Hierarchical Tilings

$$\supsetneq$$

("supertilings" may be different)

Substitution tilings

$$\overset{?}{\supsetneq}$$

(combinatorial, e.g.

Fibonacci, arbitrary lengths)

Pseudo-self-affine tilings

Def: $\mathcal{T} \xleftrightarrow{MLD} \varphi\mathcal{T}$

$$\supsetneq$$

Self-affine tilings

$\varphi\mathcal{T}$ subdivides to $\mathcal{T}$ <u>exactly</u>

II. Self-affine tilings

Tilings

Fix a set of **types** (or **colors**) labeled by $\{1, \ldots, m\}$.

Tile: $T = (A, i)$ where $A = \text{supp}(T)$ is a compact set in $\mathbf{R}^d$ which is the closure of its interior, $i = \ell(T) \leq m$ is the type of T.

Tiling: a set of tiles $\mathcal{T}$ such that

$$\mathbf{R}^d = \bigcup \{\text{supp}(T) : \ T \in \mathcal{T}\}$$

and distinct tiles have disjoint interiors.

$\mathcal{T}$**-patch:** a finite subset of $\mathcal{T}$.

Translation: $(A, i) + g = (A + g, i)$ for $g \in \mathbf{R}^d$

$$\mathcal{T} + g = \{T + g : \ T \in \mathcal{T}\}$$

We assume that:

- Any two $\mathcal{T}$-tiles with the same type (color) are translationally equivalent. (Hence there are finitely many $\mathcal{T}$-tiles up to translation.)

- the tiling $\mathcal{T}$ has **finite local complexity** (FLC), that is, for any $R > 0$ there are finitely many $\mathcal{T}$-patches of diameter less than R up to translation.

- the tiling $\mathcal{T}$ is **repetitive**, that is, for any $\mathcal{T}$-patch P there exists $R > 0$ such that every ball of radius R contains a translated copy of P.

Tile-substitutions

Let $\phi : \mathbf{R}^d \to \mathbf{R}^d$ be an expanding linear map, that is, all its eigenvalues are greater than 1 in modulus.

Definition. Let $\{T_1, \ldots, T_m\}$ be a finite **prototile set**. A **tile-substitution** with expansion ϕ is a map $T_i \mapsto \omega(T_i)$, $i = 1, \ldots, m$, where each $\omega(T_i)$ is a patch made of translates of T_j, such that

$$\mathrm{supp}(\omega(T_i)) = \phi(\mathrm{supp}(T_i)).$$

The substitution is extended to all translates of prototiles by $\omega(x + T_j) = \phi x + \omega(T_j)$, and to patches and tilings by $\omega(P) = \bigcup\{\omega(T) : T \in P\}$. We say that $\mathcal{T}$ is a fixed point of a substitution if $\omega(\mathcal{T}) = \mathcal{T}$.

Self-affine tilings

The substitution ω is **primitive** there exists $k \in \mathbf{N}$ such that $\omega^k(T_i)$ contains a translate of T_j for all i, j (equivalently, the substitution matrix is primitive).

$\mathcal{T}$ is **self-affine** if it is an FLC repetitive fixed point of a primitive substitution. $\mathcal{T}$ is **self-similar** if ϕ is a similitude, i.e.

$$|\phi(x)| = r|x|, \ \forall\, x \in \mathbf{R}^d.$$

History: "fractiles" ($m = 1$), Gilbert, Penrose, Dekking, Rauzy, Thurston, Lunnon & Pleasants,...

Connections: Markov partitions, numeration systems, wavelets,...

$\cancel{\lambda^3 = \lambda^2 + \lambda + 1}$ (Pisot)

