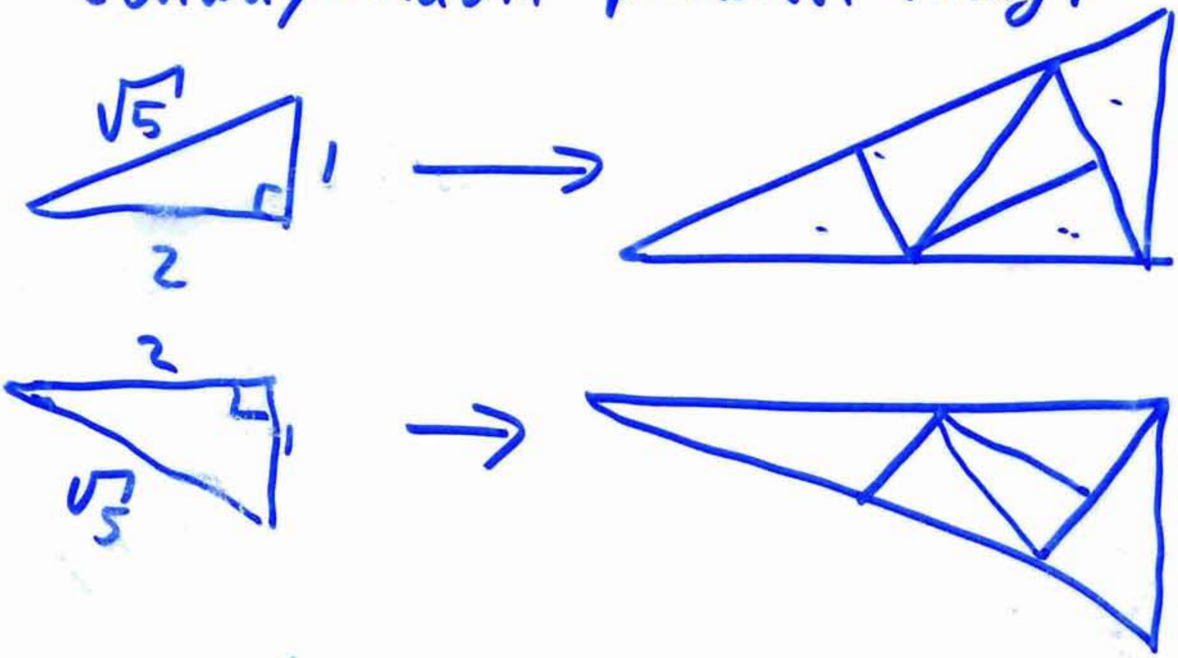
Tilings and Rotations

Conway-Radin pinwheel tiling:



$$M = \begin{pmatrix} r + rs^2 & 2r^{-1} + r^{-1}s^3 \\ 2r + rs & r^{-1} + r^{-1}s^2 \end{pmatrix}, \text{where}$$

