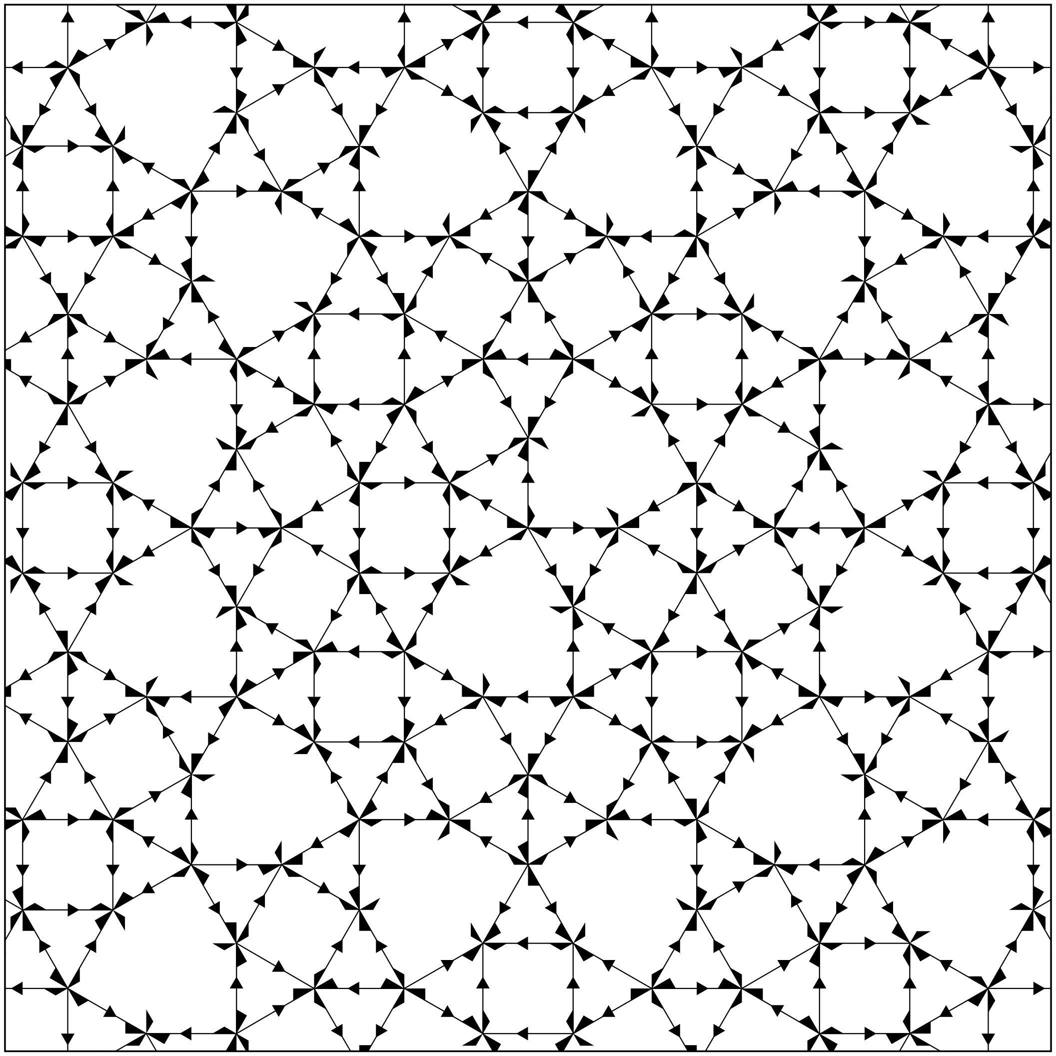


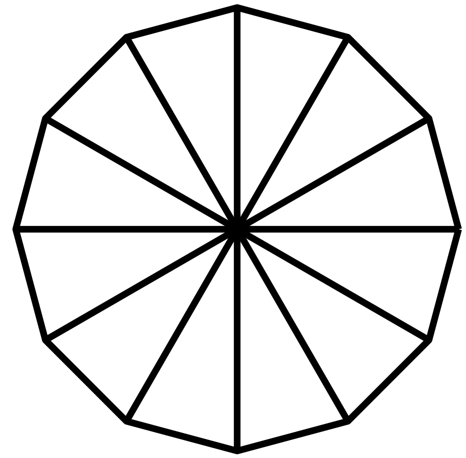


Extra window subdivision is needed for decoration.