$$\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

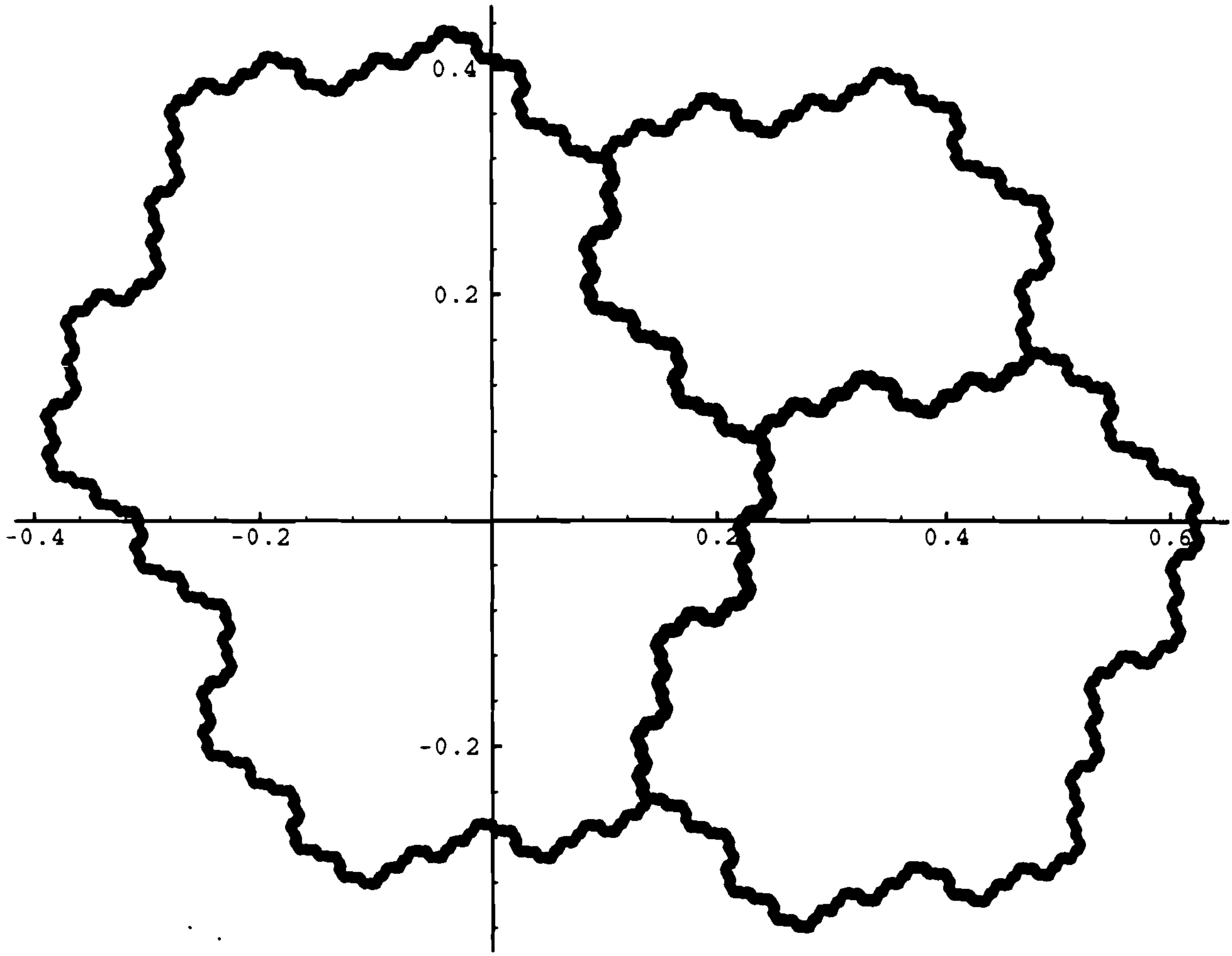
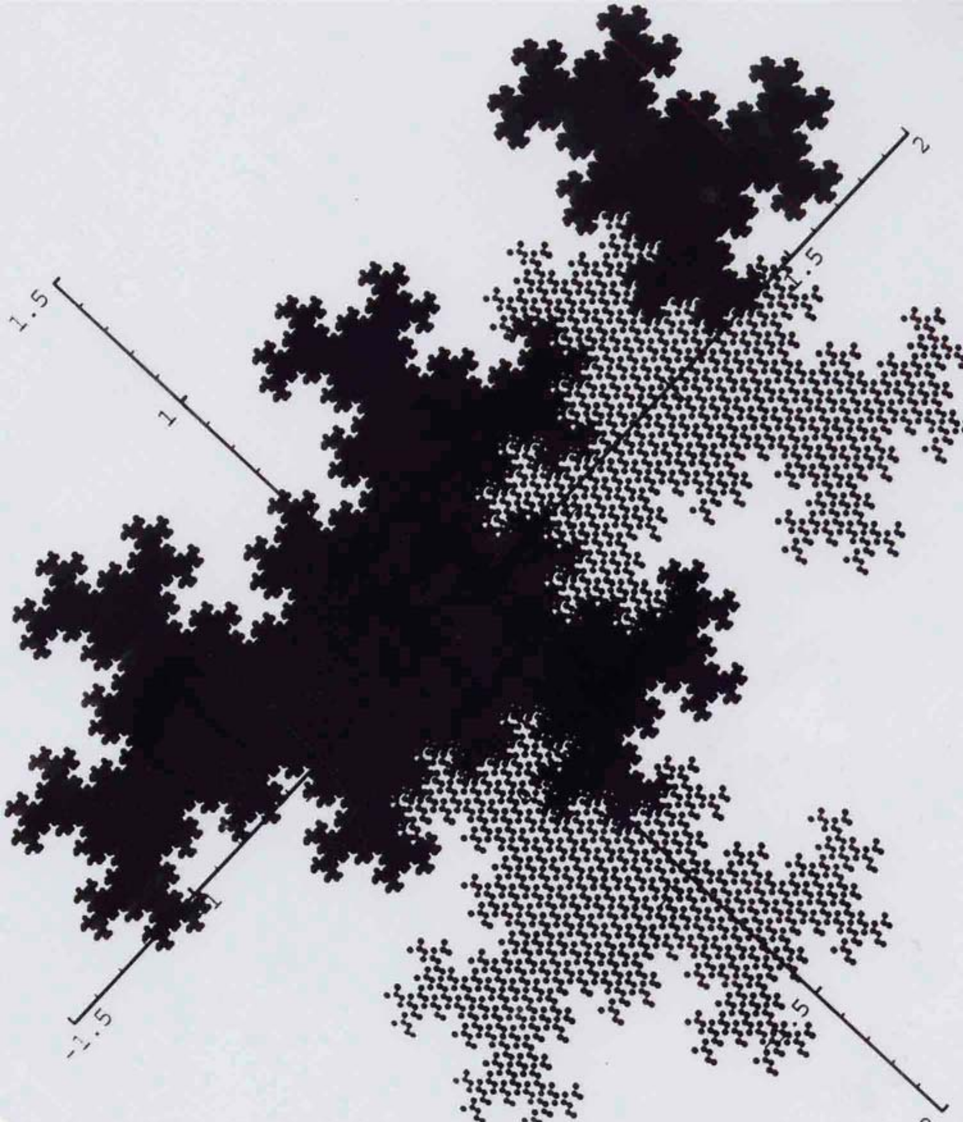