$r =$ rotation by $\tan^{-1}(1/2) =$ irrational angle

$s =$ rotation by $\pi/2$

$M^n = \left(\text{terms with } r^n, r^{n-2}, \ldots r^{-n}\right)$

In limit, uniformly distributed in $O(2)$

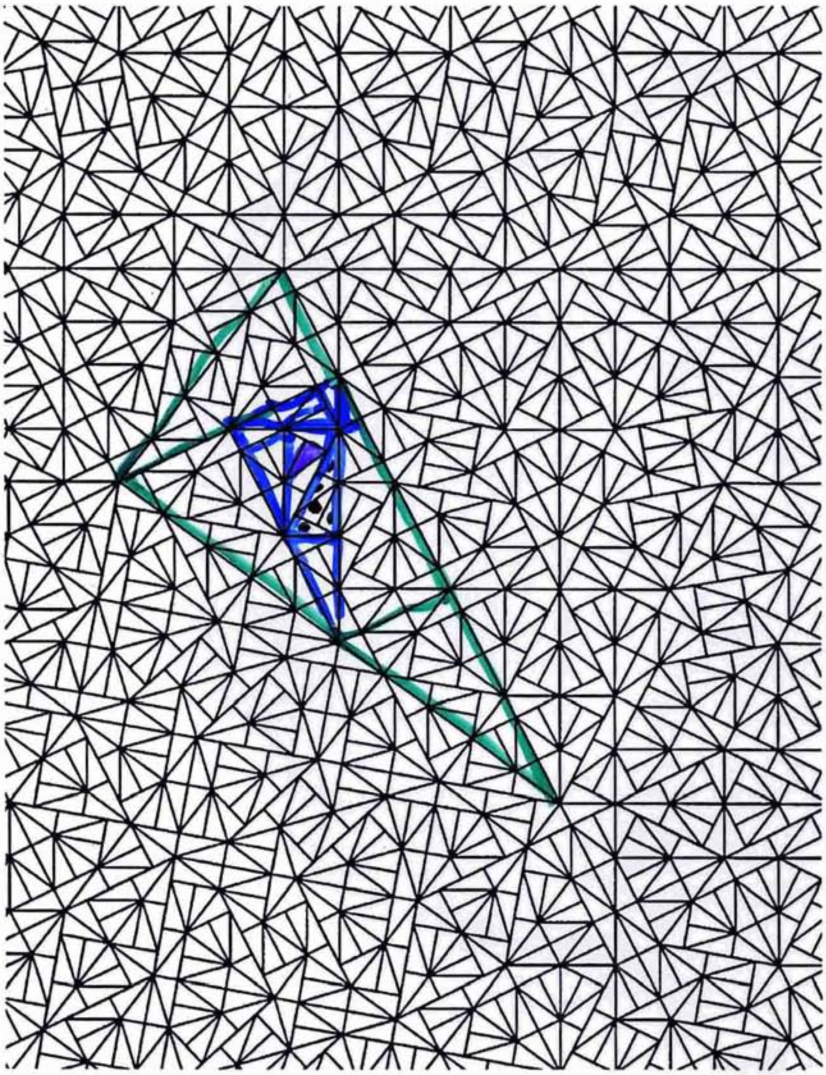


Figure 6. A pinwheel tiling.

ant pinwheel tilings also necessarily have tiles in an infinite number of distinct orientations. In fact, the relative orientation groups for all pinwheel tilings are algebraically isomorphic. Theorem 1 shows that the tiling spaces for the pinwheel and (2,3)-pinwheel are not homeomorphic.

In all the above cases, it is easy to construct explicit examples of tilings. Pick a tile to include the origin of the plane. Embed this tile in a tile of level 1 (there are several ways to do this). Embed that tile of level 1 in a tile of level 2, embed that in a tile of

$\Omega_{pinwheel}$ not compact in original topology. Need new tiling metric:

T and T' are ϵ-close if they agree on $B_{1/\epsilon}(0)$, up to ϵ-small element of Euclidean group (ϵ-translation and ϵ-rotation)

New assumptions for tilings

1) Finite # of tile types up to Euclidean motion
2) Tiles are polygons
3) Tiles meet full-edge to full-edge

(Almost) equivalently, for each R $\exists$ only finitely many patches of size R, up to Euclidean motion.

Local structure & global topology

What does ϵ-nbhd of T look like?

3 continuous degrees of freedom:
2 translations + 1 rotation.

(In d-dimensions, $d(d+1)/2$ degrees of freedom)

Discrete degrees of freedom still
Cantor-like.

locally, $\Omega_{pinwheel} \simeq \mathbb{R}^3 \times \text{Cantor}$

As before, Ω_σ connected but not
path-connected

Uncountably many path components

Each path component = Euclidean orbit
$\simeq G$ or G/finite

Inverse limit structures much as before

1) Gähler-like construction.

K_n = instructions for placing n-collared tile at origin

Each cell $\simeq$ tile $\times S^1$ or tile $\times S^1$/finite

AP-like construction (Ormes, Radin, S)

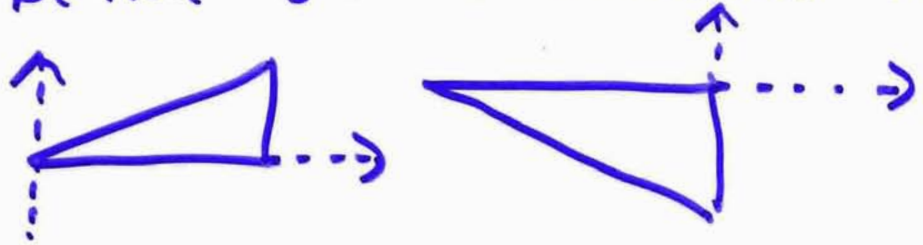
K_n = instructions for placing n-th order supertile at origin

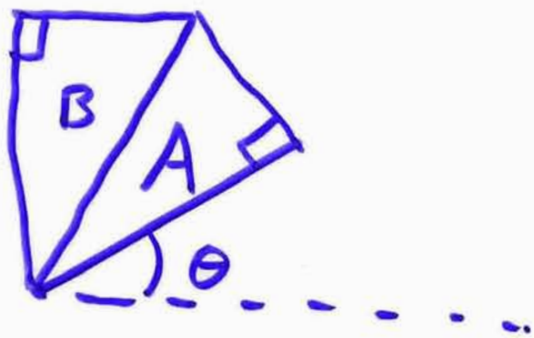
Each cell $\simeq (\lambda^n \cdot \text{tile}) \times S^1$ or $(\lambda^n t \times S^1)$/finite.

