Tiling Cohomology

Why compute tiling cohomology?

1) Because we can.

2) Simpler invariants don't work

⋮

3) H^* gives interesting information.

Why not fundamental group, or π_n, or H_n?

$\pi_0 = H_0$ counts path components.
But for Ω_T, path component = orbit, and $\exists$ uncountably many orbits.
$\pi_0(\Omega_T) = H_0(\Omega_T) = \mathbb{Z}^{\aleph_1}$

Higher homotopy groups probe a single path component. But each orbit is (typically) contractible, so $\pi_n(\Omega_T) = 0$ for $n > 0$. Likewise $H_n(\Omega_T) = 0$.

Čech cohomology does better

$\check{H}^0$ measures <u>connected</u> components, not path components, so $\check{H}^0(\Omega_T) = \mathbb{Z}$

Čech cohomology behaves well under inverse limits:

$$\boxed{\check{H}^*(\varprojlim_{\sigma} K) = \varinjlim_{\sigma^*} \check{H}^*(K) = \varinjlim_{\sigma^*} H^*(K)}$$

So what does $\check{H}^*$ mean?

So what does $\varinjlim$ mean?

Direct Limits

Given: a family of groups G_α indexed by partially ordered set I.

If $\alpha, \beta \in I$, $\exists\ \gamma \in I$ such that $\alpha < \gamma$, $\beta < \gamma$.
(directed set)

If $\alpha < \gamma$, $\exists$ homomorphism $\rho_{\gamma\alpha} : G_\alpha \to G_\gamma$

If $\alpha < \beta < \gamma$, $\rho_{\gamma\alpha} = \rho_{\gamma\beta} \circ \rho_{\beta\alpha}$

$\varinjlim_{\rho} G_\alpha = \coprod G_\alpha / \sim$

If $\alpha < \gamma$, $x \in G_\alpha$, $x \sim \rho_{\gamma\alpha}(x) \in G_\gamma$

[I]f $x \in G_\alpha$, $y \in G_\beta$, $xy = \rho_{\gamma\alpha}(x)\,\rho_{\gamma\beta}(y) \in G_\gamma$

[H]wk: Show this is well-defined, and that $\varinjlim_{\rho} G_\alpha$ is a group

[N]ote: Same construction works for rings

Examples of Direct Limits

1) $I = \mathbb{N}$, $G_\alpha = \mathbb{Z}$

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots$$

$(n,\alpha) \sim (2n, \alpha+1) \to$ identify with $\frac{n}{2^\alpha}$

$$\varinjlim_{\times 2} \mathbb{Z} = \mathbb{Z}[\tfrac{1}{2}] = \left\{ \frac{n}{2^m} \right\}$$

2) $I = \mathbb{N}$, $G_\alpha = \mathbb{Z}$, $\rho_{\gamma\alpha} = 0$ } everything is identified to zero

$\varinjlim_{\times 0} \mathbb{Z} = 0.$

3) $I = \mathbb{N}$, $G_\alpha = \mathbb{Z}^2$

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow \cdots$$

Each matrix is isomorphism, so $\varinjlim \mathbb{Z}^2 = \mathbb{Z}^2$.

Also have order structure $\simeq$ embedding in $\mathbb{R}$.

Let $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} > 0$ if $n_1 + \tau n_2 > 0$. Then

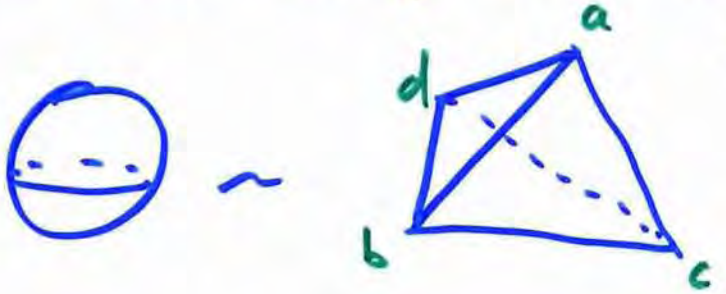
$\mathbb{Z}^2 \simeq \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{R}.$

Hw: Show this positivity is the same as having $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ have positive entries for m large enough

What the Čech is cohomology??

