Tilings and Topology

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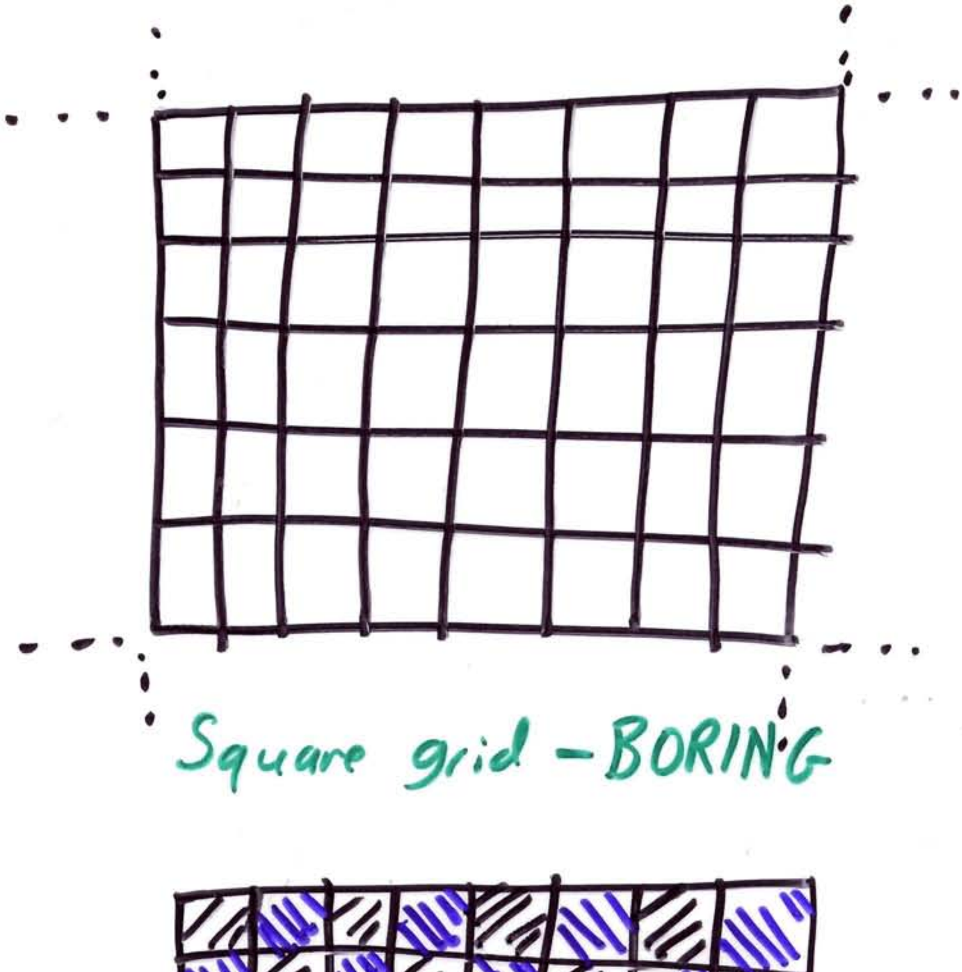
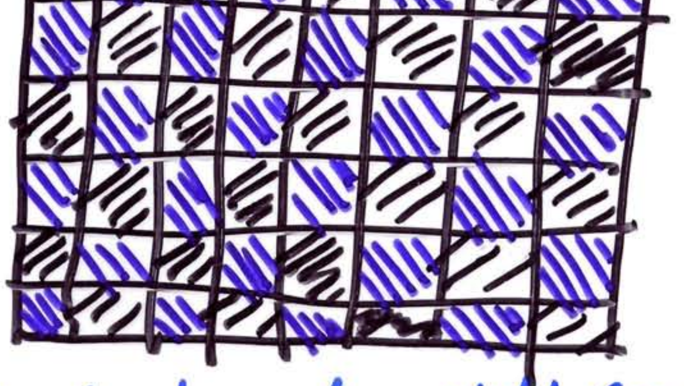
Lecture 1: Basic notions

What is a tiling?

What makes one interesting?



