Tilings and Dynamics

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Plan of the Talks

- Delone sets and associated dynamical systems: finite local complexity, repetitivity, continuous eigenfunctions, uniform cluster frequencies, measure-preserving systems, pure point spectrum.

- Self-affine tilings: tile-substitutions, characterization of expansion maps, substitution Delone sets.

- Eigenvalues for substitution tiling systems: description of eigenvalues for measure-preserving substitution tiling systems, Pisot numbers, Meyer sets.

- Toward complete spectral analysis of substitution systems: spectral measures, pure point spectrum, coincidence conditions, singular component, Lebesgue component, open problems.

I. Delone sets

Definition. $\Lambda \subset \mathbf{R}^d$ is a Delone set if $\exists\, r, R > 0$ with

$$\forall\, y \in \mathbf{R}^d,\ \#(\Lambda \cap B_r(y)) \leq 1;$$

$$\forall\, y \in \mathbf{R}^d,\ \Lambda \cap B_R(y) \neq \emptyset.$$

Λ-*cluster* is a finite subset $P \subset \Lambda$.

Two clusters P and P are *translationally equivalent* if $P = x + P'$ for some $x \in \mathbf{R}^d$.

Finite Local Complexity

Definition. The Delone set Λ has *finite local complexity (FLC)* if for every $R > 0$ there are finitely many translational classes of clusters among $\{(\Lambda - x) \cap B_R(0) : \; x \in \mathbf{R}^d\}$.

This is equivalent to $\Lambda - \Lambda$ being closed and discrete (Exercise).

FLC will be a standing assumption

Metric on the space of Delone sets

$$d(\Lambda_1, \Lambda_2) := \min\{\tilde{d}(\Lambda_1, \Lambda_2), 2^{-1/2}\},$$

where

$$\tilde{d}(\Lambda_1, \Lambda_2) = \inf\{\varepsilon > 0 : \exists\, x, y \in B_\varepsilon(0),$$

$$B_{1/\varepsilon}(0) \cap (-x + \Lambda_1) = B_{1/\varepsilon}(0) \cap (-y + \Lambda_2)\}.$$

Λ_1 and Λ_2 are close if they agree on a large neighborhood around the origin after a small translation.

Exercise. Prove that d is a metric.

Dynamical system from a Delone set

$X_\Lambda := \overline{\{-h + \Lambda : h \in \mathbf{R}^d\}}$ with the metric d.

THEOREM 1. *The metric space X_Λ is compact iff Λ has FLC.*

The group $\mathbf{R}^d$ acts on X_Λ by translations which are homeomorphisms, and we get a topological dynamical system $(X_\Lambda, \mathbf{R}^d)$.

Repetitivity

Definition. Delone set Λ is *repetitive* if for all $R > 0$ there is $C(R) > 0$ such that every ball of radius $C(R)$ contains translates of all clusters of diameter R. (Recall that FLC is always assumed.)

Λ is *linearly repetitive* (LR) if $C(R) \le \text{const} \cdot R$.

THEOREM 2. *Λ is repetitive iff $(X_\Lambda, \mathbf{R}^d)$ is minimal, that is,*

$$\overline{\{-h + \Gamma : h \in \mathbf{R}^d\}} = X_\Lambda, \ \forall \, \Gamma \in X_\Lambda.$$

Locator sets

$$\Psi_r(\Lambda) := \{x \in \mathbf{R}^d : \ \Lambda \cap B_r(0) = (\Lambda - x) \cap B_r(0)\}$$

Properties (exercise) (i) $\Psi_r(\Lambda) \subset \Psi_s(\Lambda), \ r < s$;

(ii) $\Psi_r(\Lambda)$ is uniformly discrete;

(iii) Λ is repetitive iff $\Psi_r(\Lambda)$ is relatively dense for all $r > 0$;

(iv) Λ is non-periodic (i.e. $\Lambda - h \neq \Lambda$ for $h \neq 0$) iff

$$\inf\{|x - y| : \ x, y \in \Psi_r(\Lambda), x \neq y\} \to \infty, \ r \to \infty.$$

Remark. [N. Priebe Frank '00] defined *derived Voronoi tilings* using the locator sets (the tiles are the Voronoi cells, and the labels are equivalence classes of appropriate clusters)