First consider simple space, like S^2.

View as simplicial complex:



Edges $A = \overrightarrow{ab}$ $E = \overrightarrow{bc}$
$B = \overrightarrow{ad}$ $F = \overrightarrow{cd}$
$C = \overrightarrow{ac}$
$D = \overrightarrow{db}$

Faces $\alpha = bdc$, $\beta = acd$, $\gamma = adb$, $\delta = abc$

Chains: $C_0 = \mathbb{Z}^4 \xleftarrow{\partial_1} C_1 = \mathbb{Z}^6 \xleftarrow{\partial_2} C_2 = \mathbb{Z}^4$

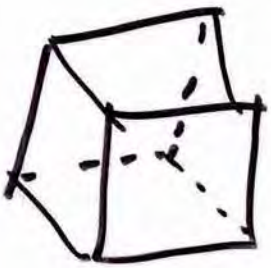
Cochains $C^0 = \mathbb{Z}^4 \xrightarrow{\delta_0} C^1 = \mathbb{Z}^6 \xrightarrow{\delta_1} C^2 = \mathbb{Z}^4$

$$\delta_0 = \partial_1^T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad \delta_1 = \partial_2^T = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$H^0 = \operatorname{Ker} \delta_0 = \mathbb{Z}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad H^1 = \frac{\operatorname{Ker} \delta_1}{\operatorname{Im} \delta_0} = 0$$

$$H^2 = \frac{C^2}{\operatorname{Im} \delta_1} = \mathbb{Z}$$

Why  rather than

 or  or ....?

Why not consider ALL simplicial decompositions Σ_α of S^2?

Let $\alpha < \beta$ if Σ_β is a refinement of Σ_α, in which case $\exists$ a simplicial map $\Sigma_\beta \to \Sigma_\alpha$ (identity on vertices of Σ_α), inducing chain map $C^*(\Sigma_\alpha) \to C^*(\Sigma_\beta)$, inducing $\rho_{\beta\alpha} : H^*(\Sigma_\alpha) \to H^*(\Sigma_\beta)$

Could define

$$\overset{s}{H}{}^*(S^2) = \varinjlim_{\rho} H^*(\Sigma_\alpha)$$

Usual procedure is to show that $\rho_{\beta\alpha}$ is isomorphism, so all simplicial complexes give same answer, so don't need direct limit. But in that case the common answer IS the direct limit. So

$$\overset{S}{H}{}^*(S^2) = H^*_{\text{Singular}}(S^2) = H^*_{\text{Simplicial}}(\Sigma_\alpha)$$

$$= H^*_{\text{Cellular}}(\text{CW-decomposition of } S^2)$$

Čech cohomology of an open cover

Let X = topological space

$\mathcal{U} = \{U_\alpha\}$ an open cover

Nerve of $\mathcal{U} \equiv N(\mathcal{U})$ = simplicial complex with

1) Vertex α for each open set U_α
2) Edge $\alpha\beta$ for each nonempty $U_\alpha \cap U_\beta$
3) Face $\alpha\beta\gamma$ for each nonempty $U_\alpha \cap U_\beta \cap U_\gamma$

Def: $\check{H}^*(\mathcal{U}) = H^*_{\text{simplicial}}(N(\mathcal{U}))$

(N.B. $\check{H}^*(\mathcal{U})$ is often defined using presheaves - not needed for basic case)

Warning – open cover matters!

If $X = S^1$, can take open cover w/ one open set: $N(\mathcal{U}) = \bullet$

Or open cover with 2 open sets

$N(\mathcal{U}) =$ 

Or open cover with 3 open sets

$N(\mathcal{U}) =$ 

If $\mathcal{V}$ is refinement of $\mathcal{U}$, (See Bott & Tu)
$\exists$ canonical map $\rho_{\mathcal{V}\mathcal{U}} : \check{H}^*(\mathcal{U}) \to \check{H}^*(\mathcal{V})$

Def: $\check{H}^*(X) = \varinjlim_{\rho} \check{H}^*(\mathcal{U})$

Open Stars + good covers

In practice, $\check{H}^*$ is often computable from finite calculation. If $\mathcal{U}$ is a <u>good cover</u> (Def: every U_α contractible, every non-empty $U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ contractible), then $\check{H}^*(X) = \check{H}^*(\mathcal{U})$.