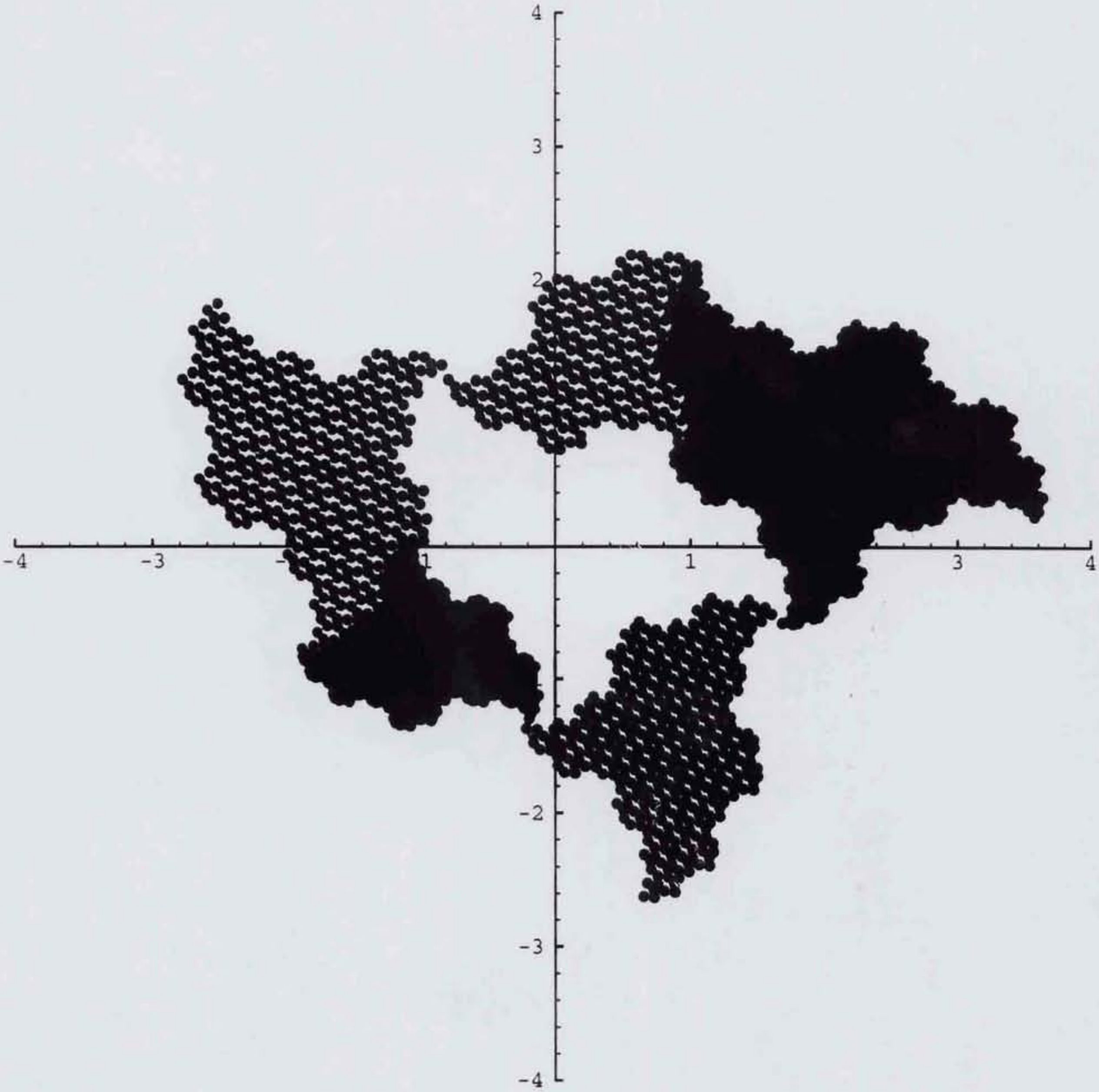
Figure 5

Rauzy tiling ; pure discrete spectrum

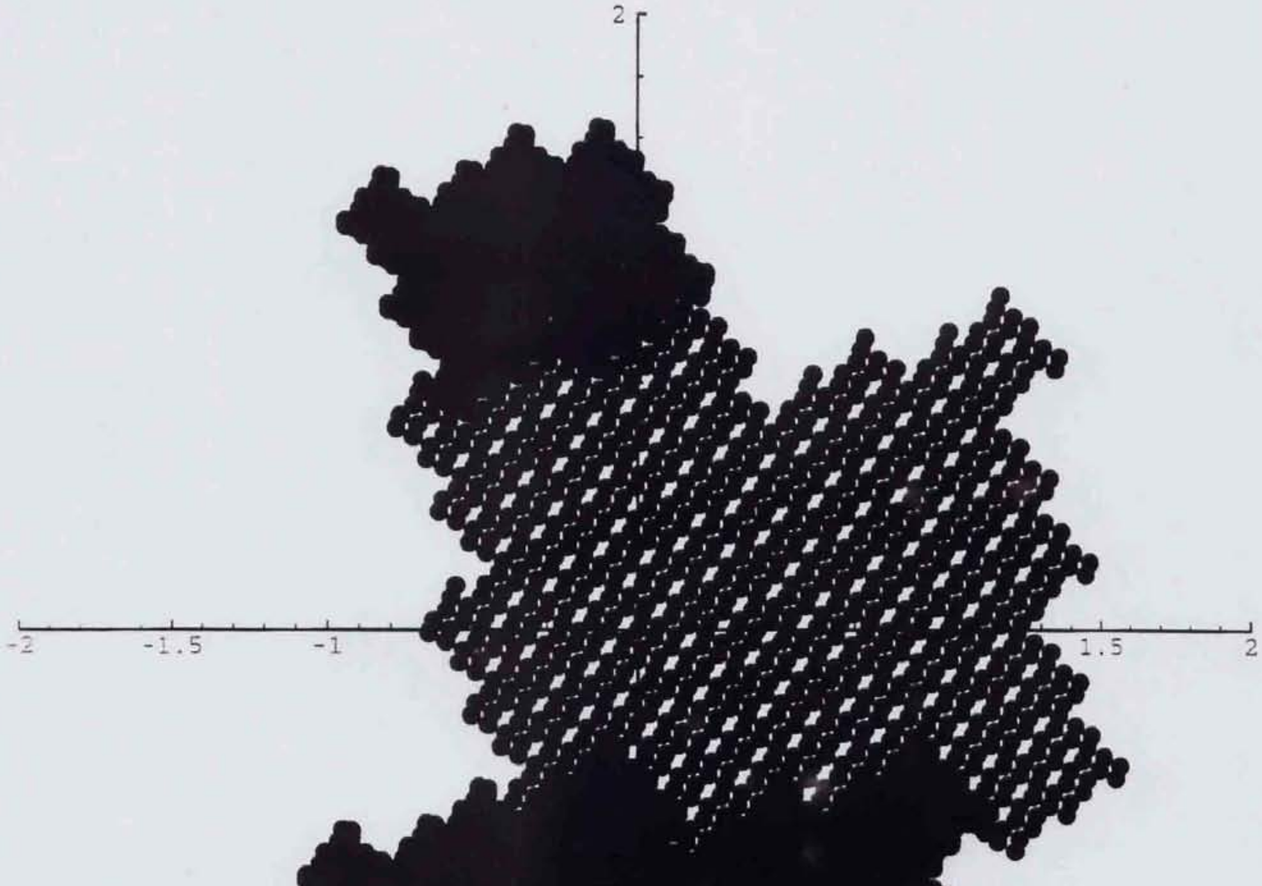


2
1.5
1.5
1
-1.5
1
1.5

$$\lambda^3 = \lambda + 1 = \lambda^2 + \lambda^{-2}$$



$\lambda^3 = 2\lambda^2 + 1$



-~~37~~ A- from free group endo:

$a \to b,\ b \to c,\ c \to a^{-3}b$

$\lambda:\ \lambda^3+\lambda+3=0$

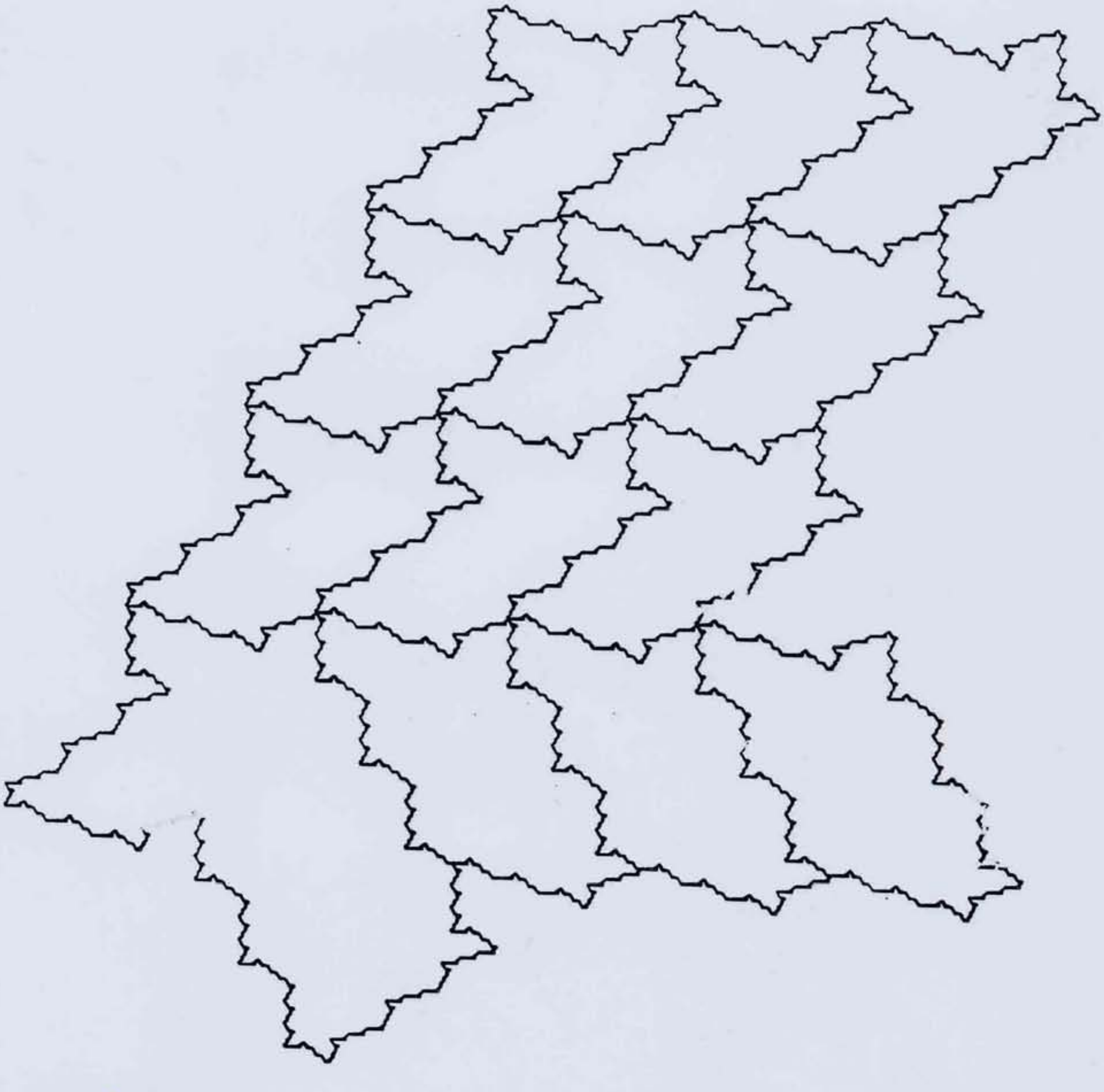


Figure ~~7~~ 4.8. Patch of a self-similar tiling (non-Pisot case)

no discrete spectrum (weak mixing)

Self-affine tilings and IFS

There exist finite sets $\mathcal{D}_{ij} \subset \mathbf{R}^d$, $i, j \leq m$:

$$\omega(T_j) = \{u + T_i : \ u \in \mathcal{D}_{ij}, \ i = 1, \ldots, m\}, \ j \leq m,$$

with

$$\phi A_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \ j \leq m. \tag{2}$$

Here all the sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

Rewrite the system of set equations (2):

$$A_j = \bigcup_{i=1}^{m} (\phi^{-1} A_i + \phi^{-1} \mathcal{D}_{ij}), \ j \leq m.$$

ϕ^{-1} is a contraction, so there is always a unique nonempty compact solution $\{A_1, \ldots, A_m\}$ (attractor of a graph-directed IFS). The difficulty is to have A_j with nonempty interiors.

Substitution matrix

$S_{ij} = \#\mathcal{D}_{ij}$, S is $m \times m$ non-negative integer matrix

primitive $\Leftrightarrow$ S^k has no zero entries for some k.

$\{\mathcal{L}^d(A_j)\}_{j=1}^m$ is a positive row eigenvector for S, with the eigenvalue $|\det(\phi)|$. (Here $\mathcal{L}^d$ is Lebesgue measure in $\mathbf{R}^d$.)

PERRON-FROBENIUS THEORY

COROLLARY 9. $|\det \phi|$ *is a* **Perron number**, *i.e. an algebraic integer* > 1 *whose Galois conjugates are strictly less in modulus.*

Characterization of expansions

THEOREM 10 (essentially [Lind '84]) *In* $\mathbf{R}^1$, $\phi(x) = \lambda x$ *is an expansion of a self-affine tiling iff* $|\lambda|$ *is a Perron number.*

THEOREM 11 [Thurston '89],[Kenyon '96] *In* $\mathbf{R}^2 \equiv \mathbf{C}$, $\phi(z) = \lambda z$ *with complex* λ, *is an expansion of a self-similar tiling iff* λ *is a complex Perron number, i.e. an algebraic integer of modulus* > 1 *whose Galois conjugates, except the complex conjugate,* $\overline{\lambda}$, *are strictly less in modulus.*

__Lemma 12__ [Kenyon '90] [Thurston '89]
$\mathcal{T}$ - self-affine tiling of $\mathbb{R}^d$ with expansion φ. Then the eigenvalues of φ are algebraic integers.