ther way: $\Omega = \varprojlim K$

Stitching the tiles

Define std orientations for each tile

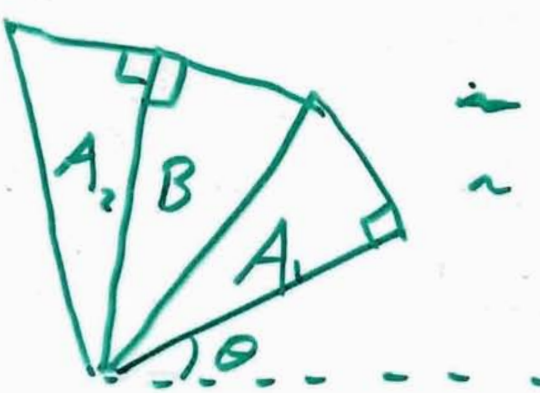




Hypotenuse of A at angle θ

$\sim$ Hypotenuse of B at angle $\theta + 2\tan^{-1}(\frac{1}{2})$

Warning: Must collar at least once to avoid bad fibers!



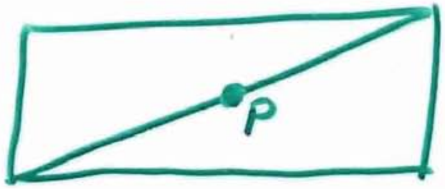
Vertex of A_1 at θ

$\sim$ Vertex of B at $\theta + 2\tan^{-1}(\frac{1}{2})$

$\sim$ Vertex of A_2 at $\theta + 2\tan^{-1}(\frac{1}{2})$

No problem if collared since $A_1 \not\sim A_2$

Even with collaring, can have finite degeneration of fibers:



$(\rho, \theta) \sim (\rho, \theta+\pi)$

Situation just like Seifert-fibered 3-manifolds: 3d structure is smooth, but quotient by S^1 has cone points.

Local model for singular fibers

$(\text{Disk} \times S^1)/\mathbb{Z}_2 \quad : \quad (r, \theta, \phi) \sim (r, \theta+\pi, \phi+\pi)$

No fixed pt of $\mathbb{Z}_2$ action, so quotient is smooth 3-manifold

S^1 action: $(r, \theta, \phi) \to (r, \theta, \phi+\alpha)$

Almost an S^1 bundle over $\text{Disk}/\mathbb{Z}_2$, but fiber over 0 is small (π vs 2π).

S^1-quotient space

In calculations, S^1 factor mostly comes along for the ride, so look at

$$\Omega_0 = \Omega_T / S^1 = \varprojlim K_n / S^1$$

$K_n / S^1 =$ branched 2-orbifold
(cone points)

Thm (?) (Hunton, Kellendonk, S-)

$$H^*(\Omega_T, \mathbb{R}) = H^*(\Omega_0 \times S^1, \mathbb{R}) = H^*(\Omega_0, \mathbb{R}) \otimes H^*(S^1, \mathbb{R}),$$

but

$$H^*(\Omega_T, \mathbb{Z}) = H^*(\Omega_0 \times S^1, \mathbb{Z}) \oplus \text{torsion in } H^2$$

with torsion coming from singular fibers

If $2a = 2c = b$, then $2(a-c) = 0$

Moral: Finite rotational symmetry matters, but infinite rotations don't!

So apply rotational techniques to tilings with finite rotational symmetry, like chair ($\mathbb{Z}_4$) and Penrose ($\mathbb{Z}_{10}$)

3 spaces of chairs

Ω_1 = "translational" space
= {all chair tilings with edges parallel to x, y axes}
= closure of <u>translational</u> orbit of one tiling (any tiling)

Ω_{rot} = "rotational" space
= {all chair tilings in any orientation}
= closure of Euclidean orbit

$\Omega_0 = \Omega_1 / \mathbb{Z}_4 = \Omega_{rot} / S^1$
= {all chair tilings} / rotation about 0.

Similar story for Penrose, only with $\mathbb{Z}_{10}$ instead of $\mathbb{Z}_4$.