Square grid – BORING



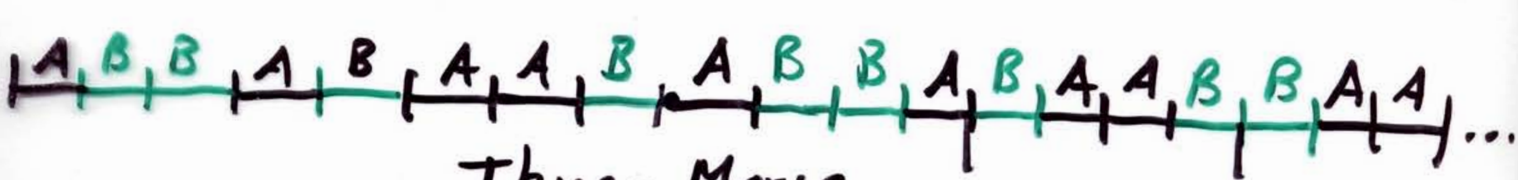
Checkerboard – still periodic (yawn)

Some 1-D tilings

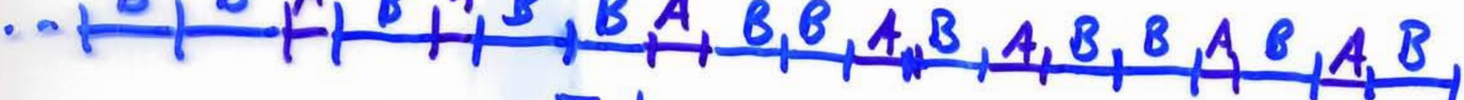
One black tile

Half and half