Decoration non-local!

The Dodecagonal Shield Tiling





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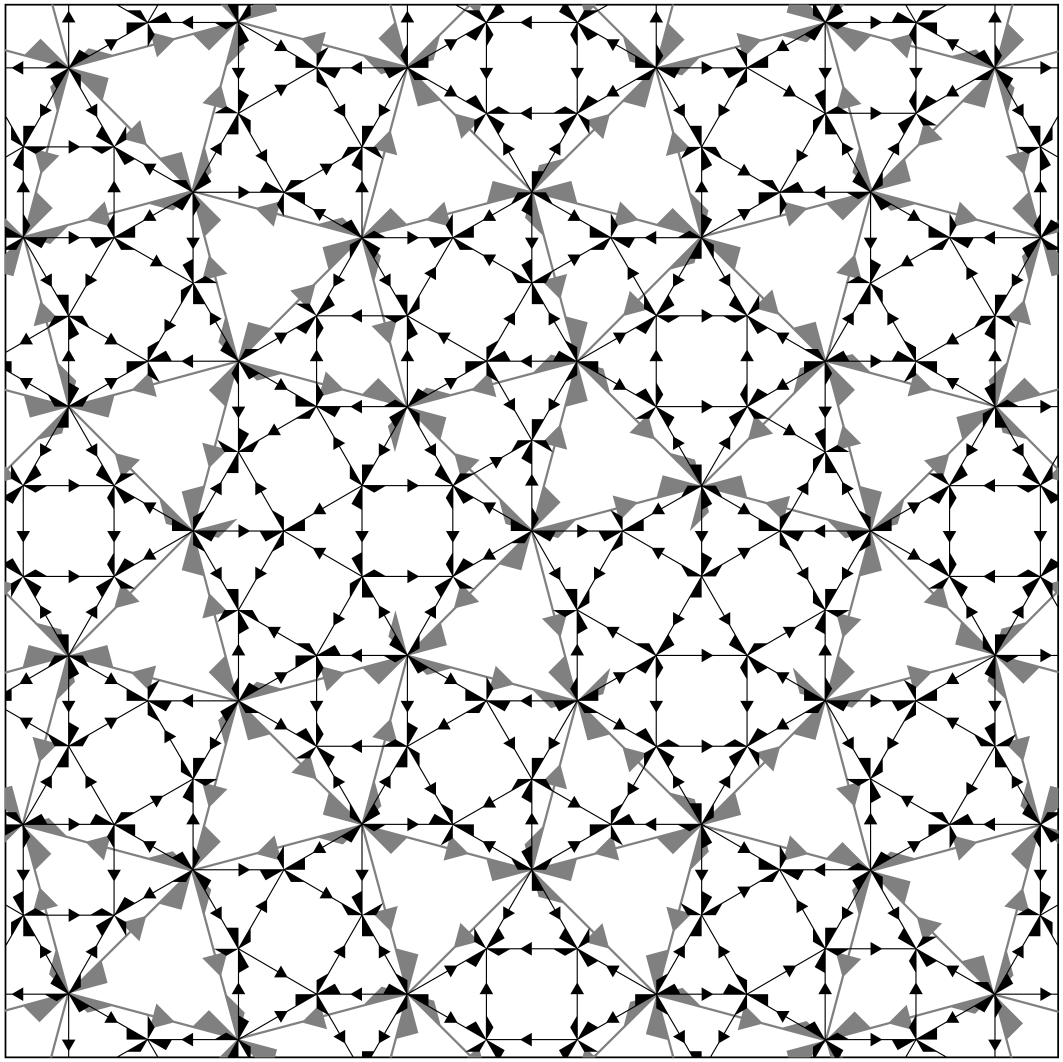
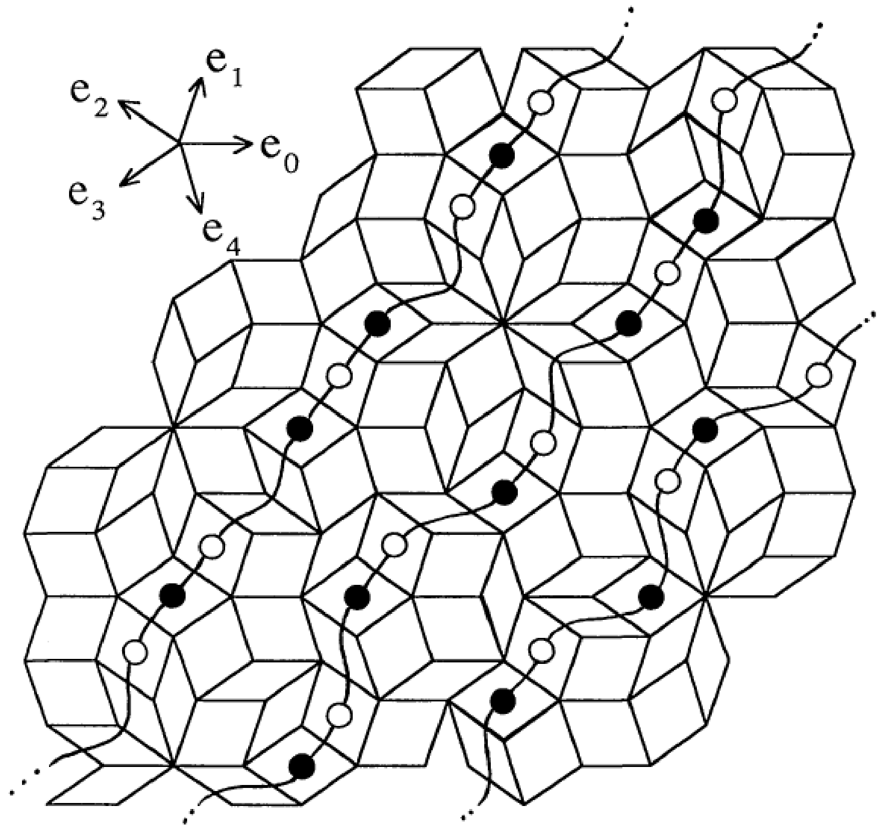