If X is a simplicial complex, can find good cover whose nerve is X,

so $\check{H}^* = H^*_{\text{simplicial}} = H^*_{\text{singular}} = H^*_{\text{cellular}}$

For each vertex α, let

$$U_\alpha = \alpha \cup \left\{ \begin{array}{l} \text{interiors of all cells} \\ \text{with } \alpha \text{ as a vertex} \end{array} \right\}$$

$U_\alpha \cap U_\beta \neq \emptyset \iff \alpha$ and β share an edge

$U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset \iff \alpha, \beta, \gamma$ share a face.

etc.

Meanwhile, back at the inverse limit

$$K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \leftarrow \cdots$$

$X = \varprojlim K_i \subset \prod K_n$ so basis for open sets in X is pullbacks of open sets in K_n.

Let $\pi_n : X \to K_n$ be natural projection

$\mathcal{U}_0 =$ good open cover of K_0

$\mathcal{U}_1 =$ good open cover of K_1, refinement of $f_1^{-1}(\mathcal{U}_0)$

$\vdots$

$\mathcal{U}_{n+1} =$ good open cover of K_n, refining $f_{n+1}^{-1}(\mathcal{U}_n)$

$\mathcal{V}_n = \pi_n^{-1}(\mathcal{U}_n) =$ (not good) open cover of X.

Claim: $\check{H}^*(X) = \varinjlim_n \check{H}^*(\mathcal{V}_n)$

HW: Show $N(\mathcal{V}_n) = N(\mathcal{U}_n)$

Since $\mathcal{U}_n$ is good, $\check{H}^*(\mathcal{U}_n) = H^*(K_n)$,

so

$$\begin{aligned}
\check{H}^*(X) &= \varinjlim_n \check{H}^*(\mathcal{V}_n) \\
&= \varinjlim H^*_{\text{simplicial}} N(\mathcal{V}_n) \\
&= \varinjlim H^*_{\text{simplicial}} N(\mathcal{U}_n) \\
&= \varinjlim H^*_{\text{any}} (K_n)
\end{aligned}$$

Examples

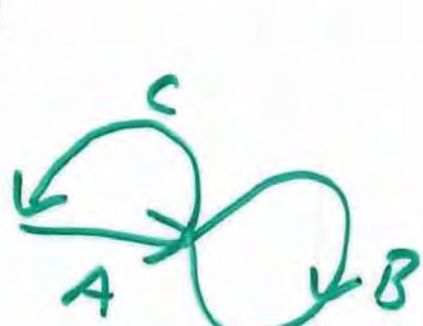
1) Dyadic solenoid

$$X = S^1 \xleftarrow{\times 2} S^1 \xleftarrow{\times 2} S^1 \leftarrow \cdots\cdots$$

$$H^*(S^1) = \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \qquad \sigma^* = \begin{cases} 1 & \text{on } H^0 \\ 2 & \text{on } H^1 \end{cases}$$

$$H^0(X) = \mathbb{Z}, \quad H^1(X) = \operatorname{Lim} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \cdots$$
$$= \mathbb{Z}[1/2]$$

2) Fibonacci: $K_n =$



$\sigma(A) = C$
$\sigma(B) = AC$
$\sigma(C) = CAB$
$\sigma(AC) = CAB$

$$H^1(K_n) = \mathbb{Z}^2 \qquad \sigma^* = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