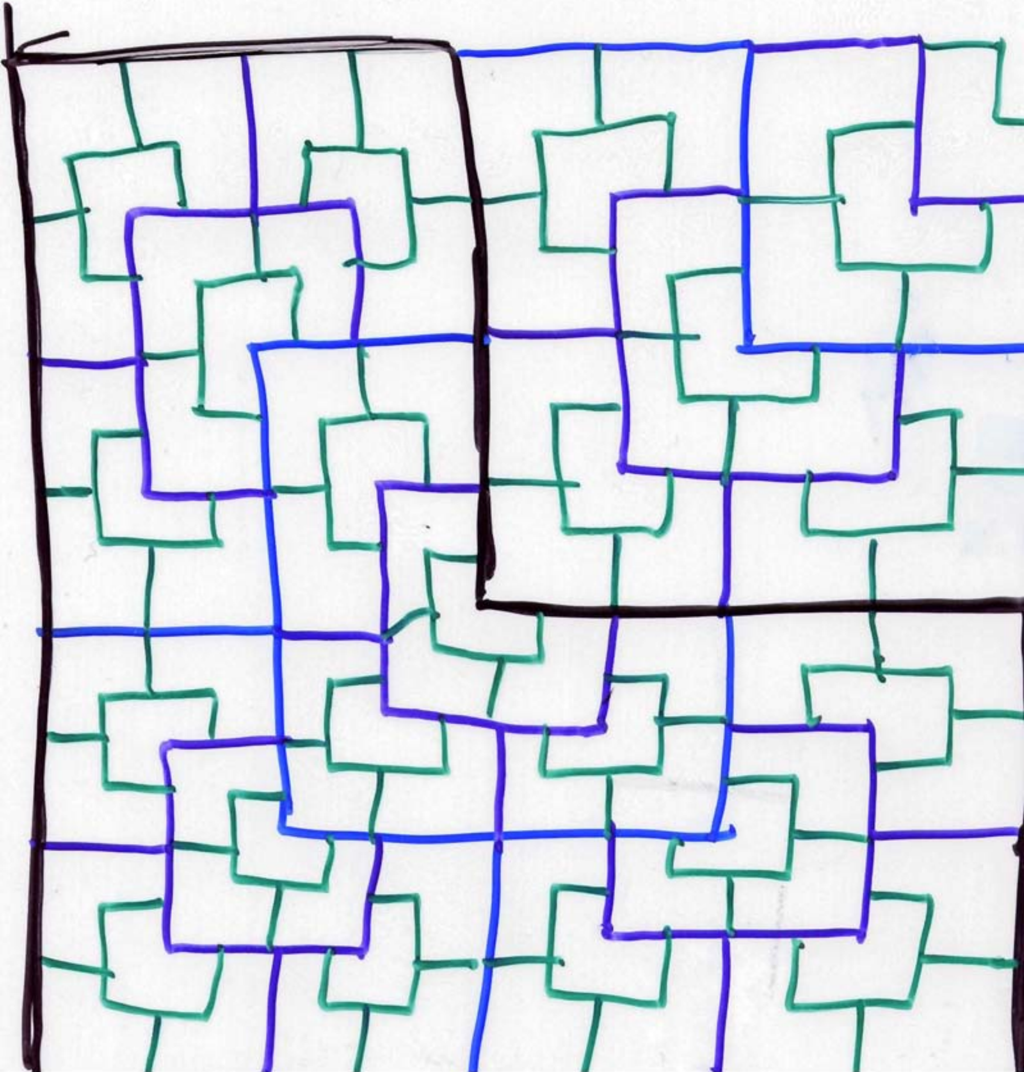
Random tiling – too little structure



Thue-Morse



Fibonacci



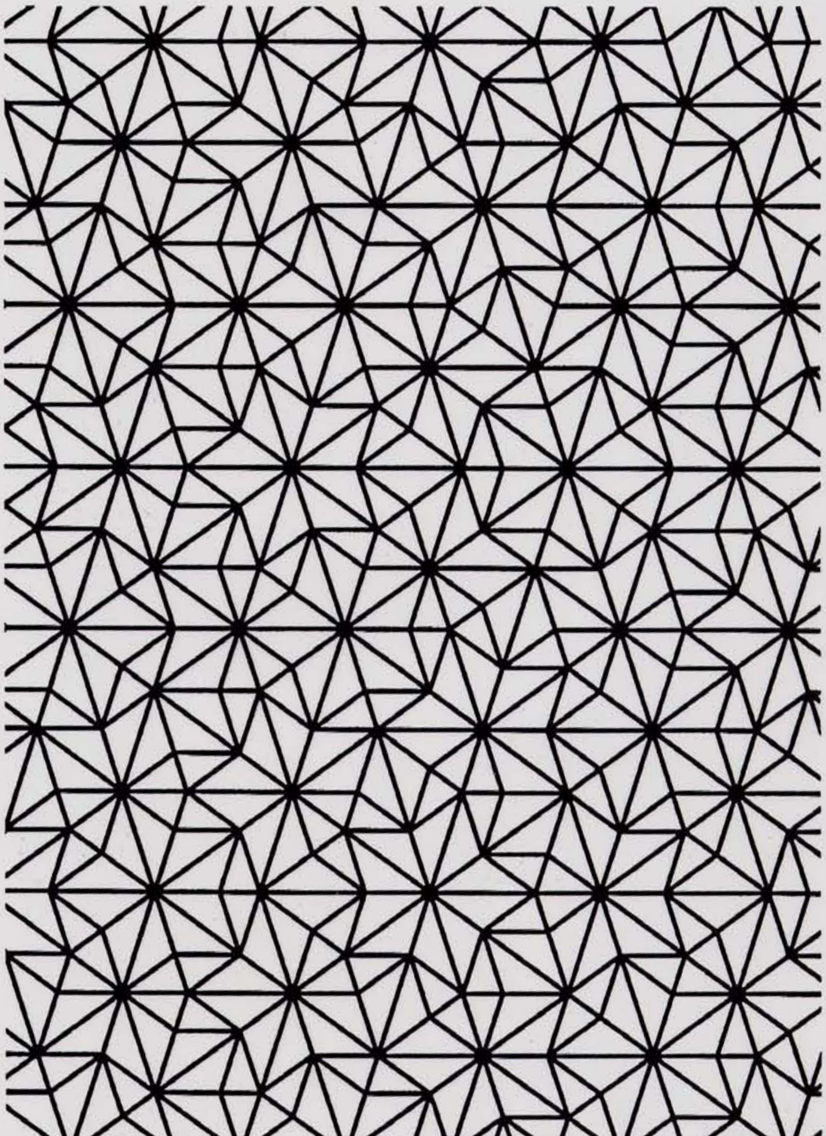


Figure 4. A Penrose tiling.