The Alternation Condition

J. Socolar, Commun. Math. Phys. **129**, 599 (1990).



Along any lane, tiles of the same shape, but different orientation, must alternate.

The Alternation Condition – Results

For N-fold rhombus tilings, N an odd or twice an odd number, the alternation condition represents weak matching rules. All rhombi with edges from an N-fold star must be used.

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However, for $N = 8$ Katz has shown that all admitted tilings are in fact model sets. The periodic ones may contain fault lines, though. $N = 12$ works the same way. Kind of strong matching rules.

Matching Rules for Projection Tilings

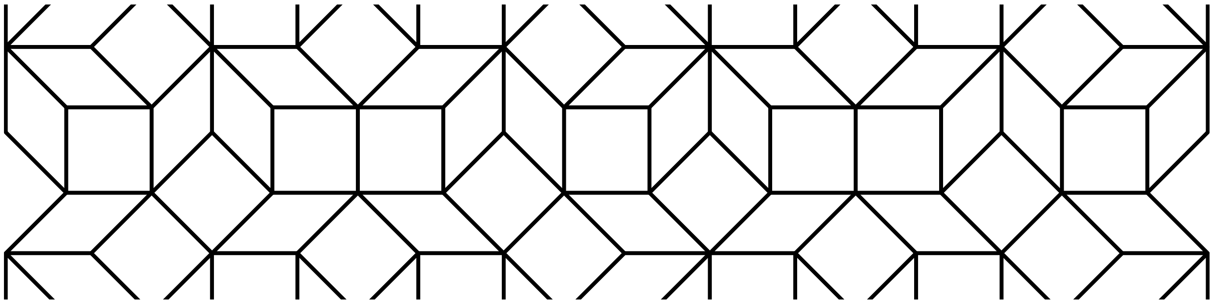
To prove matching rules, the composition-decomposition method can be used only on a case-by-case bases, after having guessed the right matching rules.

Can one derive perfect matching rules directly for canonical projection tilings?

What does it take for perfect matching rules to exist?

The following ideas are due to A. Katz and L. Levitov.

