Pattern-Equivariant Cohomology

(for now, we're back to translations only. We'll do rotations later)

Consider a tiling T, viewed as a marked $\mathbb{R}^2$. A function f on $\mathbb{R}^2$ is P-equivariant with radius R if, whenever $T-x$ and $T-y$ agree on $B_R(0)$, $f(x) = f(y)$

$C^\infty_{P,R}$ = P-equivariant functions of radius R

$$C^\infty_P = \bigcup_R C^\infty_{P,R}$$ (R is arbitrarily large, but finite)

P-equivariant 1-forms

$$f_1(x)\,dx + f_2(x)\,dy, \qquad f_1, f_2 \in C_P^\infty$$

2 forms $\quad f(x)\,dx\,dy \qquad f \in C_P^\infty$

Let $\Lambda_P^k = \{P\text{-eq } k\text{-forms}\}$

Lemma: If $\omega \in \Lambda_P^k$, $d\omega \in \Lambda_P^{k+1}$

$d^2 = 0$ as usual, so we have a complex

$$0 \to \Lambda_P^0 \to \Lambda_P^1 \to \Lambda_P^2 \to 0$$

$$H_P^*(T) = \frac{\text{closed}}{\text{exact}}$$

N.B. "exact" means d of something P-equivariant. On $\mathbb{R}^2$ every closed form is d of something.

<u>Thm</u> (Kellendonk-Putnam)

$$H^k_P(T) \cong \check{H}^k(\Omega_T, \mathbb{R})$$

<u>Cor</u> If Ω_T minimal (T repetitive), all tilings $T' \in \Omega_T$ give the same $H^k_P(T')$

How to view P-equivariance.

Functions on Gähler approximants.

Let a = inner diameter of smallest tile
A = outer diameter of biggest tile

Functions on K_n see n layers
$\Rightarrow$ are P-equivariant with $R = (n+1)A$

P-equivariant functions w/ radius R see at most R/a layers, are pullbacks of functions on $K_{\lceil \frac{R}{a} \rceil +1}$

In other words,

$$\Lambda^0_P = \bigcup_n \pi_n^* C^\infty(K_n)$$

likewise for forms, so

$$H^k_P(T) = \frac{\text{closed } P\text{-eq } k\text{-forms on } T}{d\,(P\text{-eq } k\text{-1 forms on } T)}$$

$$= \frac{\varinjlim \text{ closed forms on } K_n}{\varinjlim \text{ exact forms on } K_n}$$

$$= \varinjlim H^*_{DR}(K_n) = \varinjlim \check{H}^*(K_n, \mathbb{R})$$

$$= \check{H}^*(\Omega_T, \mathbb{R})$$

Hold it! What does smooth mean on a branched manifold?!

Near a branch, mfld is described by several charts

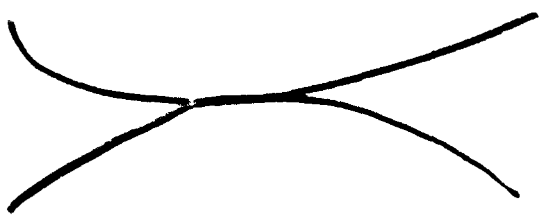
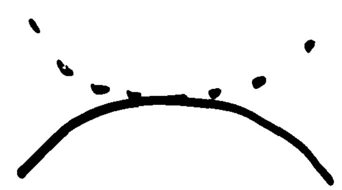


Chart 1

2:



3:

4

"Smooth" means C^∞ w.r.t. all charts.

This means all derivatives <u>uniquely</u> defined and continuous at branch set.

Forms are well-defined and de-Rham thm holds: $H^*_{DR}(K_n) = H^*_{\text{any}}(K_n, \mathbb{R})$

What if we want $\mathbb{Z}$ coefficients?

T isn't just a point pattern – it's a TILING, a cell decomposition of $\mathbb{R}^2$ with natural faces, edges, vertices.