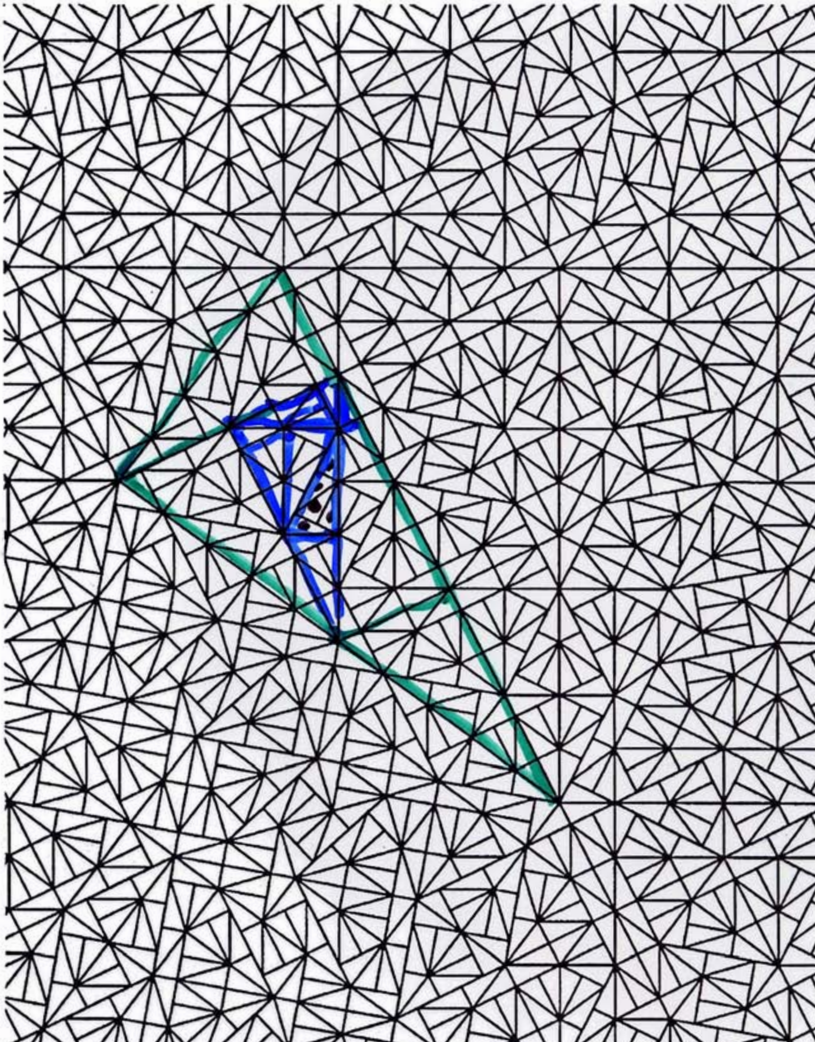


Figure 6. A pinwheel tiling.

ant pinwheel tilings also necessarily have tiles in an infinite number of distinct orientations. In fact, the relative orientation groups for all pinwheel tilings are algebraically isomorphic. Theorem 1 shows that the tiling spaces for the pinwheel and (2,3)-pinwheel are not homeomorphic.

In all the above cases, it is easy to construct explicit examples of tilings. Pick a tile to include the origin of the plane. Embed this tile in a tile of level 1 (there are several ways to do this). Embed that tile of level 1 in a tile of level 2, embed that in a tile of

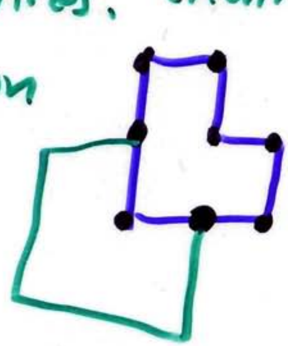
Assumptions (for now)

H1. Finite # of tile types up to translation
H2. Tiles are polygons (polyhedra)
H3. Tiles meet full-edge to full-edge.

Note:
H1. will be relaxed later this week to allow rotations (pinwheel)

H2 is for convenience only. Boris Solomyak will explain how to convert tiling $\Rightarrow$ point pattern $\Rightarrow$ polygonal tiling

H3. May require extra vertices. Chair is really an octagon



Where's the topology?

$\mathbb{R}^d$ is contractible!

Need to construct space of tilings and study topology of tiling <u>space</u>.

Step 1: Define metric on tilings:

Tilings T and T' are ϵ-close if they agree on $B_{1/\epsilon}(0)$, up to translation of size $\leq \epsilon$.

Definition: The <u>orbit</u> of a tiling T is the set $\mathcal{O}(T) = \{T - x \mid x \in \mathbb{R}^d\}$ of translates of T.

Tiling Spaces

Def A tiling space is a set Ω of tilings that is

1) Closed under translation (if $T \in \Omega$, then $\mathcal{O}(T) \subset \Omega$)
2) Complete in the tiling metric. (every Cauchy sequence converges)

Def The <u>hull</u> of a tiling T is $\Omega_T = \overline{\mathcal{O}(T)}$, the smallest tiling space containing T.

$T' \in \Omega_T \iff$ Every patch of T' is found some where in T

$\Omega_T =$ tilings that look locally like translates of T

$=$ "Local Isomorphism class" of T.

