Local structure of Ω_T

Let $T' \in \Omega_T$ be arbitrary tiling.
What does ϵ-neighborhood of T' look like?

Take central $1/\epsilon$-patch of T'

1) Wiggle it by $< \epsilon$: Continuous degrees of freedom.
2) Extend it: infinitely many discrete choices $\Rightarrow$ Cantor set.

Locally, $\Omega_{T'} \cong$ disk $\times$ Cantor set

Old hat to dynamical systems folks.
Pretty weird to the rest of us!

Inverse Limit Spaces

Let $K_1 \xleftarrow{f_1} K_2 \xleftarrow{f_2} K_3 \cdots \cdots \leftarrow$

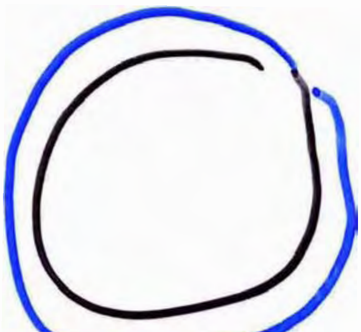
<u>Def</u> $\varprojlim K_i = \{ (x_1, x_2, \ldots) \in \prod K_i \mid x_i = f_i(x_{i+1}) \}$

Example (dyadic solenoid)

$K_1 = K_2 = \cdots = S^1 = \mathbb{R}/\mathbb{Z}$

$f_n = \times 2$

K_1



K_2

Tiling spaces are inverse limits

Anderson - Putnam ('98) — First construction for substitution tiling spaces

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Ormes, Radin, S- Generalized Anderson - Putnam to allow rotations

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Bellissard, Benedetti + Gambaudo — general construction, not limited to substitutions

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Gähler — Particularly simple construction

$\left.\begin{array}{l}\text{Benedetti + Gambaudo} \\ \text{Sadun}\end{array}\right\}$ Generalizations to non-Euclidean spaces

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$\left.\begin{array}{l}\text{work in} \\ \text{progress}\end{array}\right\}$ Spaces without finite local complexity

Gähler's Construction

Point in K_n = instructions for laying n layers of tiles around origin.

$$K_n \xleftarrow{f_n} K_{n+1} = \text{forgetful map.}$$

$\varprojlim K_n$ = consistent instructions for layers out to ∞

$\Longleftrightarrow$ tiling

To complete proof, need to construct K_n

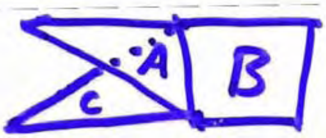
Building K_0

Point in K_0 = How to place tile at origin
= Where to place origin in tile
= point in a tile.

What if origin is on an edge?

Identify R edge of A with L edge of B!

K_0 = Union of 1 copy of each tile type with (some) edges identified
= branched surface.



Collared Tiles

Label tiles by pattern of nearest neighbors

Assigning new labels is <u>local</u> operation, so new tiling T' is MLD to old T.

Let $K_1(X_T) = K_0(X_{T'})$

= union of collared tiles w/ (some) edges identified.

Point in K_1 describes tile containing origin + nearest neighbors.

Let $K_2(X_T) = K_1(X_{T'})$

$K_3(X_T) = K_2(X_{T'})$ etc

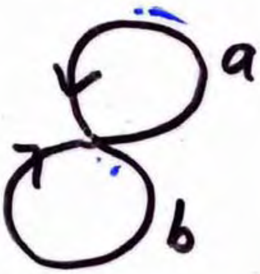
Fibonacci Tiling

...(b)abbabbababbab....

....ABCABCADABCA...

$A = (b)a(b)$
$B = (a)b(b)$
$C = (b)b(a)$
$D = (a)b(a)$

$K_0 =$



$K_1 =$

$f_1(A) = a$

$f_1(B) = f_1(C) = f_1(D) = b$

$H^1(K_0) = H^1(K_1) = \mathbb{Z}^2$; $f_1^* = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ = isomorphism

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Fact: K_n = complex with $2n+2$ edges

$H^1(K_n) = \mathbb{Z}^2$; f_n^* = isomorphism

$H^1(T) = \varinjlim \mathbb{Z}^2 \cong \mathbb{Z}^2$

Are we done yet?

Gähler construction is

1) Simple
2) Elegant, and
3) Conceptually powerful, but...
4) Computationally useless.

K_n depends on n. Need to compute K_n.