Def: A cochain is P-eq if its value on a cell depends only on its R-nbhd.

As before, $H_P^*(T,\mathbb{Z}) = \dfrac{\text{P-eq. closed cochains}}{\text{P-eq exact cochains.}}$

Inverse limit interpretation just as before.

P-eq cochains = (pullbacks of) cochains on K_n

$$H_P^*(T,\mathbb{Z}) = \varinjlim H^*(K_n) = \check{H}^*(\Omega_T)$$

Interpreting Cohomology

Look at arrow tiling $\boxed{\nearrow} \Rightarrow \begin{array}{|c|c|}\hline \searrow & \nearrow \\ \hline \nearrow & \nwarrow \\ \hline \end{array}$

Cohomology of K_1 (collared tiles) is

$$H^2(K_1) = \mathbb{Z}^3 = \text{scalar} + \text{vector}$$

$$= \underbrace{\begin{pmatrix} 1 \text{ on all} \\ \text{tiles} \end{pmatrix}}_{\omega_1} \oplus \underbrace{\begin{pmatrix} 1 \text{ on } \boxed{\nearrow} \\ -1 \text{ on } \boxed{\swarrow} \end{pmatrix}}_{\omega_2} \oplus \underbrace{\begin{pmatrix} 1 \text{ on } \boxed{\nwarrow} \\ -1 \text{ on } \boxed{\searrow} \end{pmatrix}}_{\omega_3}$$

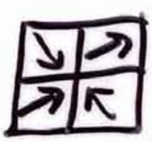
$$H^1(K_1) = \mathbb{Z}^2 = \text{vector}$$

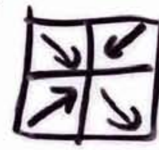
$$= \underbrace{\begin{pmatrix} 1 \text{ on } \bullet\!\!\longrightarrow\!\!\bullet \\ -1 \text{ on } \bullet\!\!\longleftarrow\!\!\bullet \end{pmatrix}}_{\mu_1} \oplus \underbrace{\begin{pmatrix} 1 \text{ on } \uparrow \\ -1 \text{ on } \downarrow \end{pmatrix}}_{\mu_2}$$

$$H^0 = \mathbb{Z} = (1 \text{ on all pts})$$

To understand substitution map, apply to supertiles

$w_1\Big($  $\Big) = 4$

$w_2\Big($  $= 2\Big)$ $\quad w_2\Big($  $\Big) = -2$

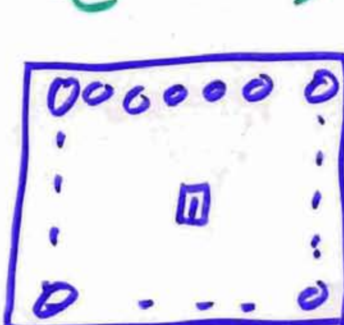
$w_2\Big($  $\Big) = 0 = w_2\Big($  $\Big)$

Likewise, w_3 gives $\begin{cases} 2 \text{ on } \nwarrow \text{ supertiles} \\ -2 \text{ on } \searrow \text{ supertiles} \\ 0 \text{ on } \nearrow \text{ or } \swarrow \text{ supertiles.} \end{cases}$

$\mu_1\Big($  $\Big) = 2 \qquad \mu_2\Big($  $\Big) = 2$

$$\check{H}^*_\Omega = \varinjlim H^*(K_n) = \begin{pmatrix} \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}[\tfrac{1}{2}]^2 \\ \mathbb{Z}[1/2]^2 \\ \mathbb{Z} \end{pmatrix}$$

$\left(\frac{1}{4}\right)^n \in H^2$ looks like



How to incorporate rotations
2 different approaches

1) Kellendonk's method

Think of functions on group G (3d!)

a function $f: G \to \mathbb{R}$ is P-eq w/ radius R if, whenever $g_1 \cdot T$ and $g_2 \cdot T$ agree on $B_R(0)$, $f(g_1) = f(g_2)$. (For higher forms, need $\omega(g_1) = (g_2 g_1^{-1})^* \omega(g_2)$)

(or maybe pull back by right-multiplication by $g_1^{-1} g_2$)

<u>Thm</u> $H^*_{P,T}(G) \simeq \check{H}^*(\Omega_{T,rot})$

Tilings of $\mathbb{R}^2$ with rotations yield H^0, H^1, H^2, H^3

Rand's method

Think of functions on TILING, not on group.