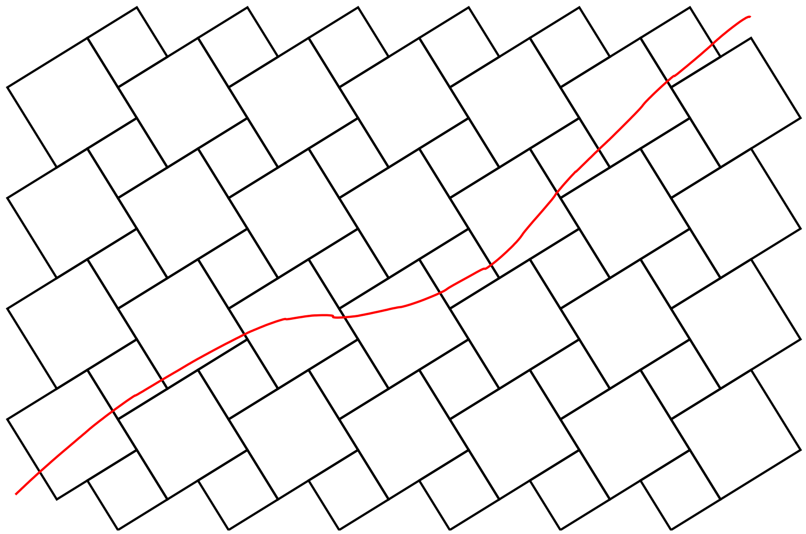
Rational Window Boundaries are Essential



Rational window boundaries make singular hexagons to form a row (called worm) and flip together.

With non-rational boundaries, flipping hexagon are isolated, and finite range matching rules cannot prevent them flipping independently.

The Meandering Section

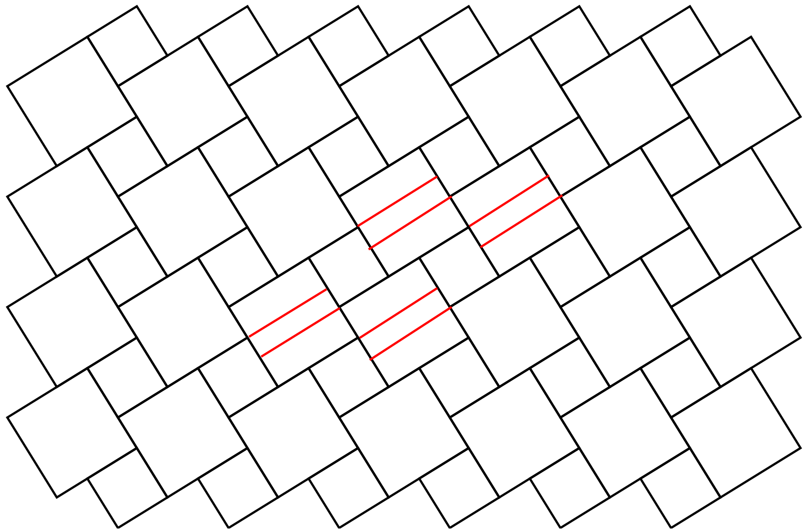


Non-quasiperiodic tilings correspond to non-planar sections of the klotz tiling, regarded as fibre bundle.

Tile boundaries parallel to physical space are forbidden.

Tilings actually correspond to homotopy classes of sections. Sections homotopic to a planar section correspond to quasiperiodic tilings.

Enlarged Forbidden Set



Idea: enlarge forbidden set to prevent meandering!

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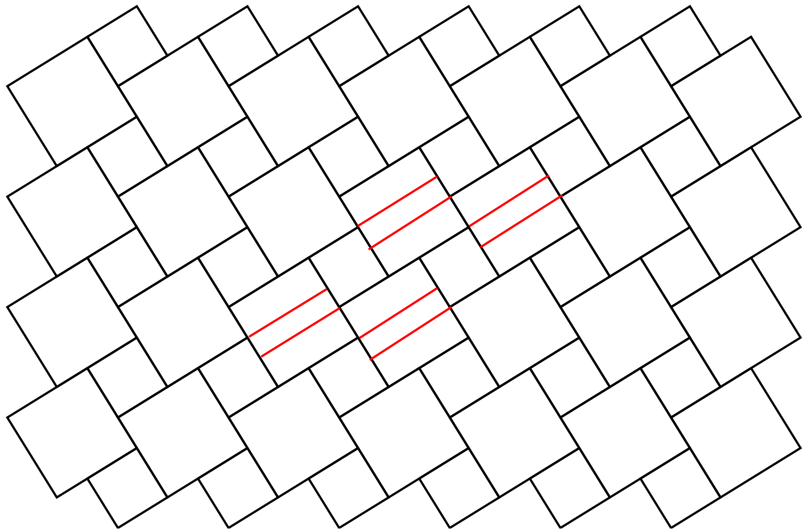