Rotations and Cohomology

$r =$ rotation by $\pi/2$ $\left(\begin{array}{l}\text{for chair, or} \\ 2\pi/10 \text{ for Penrose}\end{array}\right)$

$r : \Omega_1 \to \Omega_1$ induces

$r^* : H^*(\Omega_1) \to H^*(\Omega_1)$

How does H^* transform? Decompose $H^*(\Omega_1)$ into irreps of $\mathbb{Z}_4$. ($\mathbb{Z}_{10}$)

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r also acts on Ω_{rot}, but here $r \sim$ identity, so r^* is trivial.

$H^*(\Omega_{rot})$ is rotationally invariant

$H^*(\Omega_0) =$ rotationally invariant part of $H^*(\Omega_1)$

Richest theory is $H^*(\Omega_1)$.

As with pinwheel, cohomologies are related.

$$H^*(\Omega_0) = \text{invariant part of } H^*(\Omega_1)$$

$$H^*(\Omega_{rot}, \mathbb{R}) = H^*(\Omega_0 \times S^1, \mathbb{R}), \text{so}$$

$$H^k(\Omega_{rot}, \mathbb{R}) = H^{k-1}(\Omega_0, \mathbb{R}) \oplus H^k(\Omega_0, \mathbb{R}), \text{ but}$$

$$H^2(\Omega_{rot}, \mathbb{Z}) = H^2(\Omega_0, \mathbb{Z}) \oplus H^1(\Omega_0, \mathbb{Z})$$

$\oplus$ torsion from singular fibers.

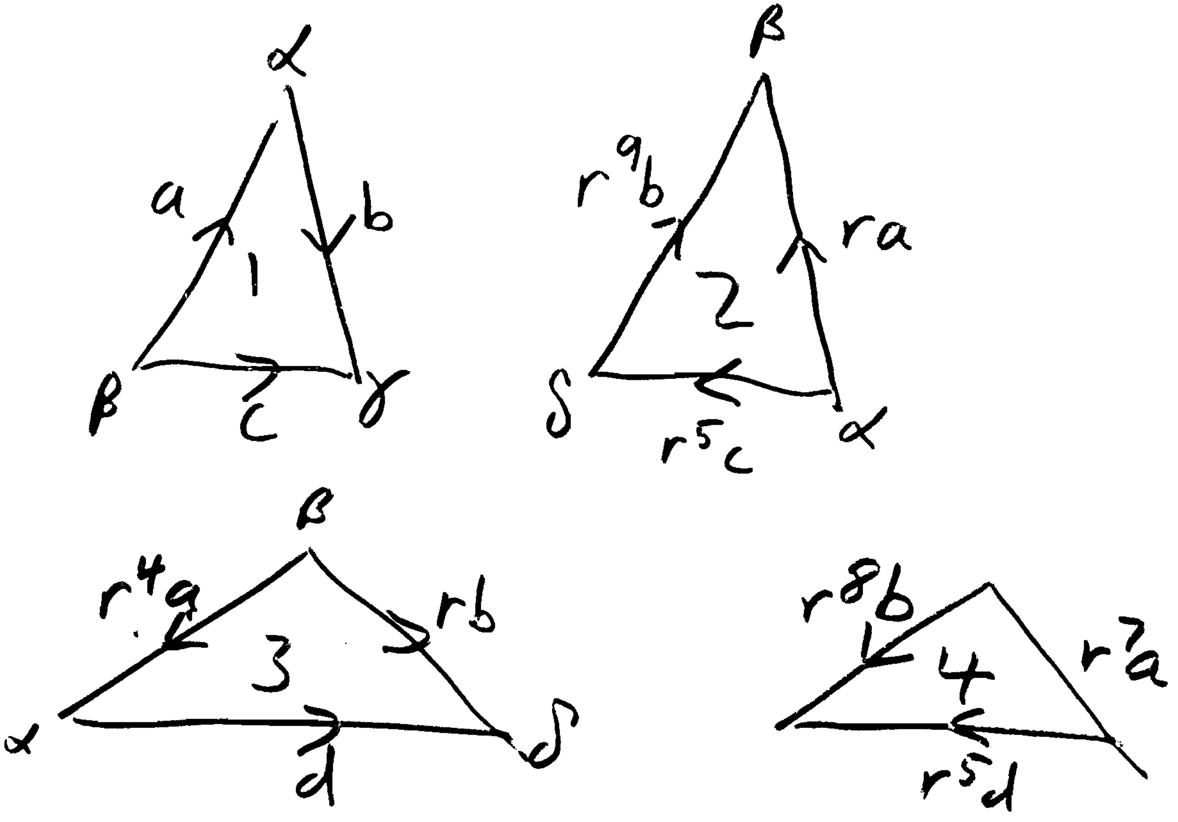
Penrose Cohomology

4 tile types, 10 orientations.