Much simpler if $\begin{cases} \text{all } K\text{'s same} \\ \text{all } f\text{'s same.} \end{cases}$

Restrict attention to

<u>substitution</u> tilings, and

go back to Anderson-Putnam

Anderson - Putnam Construction

σ = substitution, stretches by λ

K_0 as before (one copy of each tile type, stitched together at edges)

K_1 = larger copy of K_0 (stretch by λ)

K_n = " " " " (stretch by λ^n)

View K_1 as $\coprod (\sigma(\text{tiles}))/\sim$

K_n as $\coprod \sigma^n(\text{tiles})/\sim$

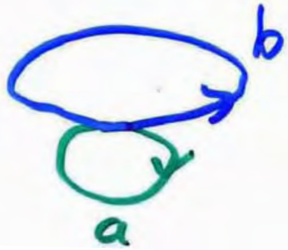
K_0 tells where origin sits in tile.

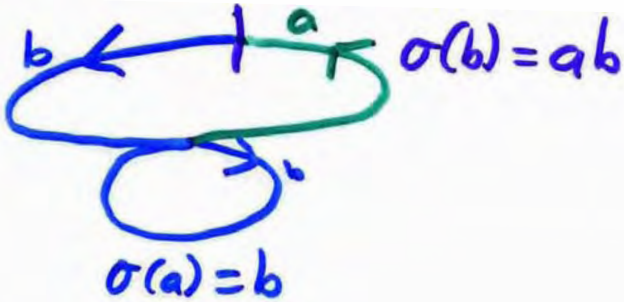
K_1 " " " " " level-1 cluster (supertile)

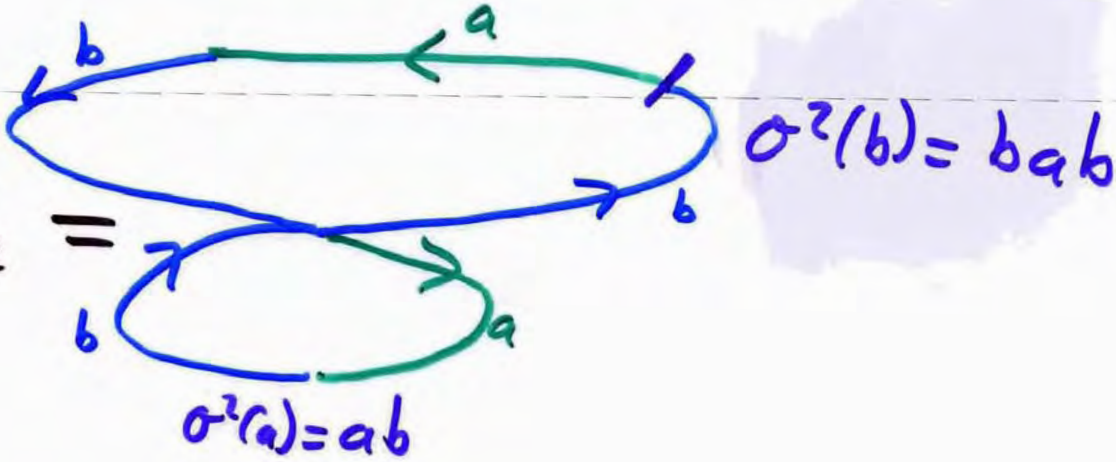
K_n " " " " " level-n supertile

Natural forgetful map $f_n : K_n \to K_{n-1}$

Fibonacci Again

$K_0 =$ 

$K_1 =$ 

$K_2 =$ 

All K_s look like K_0

All f_s look like $\sigma: K_0 \to K_0$,

sending lower loop to upper

upper to lower + upper

Now are we done?