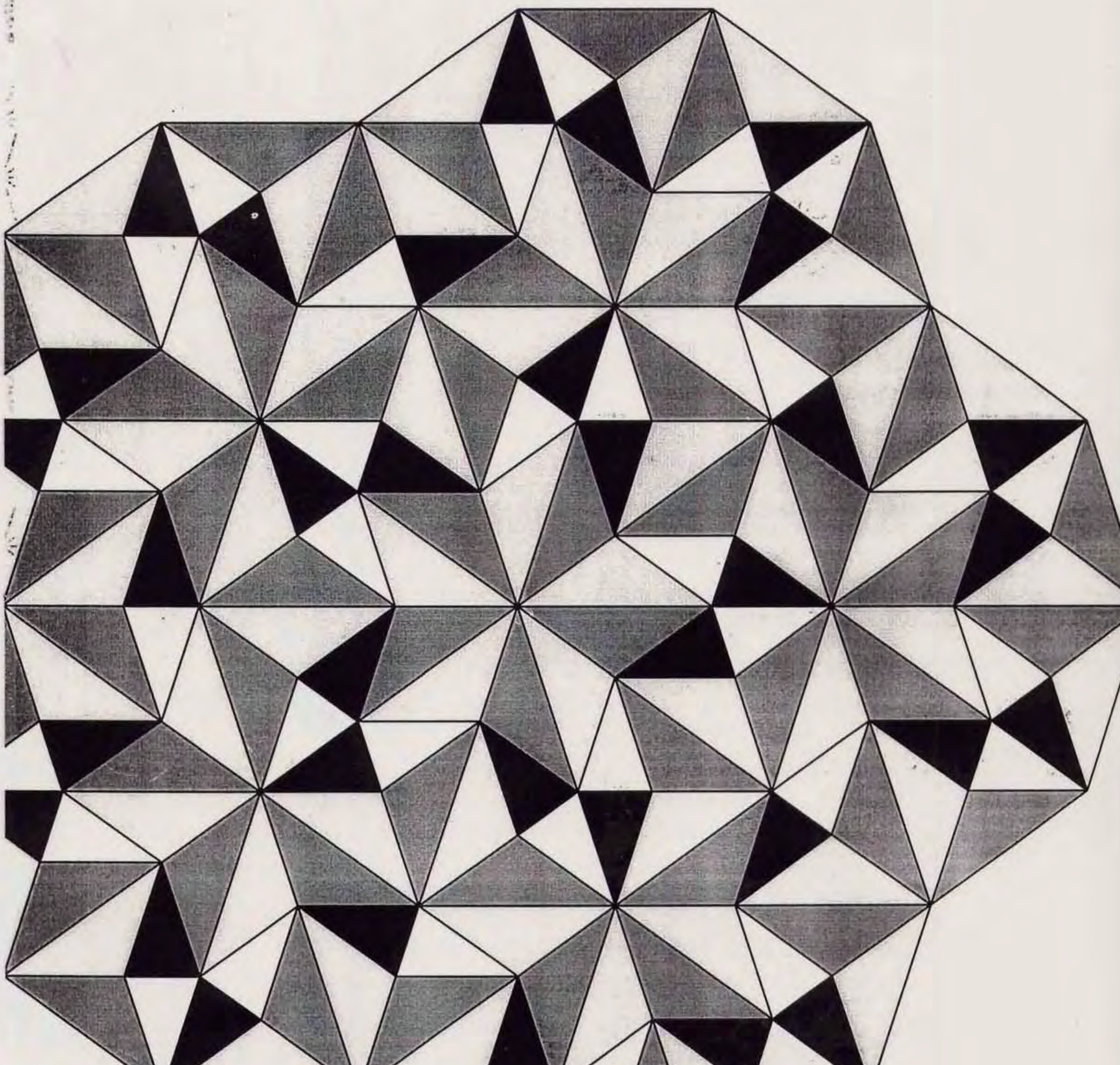
$$\check{H}^1(\text{Fibonacci}) = \mathbb{Z} \oplus \mathbb{Z}$$

So who cares?