Continuous eigenfunctions

Definition. $\alpha \in \mathbf{R}^d$ is an eigenvalue for the topological dynamical system $(X_\Lambda, \mathbf{R}^d)$ if there exists a continuous $f : X_\Lambda \to \mathbf{C}$, $f \not\equiv 0$, such that

$$f(\Gamma - x) = e^{2\pi i \langle x, \alpha \rangle} f(\Gamma), \ \forall x \in \mathbf{R}^d, \ \forall \Gamma \in X_\Lambda.$$

In general, given a locally compact Abelian group G and a G-action, the eigenvalues are the elements of the **dual group** $\widehat{G}$.

THEOREM 3. *Let Λ be a repetitive Delone set. Then $\alpha \in \mathbf{R}^d$ is an eigenvalue for $(X_\Lambda, \mathbf{R}^d)$ iff*

$$\lim_{r\to\infty} \sup_{x \in \Psi_r(\Lambda)} |e^{2\pi i \langle x, \alpha \rangle} - 1| = 0.$$

Proof Sketch of Theorem 3. ⇒

Let f be a continuous eigenfunction corresponding to α. It is uniformly continuous since X_Λ is compact. Fix $\epsilon > 0$.

$$x \in \Psi_r(\Lambda) \;\Rightarrow\; \Lambda \cap B_r(0) = (\Lambda - x) \cap B_r(0)$$

$$\Rightarrow\; d(\Lambda, \Lambda - x) \leq 1/r \text{ (for large } r)$$

$$\Rightarrow\; |f(\Lambda) - f(\Lambda - x)| \text{ is small (for large } r)$$

$$|f(\Lambda) - f(\Lambda - x)| = |f(\Lambda)|\,|e^{2\pi i \langle x, \alpha\rangle} - 1|$$

$f(\Lambda) \neq 0$ by minimality, hence

$$|e^{2\pi i \langle x, \alpha\rangle} - 1| < \epsilon \text{ (for large } r)$$

□

⇐ Define

$$f(\Lambda - x) = e^{2\pi i \langle x, \alpha \rangle}$$

Since the orbit is dense in X_Λ, it suffices to show that f is uniformly continuous on the orbit (then it extends to X_Λ and clearly satisfies the eigenfunction equation).

If $\Lambda - x$ is close to $\Lambda - y$, then they agree on a large neighborhood of the origin, after a small translation. By repetitivity,

$$x - y = x' - y' + w, \text{ where } x', y' \in \Psi_r(\Lambda),$$

with r large and $|w|$ small. Then

$$\begin{aligned} & |f(\Lambda - x) - f(\Lambda - y)| = |e^{2\pi i \langle x, \alpha \rangle} - e^{2\pi i \langle y, \alpha \rangle}| \\ = & \ |e^{2\pi i \langle x - y, \alpha \rangle} - 1| = |e^{2\pi i \langle x' - y' + w, \alpha \rangle} - 1| \\ \leq & \ |e^{2\pi i \langle x', \alpha \rangle} - 1| + |e^{2\pi i \langle y', \alpha \rangle} - 1| + |e^{2\pi i \langle w, \alpha \rangle} - 1| \end{aligned}$$

which is small. □

Uniform Cluster Frequencies

$$L_P(A) := \sharp\{x \in \mathbf{R}^d : \; x + P \subset A \cap \Lambda\}$$

where P is a non-empty cluster and $A \subset \mathbf{R}^d$ is bounded, i.e. $L_P(A)$ is the number of translates of P contained in A.

Sequence $\{F_n\}_{n\geq 1}$ of bounded measurable subsets of $\mathbf{R}^d$ is **van Hove** if

$$\lim_{n\to\infty} \mathrm{Vol}(\partial F_n + B_r(0))/\mathrm{Vol}(F_n) = 0, \text{ for all } r > 0.$$

Definition. Let $\{F_n\}_{n\geq 1}$ be a van Hove sequence. The Delone set Λ has *uniform cluster frequencies* (UCF) (relative to $\{F_n\}_{n\geq 1}$) if for any non-empty cluster P, the limit

$$\mathrm{freq}(P, \Lambda) = \lim_{n\to\infty} \frac{L_P(x + F_n)}{\mathrm{Vol}(F_n)} \geq 0$$

exists uniformly in $x \in \mathbf{R}^d$.