Pick representation $\rho: G_0 \to End(V)$
where G_0 = rotational part of symmetry group,
V = representation space

V-valued functions are ρ-eq if,
whenever $T-x$ and $g_0(T-y)$ agree on $B_R(0)$,

$$f(x) = \rho(g_0)\, f(y)$$

For forms, g_0 also acts on dx, dy in usual way.

Can also work with $V = \mathbb{Z}^d$ (not $\mathbb{R}^d$) using tiling cells.

If $T-x$ and $g_0(T-y)$ agree on $B_R(0)$, and cell α at x corresponds to β at y,

$$\omega(\alpha) = \rho(g_0)\, \omega(\beta)$$

What does this tell us?

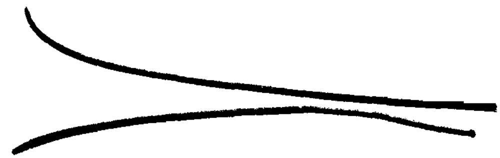
Rand's ρ-eq cohomology gives info about Ω_T and action of G_0

Guess: related to G_0-equivariant cohomology of Ω_T.

But what's the role of the representation? So far unclear.

Other approaches to tiling topology (Mostly 1D, Barge - Diamond)

Asymptotic composants

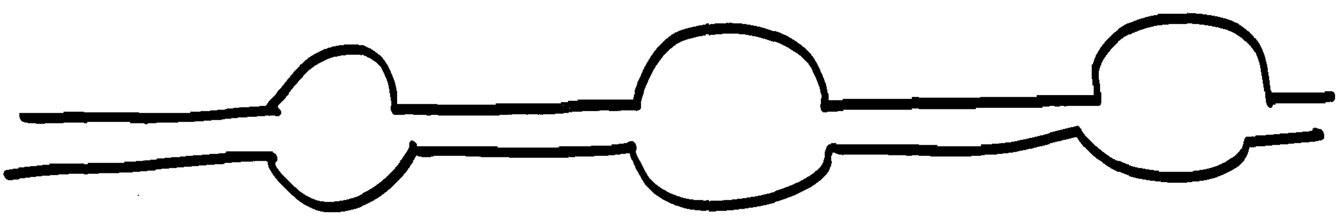


tilings identical after (before) some pt.

Finite # of such pairs. # and arrangement is topological invariant.

Augmented cohomology. Glue together some asymptotic ends and compute H^* of resulting space

Proximal ends



Open problems

1) Understand spaces w/o FLC
2) When spaces are nearly the same (differ on thin set), how do their cohomologies relate?

$$\begin{tikzcd}
\Omega_{chair} \arrow[d, "\text{a.e. } 1\text{-}1"] & (\mathbb{Z}, \mathbb{Z}[\tfrac{1}{2}]^2, \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}[\tfrac{1}{2}]^2) \\
(\text{Dyadic solenoid})^2 & (\mathbb{Z}, \mathbb{Z}[\tfrac{1}{2}]^2, \mathbb{Z}[1/4]) \arrow[u]
\end{tikzcd}$$

3) Understand asymptotic ends in $d>1$
4) Rotations in $d>2$ (inverse limit structure is clear. H^* is not)
5) Other topologies (auto-correlation)
6) Tilings of other than $\mathbb{R}^d$ (e.g. hyperbolic space)
7) How to pronounce "Meyer"

Finally, I'd like to thank the organizers for giving me the opportunity to (finally) stop speaking.