Proposition 13 [Kenyon '90] [Thurston '89]
$\mathcal{T}$ - self-similar tiling of $\mathbb{R}^d$ with expansion $\varphi = \eta O$, where $\eta > 1$, O - orthogonal matrix. Let λ be an eigenvalue of φ. Then every Galois conjugate of λ is either an eigenvalue of φ or strictly less than $|\lambda| = \eta$ in modulus

Control Points

Want to define "reference points" $c(T) \in \mathrm{supp}(T)$ $\forall T \in \mathcal{T}$ so that

(i) $T' = T + x \implies c(T') = c(T) + x$

(ii) $\varphi \mathcal{C} \subset \mathcal{C}$, $\mathcal{C} = \{c(T) : T \in \mathcal{T}\}$

$\{T_1, \dots, T_m\}$, choose $\gamma(T_j) \in \omega(T_j)$

$$\gamma(T_j + x) = \omega(T_j) + \varphi x$$

$$c(T) = \bigcap_{n=0}^{\infty} \varphi^{-n}(\gamma^n(T))$$

Address Map

$J = \langle C \rangle =$ $\mathbb{Z}$-module generated by C ✓

= all integer linear comb.

Lemma. J is finitely generated.

Proof: FLC, can jump from neighbor to neighbor

$J \approx \mathbb{Z}^N, \quad N \geq d$

Let $\{v_1, \dots, v_N\}$ be generators

(need not be in C)

$$\forall \xi \in J \ \exists! a = a(\xi) \in \mathbb{Z}^N : \xi = \sum_{j=1}^{N} a_j v_j$$

$\boxed{\xi \mapsto a(\xi)}$ address map

$$V = [v_1, \dots, v_N], \quad d \times N \text{ matrix}$$
$$\operatorname{rank} V = d$$

$$\boxed{\xi = V a(\xi)}$$

PROOF OF LEMMA 12

$$\varphi C \subset C \Rightarrow \varphi J \subset J$$

integer matrix

$$\varphi v_j = \sum_{i=1}^{N} b_{ij} v_i \qquad B = [b_{ij}]_{N \times N}$$

$$\boxed{\varphi V = VB}$$

λ e/value of $\varphi \Rightarrow \exists\, x \in \mathbb{C}^d$, $x \neq 0$, $\varphi^T x = \lambda x$

$$B^T V^T x = V^T \lambda x = \lambda \underbrace{V^T x}_{\neq 0}$$

$\Rightarrow \lambda$ is an e/value of B

$\Rightarrow \lambda$ is an algebraic integer.

$\blacksquare$

What are the Galois conjugates of λ? They are eigenvalues of B

Claim. $\rho(B) \leq \rho(\varphi)$, ρ = spectral radius

Proof $VBa(\xi) = \varphi Va(\xi) = \varphi\xi$

$$\Rightarrow \boxed{a(\varphi\xi) = Ba(\xi)}$$

Sublemma $\xi \mapsto a(\xi)$ is Lipschitz on C

(caution: NOT Lip on $J = \langle C \rangle$ in general)

$$|B^n a(\xi)| = |a(\varphi^n \xi)| \leq L \cdot |\varphi^n \xi|$$

$\sim \rho(B)^n$ for some ξ $\qquad \lesssim \rho(\varphi)^n$

Proof of Prop 13 (self-similar case)

λ - e/value of φ, $|\lambda| = \eta > 1$.