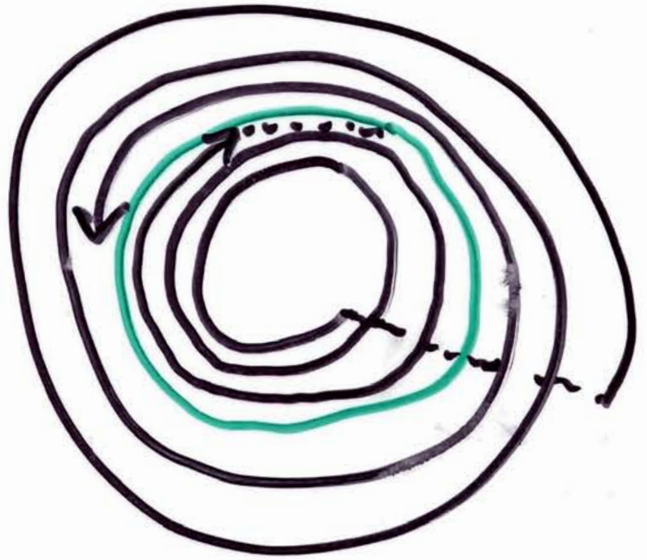
Examples of hulls

1) $T = \cdots \vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash \cdots$ unit grid in $\mathbb{R}^1$. (periodic)

$\Omega_T = O(T) \approx \mathbb{R}/\mathbb{Z} \simeq S^1$

2) $T = \cdots \vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash\!\vdash \cdots$ one black tile

$\Omega_T =$ tilings with 0 or 1 black tile.



— $= \mathbb{R} =$ tilings with one black tile

$O = S^1 =$ tilings with no black tiles

$= \lim_{x \to \infty} T - x = \lim_{x \to -\infty} T - x$

HW: What is Ω_T for

a) half-and-half b) random tiling (answer should apply to almost every tiling T)

Properties of tiling spaces

1) Ω is compact. [In any finite ball, in $\mathbb{R}^d$, $\exists$ only finitely many patterns + bounded set of translations. So every sequence has subsequence that converges on $B_R(0)$. Now apply Cantor trick]

2) ϵ-nbhd of T in $\Omega \approx$ (ϵ-disk in $\mathbb{R}^d$) $\times$ totally disconnected set.

Take central $1/\epsilon$-size patch

Wiggle by up to ϵ — continuous degrees of freedom

Extend patch to ~~∞~~ — (infinitely many?) discrete choices.

3) Thm (S – Williams) Ω is a fiber bundle over $\mathbb{R}^d/\mathbb{Z}^d$ with totally disconnected fiber

When are two tiling spaces Ω_{T_1}, Ω_{T_2} equivalent? And what does "equivalent" mean, anyway?

1) Homeomorphism: A continuous, 1-1, onto map $f: \Omega_{T_1} \to \Omega_{T_2}$. Since Ω_{T_i} is compact, f^{-1} is automatically continuous.

Homeomorphisms preserve topology but little else.

2) Topological conjugacy: A homeomorphism that commutes with translations:
$f(T-x) = f(T)-x$. Preserves dynamics (invariant measures, dynamical spectrum, mixing, etc.)

3) Mutual Local Derivability (MLD): Topological conjugacy that is defined <u>locally</u>. $\exists R$ s.t., if T and T' agree (exactly) on $B_R(0)$, then $f(T)$ and $f(T')$ agree (exactly) on $B_1(0)$. Preserves locality.

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How to create interesting tilings

1. Substitution tilings

$\approx$ self-similar tilings

$\approx$ pseudo-self-similar tilings

$\approx$ (pseudo)-self-affine tilings.

Ex: Thue-Morse, Fibonacci, Period-doubling 1-D
Chair, table, half-hex, Penrose 2-D

2. Cut-and-project tilings

$\approx$ model sets

$\approx$ canonical projection tilings

$\approx$ diffractive sets

Ex: Fibonacci, Penrose, octagonal, icosahedral

3. Local matching rules

Ex: Penrose

Substitutions in 1D

1) Pick an "alphabet" $\mathcal{A}$ with finite # of letters (e.g $\mathcal{A} = \{a, b\}$)

2) Associate a word to each letter (say $a \to ab$, $b \to ba$) $\Leftarrow$ Thue-Morse

3) The substitution σ replaces each letter by its word:

$$\sigma(a) = ab \qquad \sigma(b) = ba$$
$$\sigma(ab) = abba \qquad \sigma(ba) = baab$$
$$\sigma(abba) = abbabaab \qquad \sigma(baab) = baababba$$

4) The substitution matrix M counts letters: $M_{ij} = \#(i\text{-th letter in } \sigma(j\text{-th letter}))$

For Thue-Morse, $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Primitivity and Perron-Frobenius

Def A square matrix M with non-neg entries is <u>primitive</u> if $\exists\, n$ s.t. M^n has all entries strictly > 0.