Unique Ergodicity

A topological dynamical system is *uniquely ergodic* if it has a unique invariant Borel probability measure.

THEOREM 4. *Let Λ be a Delone set. Then the dynamical system $(X_\Lambda, \mathbf{R}^d)$ is uniquely ergodic iff Λ has UCF.*

COROLLARY 5. *The existence of UCF does not depend on the van Hove sequence.*

THEOREM 6. [Lagarias & Pleasants '03] *Any linearly repetitive Delone set has UCF.*

Proof Sketch of UCF $\Rightarrow$ Unique Ergodicity. Using approximation by step-functions, we obtain from UCF

$$(I_n)(\Gamma, f) := \frac{1}{\mathrm{Vol}(F_n)} \int_{F_n} f(-g + \Gamma)\, dg \to \text{ const}, \tag{1}$$

as $n \to \infty$, for any continuous $f : X_\Lambda \to \mathbf{C}$, uniformly in $\Gamma \in X_\Lambda$, with the constant depending on f.

For any invariant measure μ, exchanging the order of integration yields

$$\int_{X_\Lambda} I_n(\Gamma, f)\, d\mu(\Gamma) = \int_{X_\Lambda} f\, d\mu,$$

so by the Dominated Convergence Theorem, the constant in (1) is $\int_{X_\Lambda} f\, d\mu$. If there is another invariant measure ν, then $\int_{X_\Lambda} f\, d\mu = \int_{X_\Lambda} f\, d\nu$ for all $f \in C(X_\Lambda)$, hence $\mu = \nu$. $\square$

Measure-preserving systems

For Λ with UCF, $(X_\Lambda, \mathbf{R}^d)$ has a unique invariant measure μ, so we consider the measure-preserving system $(X_\Lambda, \mu, \mathbf{R}^d)$.

Tools and Concepts of Ergodic Theory:
Ergodic Theorem, mixing properties, spectral properties.

Eigenvalues

Consider the associated group of unitary operators $\{U_g\}_{g \in \mathbf{R}^d}$ on $L^2 = L^2(X_\Lambda, \mu)$:

$$U_g f(\mathcal{S}) = f(-g + \mathcal{S}).$$

A vector $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{R}^d$ is an eigenvalue for the $\mathbf{R}^d$-action if there exists an eigenfunction $f \in L^2(X_\Lambda, \mu)$, that is, $f \not\equiv 0$ and

$$U_g f = e^{2\pi i \langle g, \alpha \rangle} f, \ \forall \ g \in \mathbf{R}^d.$$

The set of eigenvalues is a **subgroup** of $\mathbf{R}^d$.

Pure point spectrum and quasicrystals

The dynamical system $(X_\Lambda, \mu, \mathbf{R}^d)$ has *pure point* (or pure discrete) *spectrum* if the linear span of the eigenfunctions is dense in L^2.

POINTS OF A DELONE SET $\longleftrightarrow$ ATOMS OF A SOLID

The X-ray diffraction spectrum is, in some sense, a "part" of the dynamical spectrum [Dworkin '93]. Bragg peaks are the sharp, bright spots in the diffraction picture, associated with crystals or quasicrystals.

EIGENVALUES $\longleftrightarrow$ BRAGG PEAKS

THEOREM 7 [Lee, Moody & S.'03], [Baake & Lenz '04], [Gouéré '05] *Pure point dynamical spectrum is equivalent to pure point diffraction.*

Conditions for Pure Point Spectrum

Suppose that Λ is a repetitive Delone set with UCF. Say that $x \in \mathbf{R}^d$ is a δ-almost period for Λ if

$$\mathrm{freq}(\Lambda \setminus (\Lambda + x)) \leq \delta$$

(exists by UCF).

THEOREM 8 [S. '98], [Baake & Moody '04], [Gouéré '05] *Suppose that Λ is a repetitive Delone set with UCF. Then $(X_\Lambda, \mu, \mathbf{R}^d)$ has pure point spectrum iff for every $\delta > 0$, the set of δ-almost periods for Λ is relatively dense in $\mathbf{R}^d$.*