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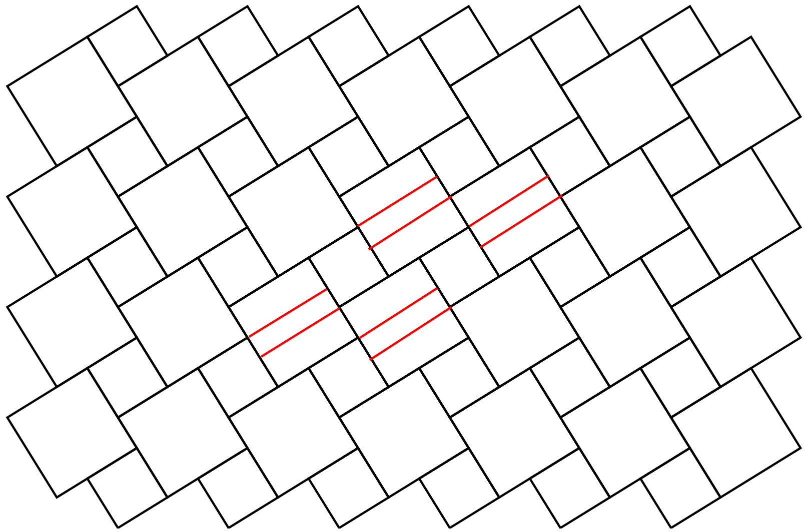

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Thin forbidden planes produce the same restrictions to the cut, if singular planes have rational positions.

Example: the Penrose tiling

Penrose tilings can be obtained from a 10-fold symmetric lattice in four dimensions.

Forbidden planes are 2D; each divides internal space into two half spaces. The fibre bundle section is always in the same half space. This induces an orientation of the forbidden plane.