Let γ be a conjugate of λ

Let U_γ be the "eigenspace" of B corr. to γ

(real; 1-dim if $\gamma \in \mathbb{R}$ (~~real~~)
2-dim if $\gamma \notin \mathbb{R}$)

P_γ - projection on U_γ : $BP_\gamma = P_\gamma B$

$$\boxed{\text{Suppose } |\gamma| = \eta. \text{ Want to show that } \gamma \text{ is an eigenvalue of } \varphi}$$

We already know that $|\gamma| \leq \eta = \rho(\varphi)$

Define $f_\gamma : C \to U_\gamma$

$$f_\gamma(\xi) = p_\gamma \, a(\xi)$$

def. $f_\gamma : \varphi^{-k} C \to U_\gamma$ by

$$f_\gamma(\varphi^{-k}\xi) = B^{-k} f_\gamma(\xi)$$

consistent

f_γ is defined on $\bigcup_{k=0}^{\infty} \varphi^{-k} C$ ← dense in $\mathbb{R}^d$

Claim f_γ is Lip. on

$$|f_\gamma(\varphi^{-k}c_1) - f_\gamma(\varphi^{-k}c_2)| = |B^{-k}(f_\gamma(c_1) - f_\gamma(c_2))|$$

$$= |\gamma|^{-k} |f_\gamma(c_1) - f_\gamma(c_2)| \le L|\gamma|^{-k}|c_1 - c_2|$$

$$= L|\varphi^{-k}(c_1 - c_2)|$$ (use that φ expands by $2 = |\gamma|$)

$\Rightarrow f_\gamma$ extends to Lip $f_\gamma : \mathbb{R}^d \to U_\gamma$

Claim

f_γ is linear (!)

Why?

- Lip => differentiable a.e
- almost flat in a small ball
  expand by φ $\Rightarrow$ almost flat on a large ball

$$\boxed{f_\gamma \circ \varphi = B \circ f_\gamma}$$

- use repetitivity & that f_γ depends only on tile type

$\Downarrow$

almost flat close to the origin

f_γ is onto ($a(\xi)$ span $\mathbb{R}^N$)

$\Downarrow$

$B|_{U_\gamma}$ is <u>linearly</u> conjugate

to $\varphi|_{\text{subspace of } \mathbb{R}^d}$

$\Downarrow$

γ is an e/value of φ

$\blacksquare$

Substitution Delone multisets

A *multiset* or m-*multiset* in $\mathbf{R}^d$ is a subset $\mathbf{\Lambda} = \Lambda_1 \times \cdots \times \Lambda_m \subset \mathbf{R}^d \times \cdots \times \mathbf{R}^d$ (m copies) where $\Lambda_i \subset \mathbf{R}^d$. ($i$ is the "color" of points in Λ_i).

We also write $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_m) = (\Lambda_i)_{i \le m}$.

$\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone multiset* in $\mathbf{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \bigcup_{i=1}^m \Lambda_i \subset \mathbf{R}^d$ is Delone.

Definition. $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *substitution Delone multiset* if $\mathbf{\Lambda}$ is a Delone multiset and there exist an expansive map $\phi : \mathbf{R}^d \to \mathbf{R}^d$ and finite sets $\mathcal{D}_{ij}$ for $i, j \le m$ such that

$$\Lambda_i = \bigcup_{j=1}^m (\phi \Lambda_j + \mathcal{D}_{ij}), \quad i \le m, \tag{3}$$

where the unions on the right-hand side are disjoint.

[Lagarias & Wang '03]

LEMMA 14. *If $\mathcal{T}$ is self-affine, $\mathcal{T} = \omega(\mathcal{T})$, then*

$$\mathcal{T} = \bigcup_{j=1}^{m} (T_j + \Lambda_j) \tag{4}$$

where $(\Lambda_i)_{i \le m}$ is a substitution Delone multiset.

Proof. We have, applying ω to both sides of (4),

$$\begin{aligned} \mathcal{T} &= \bigcup_{j=1}^{m} (\omega(T_j) + \phi\Lambda_j) \\ &= \bigcup_{j=1}^{m} \left(\bigcup_{i=1}^{m} (T_i + \mathcal{D}_{ij}) + \phi\Lambda_j \right) \\ &= \bigcup_{i=1}^{m} \left(T_i + \bigcup_{j=1}^{m} (\phi\Lambda_j + \mathcal{D}_{ij}) \right). \end{aligned}$$

Thus,

$$\Lambda_i = \bigcup_{j=1}^{m} (\phi\Lambda_j + \mathcal{D}_{ij}), \quad i \le m,$$

which is (3). $\square$