For substitutions this means than σ^n(any letter) contains every letter.

Thm (P-F) A primitive matrix has a largest positive real eigenvalue λ_{PF} of multiplicity one. The corresponding left- and right-eigenvectors have positive entries. All other eigenvalues have $|\lambda_i| < \lambda_{PF}$

For Thue-Morse, $\lambda_{PF} = 2$, $\vec{L} = (L_1, L_2) = (1,1)$

$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let an "a" tile have length L_1
"b" tile length L_2,
etc.

Hw: Show that the length of σ^n(i-th letter) is $\lambda_{PF}^n L_i$;

Hw: Show that R_i's give relative populations of letters in $\sigma^n(a)$ as $n \to \infty$

Substitution Sequences, Substitution Tilings.

A bi-infinite word is <u>σ-admissible</u> if each finite sub-word can be found in $\sigma^n(a)$, for some $n \geq 0$

Sequence space $S_\sigma = \{\sigma\text{-admissible words}\}$

tiling space $\Omega_\sigma = \{$tilings with tiles $a, b, \ldots$ of length $L_1, L_2, \ldots$ such that the corresponding sequence is σ-admissible$\}$

σ acts geometrically on Ω_σ

1) Stretch by λ_{PF} about origin
2) Replace each (stretched) tile by word.

Thm (Mossé, Solomyak) If Ω_σ contains non-periodic tilings, then $\sigma : \Omega_\sigma \to \Omega_\sigma$ is a homeomorphism.

1-D Examples

1) Thue-Morse $\quad a \to ab \to abba$ $\quad b.a$

$\qquad\qquad\qquad\quad b \to ba \to baab$

Note that

$$s_0 = \ldots\ldots abbaabbabaab.abbabaabbaababba\ldots$$

is fixed point of σ^2 and $\Omega_\sigma = \Omega_{s_0}$

2) Fibonacci $\quad a \to b$ $\quad M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \lambda_{PF} = \tau = \frac{1+\sqrt{5}}{2}$

$\qquad\qquad\qquad\quad b \to ab$

$L = (1, \tau) \quad R = \begin{pmatrix} 1 \\ \tau \end{pmatrix}$

3) Period doubling $\quad a \to bb$ $\quad M = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad \lambda_{PF} = 2$

$\qquad\qquad\qquad\quad b \to ab$

$L = (1, 1) \quad R = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

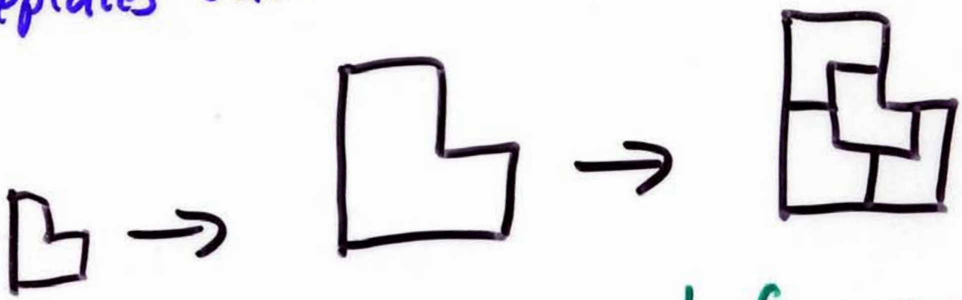
$a \to bb \to abab \to bbab\,bbab \to$

$b \to ab \to bbab \to abab\,bbab \to$

Substitutions in higher dimensions

Need a stretching factor λ and a substitution σ that:

1) Stretches each tile by a linear factor λ,
2) Replaces each stretched tile by a cluster:

e.g. 