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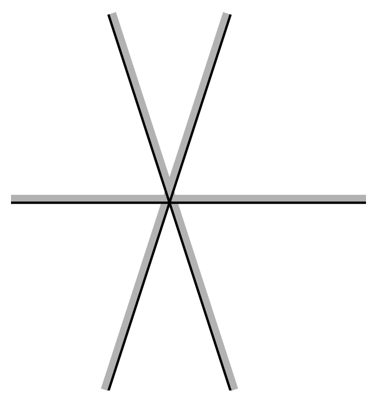
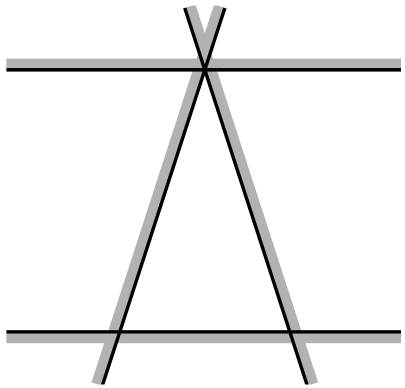
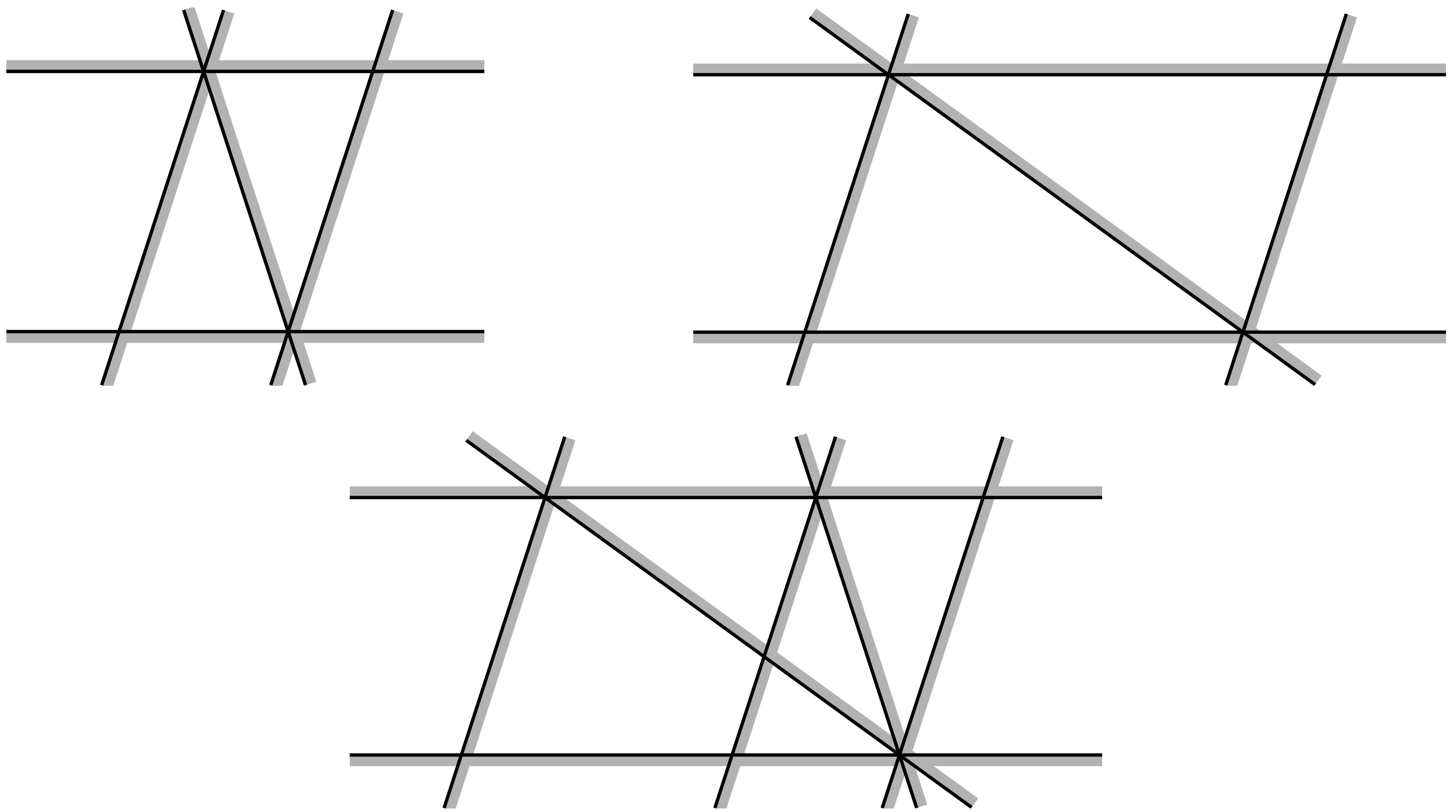
Forbidden planes cutting the same fibre have compatible orientations:



In the Penrose case, an intersection of two forbidden planes is always crossed by forbidden planes of all directions. A forbidden triangle therefore implies a forbidden strip:



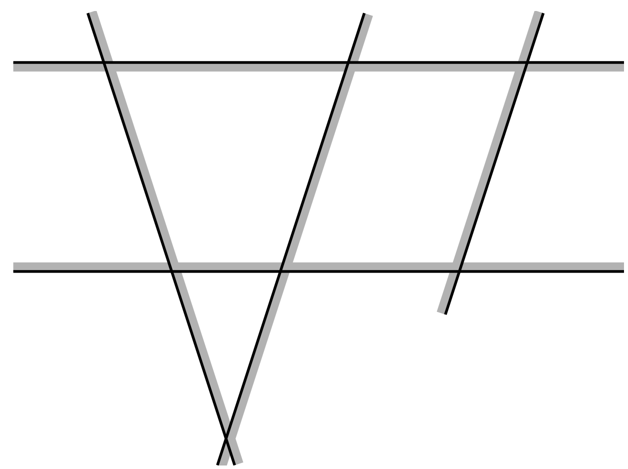
Can there be any "forbidden strips"? If so, there would be many forbidden planes having the same orientation:



From these one can choose a sequence, whose projections on tiling space converge to a limiting line, but which push the section to infinity. This is in contradiction with the continuity of the section.