DON'T need to collar.

Substitution matrices are invertible in all reps, so

$$H^*(\Omega_1) = H^*(K_0)$$



$r\alpha = \beta \quad r\gamma = \delta$
$r\beta = \alpha \quad r\delta = \gamma$

so C_0 (and C^0) only

has $r = 1$ and $r = -1$ reps.

C^1 and C^2 have all reps.

$$\partial_2 = \begin{pmatrix} -1 & r & r^4 & -r^7 \\ -1 & r^9 & -r & r^8 \\ 1 & -r^5 & 0 & 0 \\ 0 & 0 & 1 & -r^5 \end{pmatrix} = \delta_1^T$$

(all reps)

$$\partial_1 = \begin{pmatrix} 1-r & -1 & -r & -1 \\ 0 & 1 & 1 & r \end{pmatrix} = \delta_0^T$$

($r = \pm 1$) only

Analyze one irrep at a time:

$r = 1$ (rotationally invariant, or scalar)

$r = -1$ (pseudo scalar)

$r^4 + r^3 + r^2 + r + 1 = 0$ (pseudo-vector; primitive 5th root of 1)

$r^4 - r^3 + r^2 - r + 1 = 0$ (vector; primitive 10th root)

$\underline{r=1:}$

$$\mathbb{Z}^2 \xrightarrow{\delta_0} \mathbb{Z}^4 \xrightarrow{\delta_1} \mathbb{Z}^4$$

$$\delta_0 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \delta_1 = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

$H^0 = \operatorname{Ker} \delta_0 = \mathbb{Z}$

$H^1 = \dfrac{\operatorname{Ker} \delta_1}{\operatorname{Im} \delta_0} = \mathbb{Z}$

$H^2 = \mathbb{Z}^4 / \operatorname{Im} \delta_1 = \mathbb{Z}^2$

$r=-1$

$$\mathbb{Z}^2 \xrightarrow{\delta_0} \mathbb{Z}^4 \xrightarrow{\delta_1} \mathbb{Z}^4$$

$$\delta_0 = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad \delta_1 = \begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$H^0 = \operatorname{Ker} \delta_0 = 0$, $H^1 = \operatorname{Ker} \delta_1 / \operatorname{Im} \delta_0 = 0$, $H^2 = \operatorname{coker} \delta_1 = \mathbb{Z}^2$

$r^4 + r^3 + r^2 + r + 1 = 0$

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{\delta_1} \mathbb{Z}^4$$

δ_1 isomorphism, $H^0 = H^1 = H^2 = 0$

Finally, $r^4 - r^3 + r^2 - r + 1 = 0$ (vector)

$$0 \to \mathbb{Z}^4 \xrightarrow{\delta_1} \mathbb{Z}^4$$

δ_1 has rank 3, $H^1 = H^2 = \mathbb{Z} \otimes \overset{4\text{-d}}{\text{representation}}$

Summary:

$H^0(\Omega_1) = \mathbb{Z}$ (scalar)

$H^1(\Omega_1) =$ one copy of trivial rep $(\mathbb{Z})$
+ one copy of vector rep $(\mathbb{Z}^4)$ $\Big\}\ \mathbb{Z}^5$

$H^2(\Omega_1) =$ 2 copies of trivial $(\mathbb{Z}^2)$
2 copies of $r = -1$ $(\mathbb{Z}^2)$
1 copy of vector $(\mathbb{Z}^4)$ $\Big\}\ \mathbb{Z}^8$

$H^*(\Omega_0) =$ rotationally invariant part of $H^*(\Omega_1)$

$H^0(\Omega_0) = \mathbb{Z}$

$H^1(\Omega_0) = \mathbb{Z}$

$H^2(\Omega_0) = \mathbb{Z}^2$

What about $H^*(\Omega_{rot})$?

Rationally, it's $H^*(\Omega_0 \times S^1)$. But there are 2 singular ($\mathbb{Z}_5$) fibers, yielding torsion in H^2. So

$$H^0(\Omega_{rot}) = \mathbb{Z}$$

$$H^1(\Omega_{rot}) = \mathbb{Z} \oplus \mathbb{Z} = H^0(\Omega_0) \oplus H^1(\Omega_0)$$

$$H^2(\Omega_{rot}) = \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}_5$$

$$H^3(\Omega_{rot}) = \mathbb{Z}^2.$$

NB: Computation of $H^*(\Omega_1)$ and $H^*(\Omega_0)$ is old and checked. Calculation of $H^*(\Omega_{rot})$ is new and suspect.