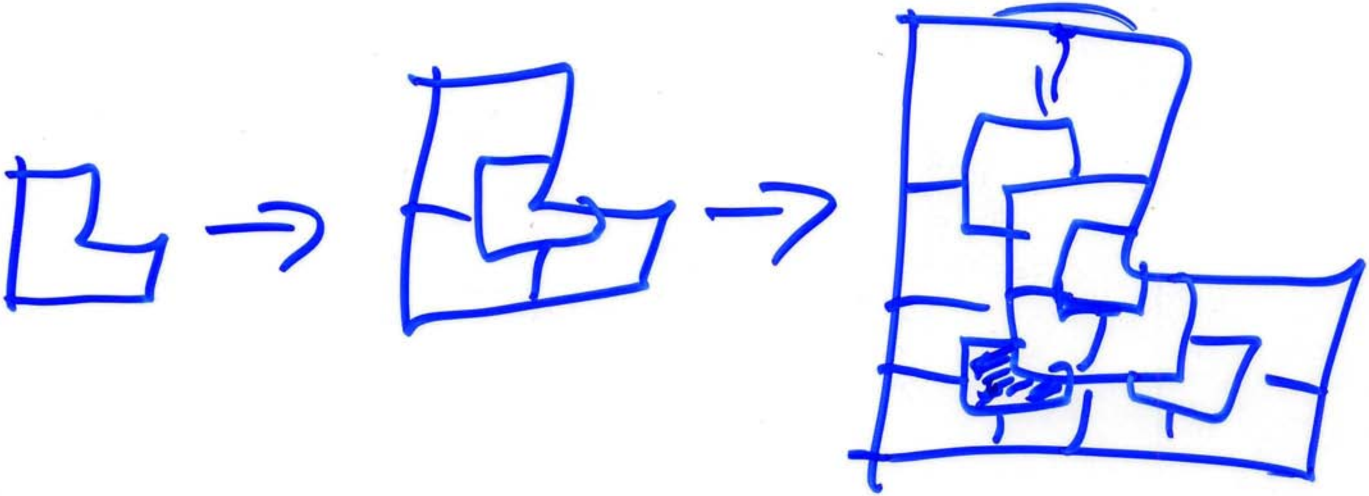
Substitution matrix M as before, assume primitive. $L=(L_1, \ldots)$ gives volumes of tiles, but not shapes.

$\Omega_\sigma = \{$Tilings T | every patch of T is found somewhere in σ^n(a tile)$\}$

Thm Ω_σ is non-empty

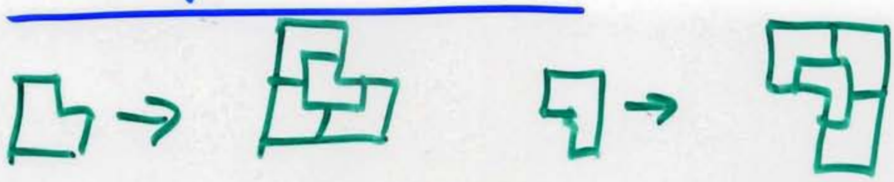
Thm (Solomyak) If Ω_σ has non-periodic tilings, $\sigma: \Omega_\sigma \to \Omega_\sigma$ is homeomorphism.

(... recognizability)



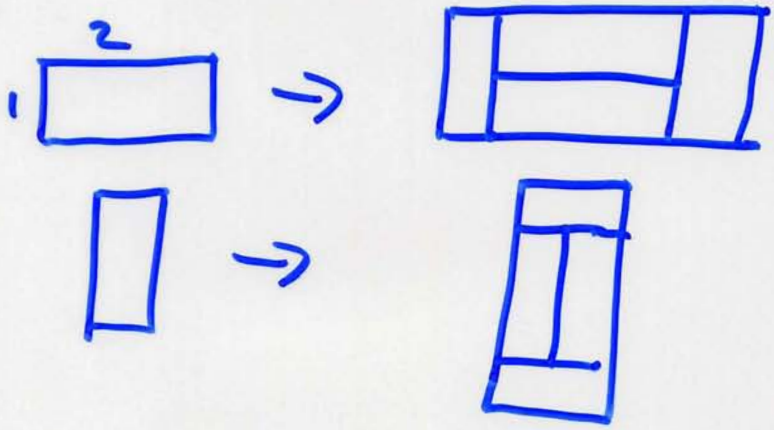
HW: Prove that, if $T_1, T_2 \in \Omega_\sigma$, then every patch in T_1 is also found in T_2 (and vice-versa). Hence $\Omega_{T_1} = \Omega_\sigma$. (minimality). [Hint: show that for each tile t and integer $n \geq 0$, $\exists$ a radius R s.t. every ball of radius R in T_2 contains a copy of $\sigma^n(t)$. Start with $n = 0$]