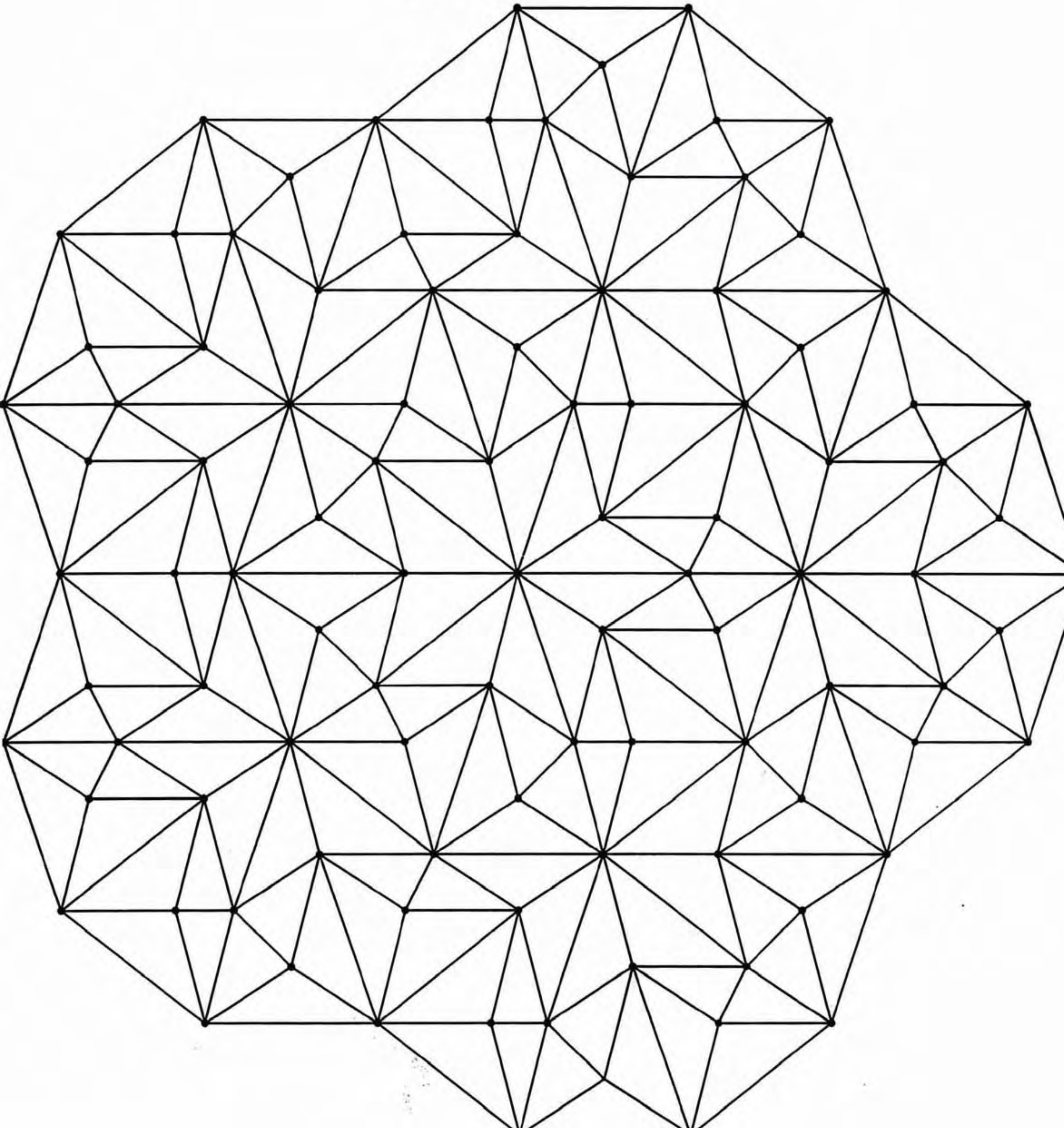
1) Cohomology helps classify spaces

2) Projection of top cohomology on $\mathbb{R}$ gives "gap labeling group".