The orientation of any three forbidden planes are therefore compatible, even if they don't intersect a common fibre.

Each forbidden plane which does not intersect a parallelogram, is compatible with the parallelogram:



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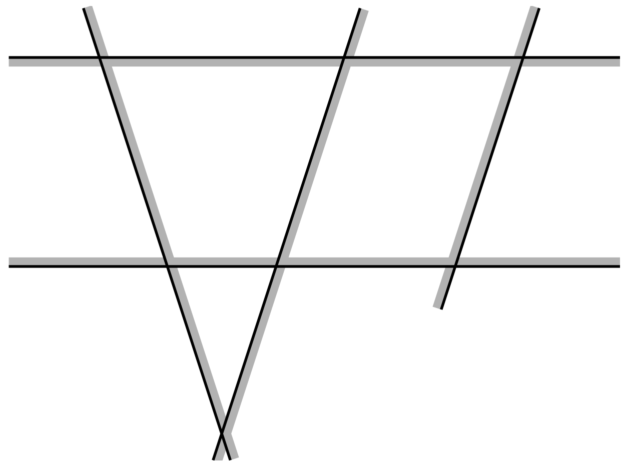


With Helly's Theorem it can be concluded that all forbidden planes have compatible orientations. Therefore, the intersection of all corresponding half spaces is non-empty, and any section is homotopic to a planar section.

Which properties of the system of forbidden planes did we exploit?
Apparently, the forbidden planes must:

- fix the orientation of the tiling space, through the requirement that they all project to codimension 1 planes in both tiling space and internal space.

- cut internal space into two half spaces

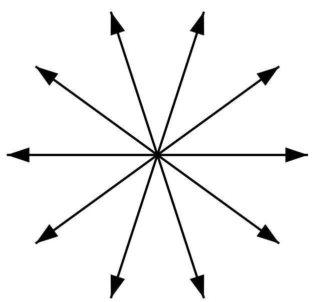
- have sufficiently many non-generic intersections

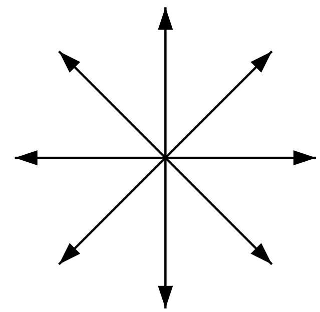
- be sufficiently rich to allow the "pushing procedure"

These are highly non-generic conditions!

Sufficient Systems of Forbidden Planes (2D)

For 10-fold tilings in 2D there are two sufficient sets of forbidden planes: between projected basis vectors, and along projected basis vectors. For 8- and 12-fold tilings in 2D both systems are needed at the same time!





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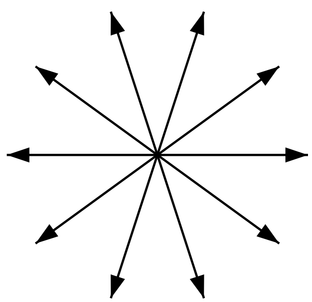

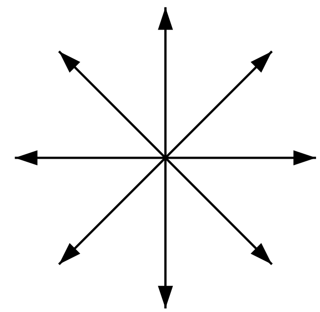

For 7-, 9-fold tilings, etc, there are no sufficient systems of forbidden planes. Candidate planes are not sufficiently rational, so that their rational part does not cut internal space into two halves.

Sufficient Systems of Forbidden Planes (3D)

For P-type icosahedral tilings in 3D the mirror planes can be used as forbidden planes. The 3D Ammann-Kramer tiling is such an example.

For F-type icosahedral tilings a set of forbidden planes perpendicular to 5-fold "axes" is sufficient. The Danzer tiling is such an example.

Conclusions

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This theory does not exclude 7-fold tilings with perfect matching rules. These cannot, however, have polyhedral acceptance domains. One should rather expect acceptance domains with fractal boundary.