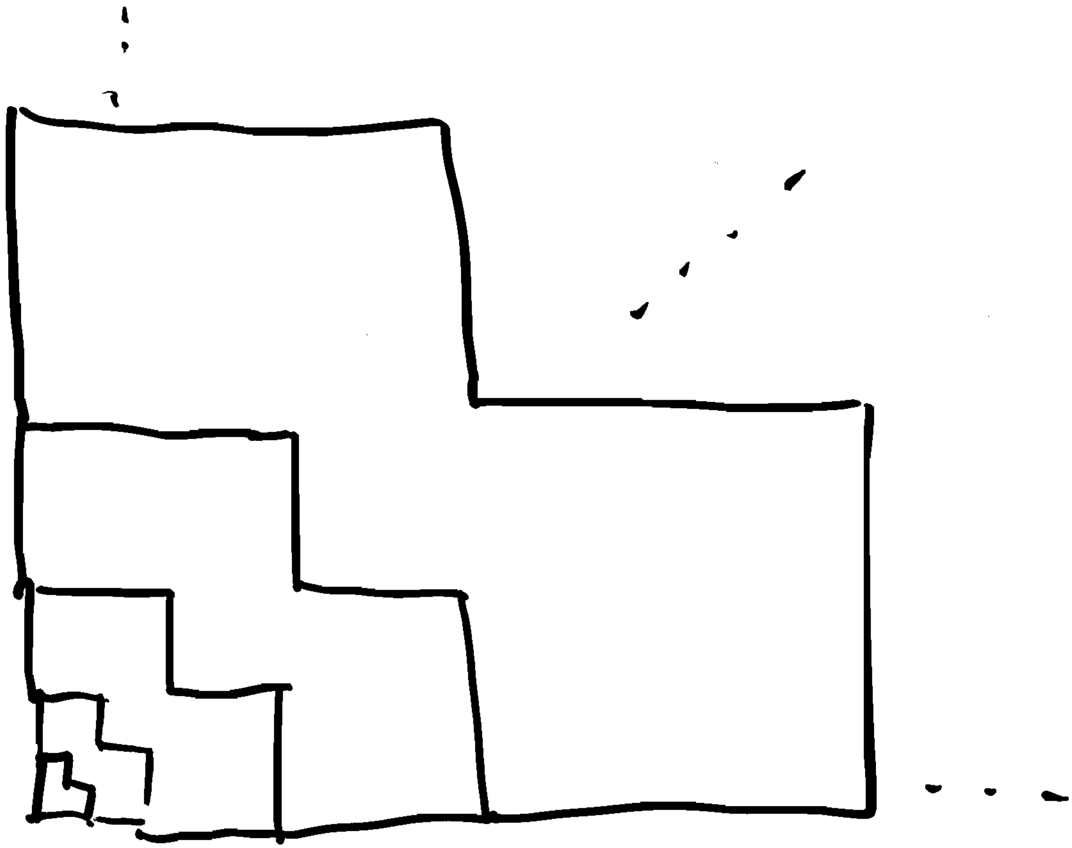
Point in $\varprojlim K_n$ gives consistent instructions for placing

$$\text{origin} \in \text{tile} \subset \text{level-1 supertile} \subset \cdots \subset \text{level-}n\text{ supertile} \subset$$

on forever.

Is that the same as "instructions for tiling the whole plane"?

Not quite!



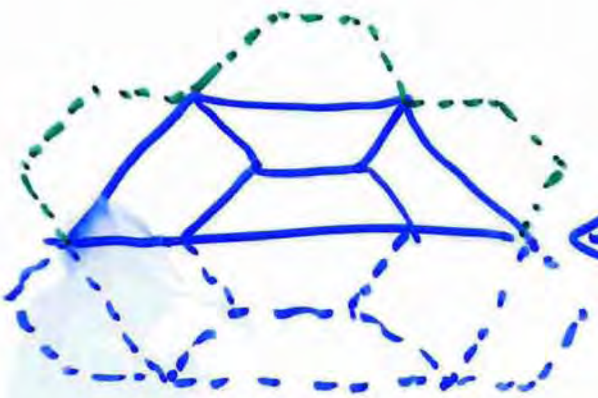
An ∞-order supertile may not cover the entire plane.

Is there an extension to the entire plane? (yes!)

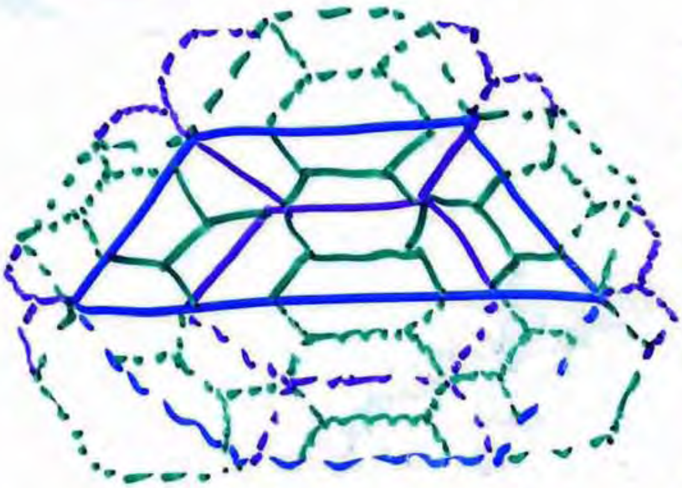
Is that extension unique?
(not always)

Half-hex

But  always faces 



So level-1 supertile looks like



2nd-order supertile determines its 0-th order neighbors

2+K-th order supertile determines K-th order neighbors

∞-order supertile determines entire plane.

Forcing the Border

Def (Kellendonk) A substitution forces the border if $\exists$ n s.t. each n-th order Supertile (of the same type) has the same 0-th order neighbors.