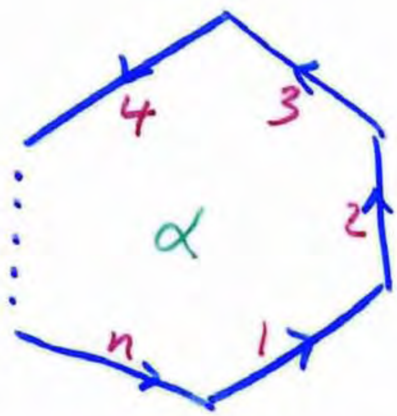
3) $\check{H}^1(\Omega_T, \mathbb{R}^d)$ parametrizes shape deformations



180 Tiles



Parametrizing Shapes



If α is an n-gon,

$$\alpha \longleftrightarrow \{\vec{V}_{\alpha 1}, \vec{V}_{\alpha 2}, \cdots \vec{V}_{\alpha n}\} \in \mathbb{R}^{2n}$$

subject to the closed condition

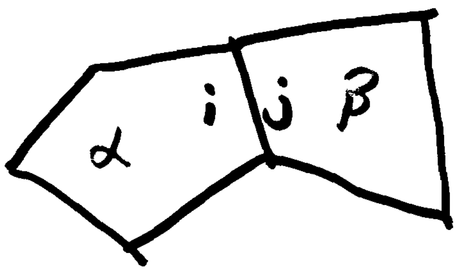
(1) $\sum_j \vec{V}_{\alpha j} = 0$

and some open conditions

($\partial\alpha$ is positively oriented closed path w/o self intersections)

Matching Conditions

If somewhere in a tiling, tile types α, β meet along edges i, j, then the edge vectors for α, i and β, j are the same (up to sign)



Shape space $\subseteq \left\{\begin{array}{l}\text{vector-valued} \\ \text{1-cochains on } \Gamma\end{array}\right\}$, where

$\Gamma = \coprod$ tile types $/$ edge identifications $= K_0$!

(More generally, deformations of n-collared tiles give cochains on K_n)

Cochain is closed, since

$$\delta f(\alpha) = f(\partial \alpha) = \sum_i V_{\alpha, i} = 0$$

Deformations/MLD = 1st cohomology

Coboundaries of 0-cochains do not affect MLD class, since 0-cochain describes local correction. Allowing collaring,

$$\text{Deformations/MLD} = \frac{\text{closed cochains on } K_n}{\text{exact cochains on } K_n}$$

$$= H^1(K_n, \mathbb{R}^2) \Longrightarrow H^1(X_T, \mathbb{R}^2)$$

Topological Conjugacy

All nonzero cohomology classes change MLD class, but some preserve topological conjugacy class (hence dynamical spectrum & K-theory). These are called asymptotically negligible.

<u>Thm ((Clark-S)</u> Let T be a substitution tiling, $\psi: X_T \to X_T$ the substitution map, inducing $\psi^*: H^1(X_T, \mathbb{R}^2) \to H^1(X_T, \mathbb{R}^2)$. Then

$$\left\{\begin{array}{c}\text{Asymptotically}\\ \text{negligible}\\ \text{classes}\end{array}\right\} = \bigoplus \begin{array}{l}\text{Eigenspaces}\\ \text{of } \psi^* \text{ with}\\ |\lambda|<1.\end{array}$$

Penrose Revisited

$$H^1(X_p, \mathbb{R}^2) = \mathbb{R}^{10} = \underset{\lambda=\tau}{\mathbb{R}^4} \oplus \underset{\lambda=1-\tau}{\mathbb{R}^4} \oplus \underset{\lambda=-1}{\mathbb{R}^2}$$

$\lambda = \tau$: dilation, rotation, 2 shears
= general linear transformations of P.

$\lambda = 1-\tau$: Asymptotically negligible.
Changes MLD class but not dynamics.

$\lambda = -1$: Breaks $180°$ rotational symmetry.
Does not appear in original Penrose or rational Penrose.

Original Penrose : shape is pure $\lambda = \tau$

Rat'l Penrose: shape is linear combination of $\lambda = \tau$ and $\lambda = 1-\tau$.

$\Longrightarrow$ Top. conj (but not MLD) to linear transformation of original.