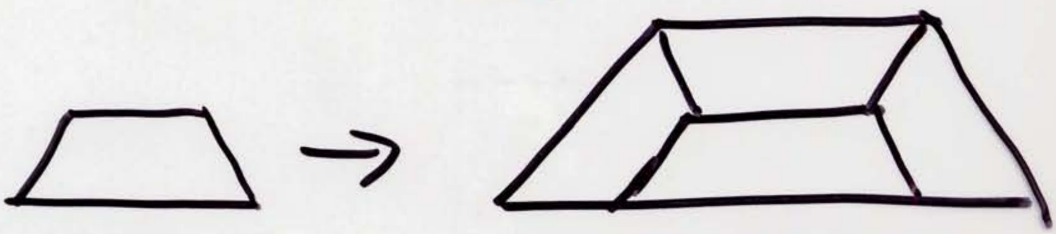
Examples in 2D

Chair:  etc

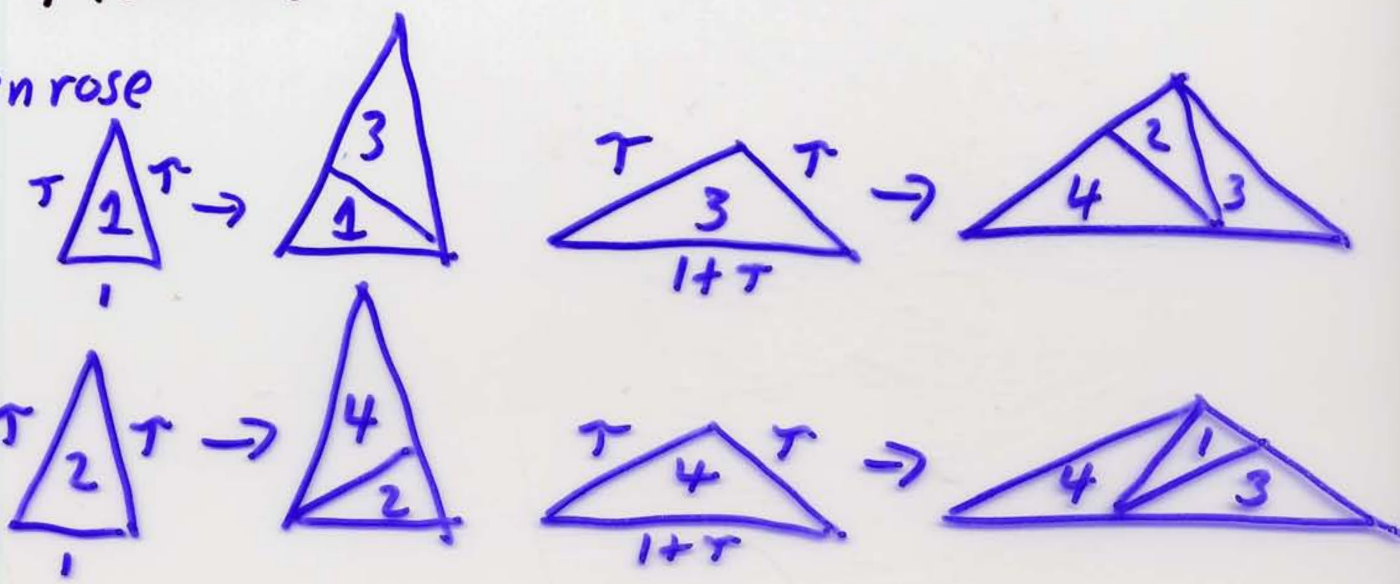
Arrow:  , etc

MLD to chair 

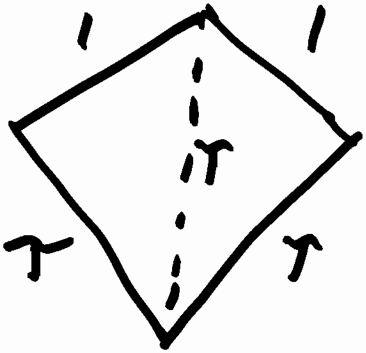
Table 

Half-hex 

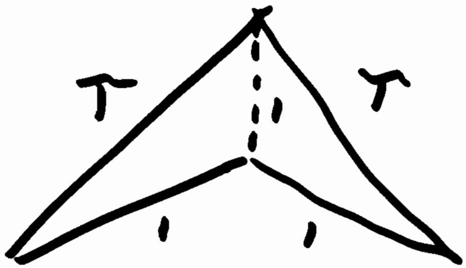
+ rotations

Penrose 

Penrose via matching rules



Kite



Dart