Thue-Morse does not force the border.

Fibonacci forces on one side only, $\begin{pmatrix} a \to b \\ b \to ab \end{pmatrix}$ since all words end in b.

Chair does not force border:

Penrose DOES force the border.

To make A-P construction work, we need to force the border

Anderson-Putnam Collaring Trick

Rewrite substitution using once-collared tiles.

This <u>always</u> forces the border!

Fibonacci yet again:

$A = (b)\,a\,(b)$

$B = (a)\,b\,(b)$

$C = (b)\,b\,(a)$

$D = (a)\,b\,(a)$

$$\sigma(A) = (ab)\,b\,(ab) = (B)\,C\,(A)$$

$$\sigma(B) = (b)\,ab\,(ab) = A\,D\,(A)$$

$$\sigma(C) = (ab)\,ab\,(b) = (D)\,A\,B$$

$$\sigma(D) = (b)\,ab\,(b) = A\,B$$

$$\sigma^2(A) = (AD)\,AB\,(C)$$

$$\sigma^2(B) = (B)\,C\,AB\,(C)$$

$$\sigma^2(C) = (AB)\,C\,AD\,(A)$$

$$\sigma^2(D) = (B)\,C\,AD\,(A)$$

Forces the border.

One example isn't a proof, so

HW: Prove (first in 1D, then in higher dimensions) that it is always possible to rewrite a substitution in terms of collared tiles, and that this forces the border.

In summary:

Thm (AP) Let K_0 be the cell complex obtained from stitching uncollared tiles, and let $\hat{K}_0$ be the complex of 1-collared tiles.

1) In all cases, $\Omega_T \simeq \varprojlim_{\sigma} \hat{K}_0$

2) If σ forces the border, then

$$\Omega_T \simeq \varprojlim_{\sigma} K_0$$

Overkill

Can also construct Ω_T as $\varprojlim$ of complexes made from twice-collared tiles, or 17-times-collared tiles, etc. but that's just computational overkill. In fact, collaring once is often overkill.

Minimal collaring

Consider all tiles t_i in a tiling T.

Let $t_i \sim t_j$ if, for some $n \geq 0$, $\sigma^n(t_i)$ and $\sigma^n(t_j)$ agree (up to translation) and have the same 0-th nbhds. This

1) Gives finer resolution than plain tiles
2) Gives coarser " " collared ",
3) Forces the border.

Fibonacci (last time
today – promise!)

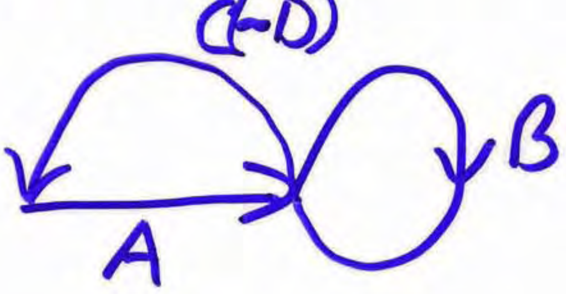
$A = (b)a(b)$
$B = (a)b(b)$
$C = (b)b(a)$
$D = (a)b(a).$

But $\sigma(C) = (ab)ab(b)$
$\sigma(D) = (b)ab(b)$,

so $C \sim D$

Only 3 minimally collared tiles, not 4.

$K =$



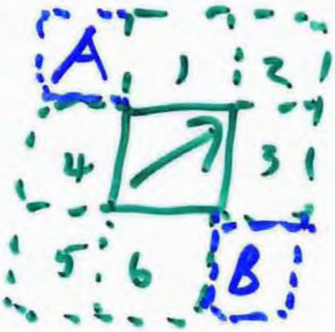
$\sigma(A) = C$
$\sigma(B) = AC$
$\sigma(C) = AB$

$\Omega_{\text{Fibonacci}} = \varprojlim_{\sigma} K$

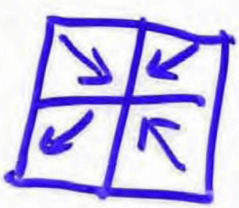
A 2-D Example

Arrow (MLD to chair)

When collaring, only care about green positions, not A or B



Why? A is either ↗ or ↙, so

$\sigma(A) =$  or . Either way, piece touching $\sigma($ $)$ is 

Ditto for B.

Fact: In arrow tiling all vertices have 0 or 3 inwards arrows, so

and

$3 \times 2 = 6$ "minimal" collarings of

, compared to 13 "full" collarings.

For original chair, 3 minimal collarings of , vs 14 "full" collarings.

(C. Holton)

Minimal collaring allows feasible hand calculations, but is not as needed for computer calculations