Some Inverse Transport Problems and their Applications

Guillaume Bal *

March 28, 2005

This set of lectures covers several inverse problems related to the propagation of particles or radiation through various underlying media and some applications in medical and geophysical imaging. We start by considering inverse transport problems in the absence of scattering (ray transforms) in Lecture 1, and consider inverse transport problems in highly scattering media (diffusion approximation) in Lecture 3. The intermediate regime is characterized by radiative transfer equations (RTE). Lecture 2 will recall some theoretical results on inverse RTE problems and will mostly focus on the many mathematically open inverse problems that are really useful in practice.

Lecture 1. This lecture will cover basic results on ray transforms and inverse source problems in the absence of scattering. I will focus on the method of the complexification of the geodesic vector field and its application in medical imaging (SPECT) and geophysical imaging (emission problem in hyperbolic geometry). Some bibliographical references include [1, 7, 8, 9, 11, 15, 18, 19, 21, 22, 23].

Lecture 2. Quite opposite to the framework covered in the first lecture, we consider here the case of highly heterogeneous media, where particles interact so much with the underlying structure that their density can be modeled by a diffusion equation. I will focus on the modeling of non-scattering inclusions in highly scattering environments and how these inclusions can be reconstructed from boundary measurements. The main application for such works is optical tomography in medical and atmospheric imaging. Bibliographical references are [2, 4, 5, 6, 10, 12, 14, 17, 24].

Lecture 3. Radiative transfer equations are the proper model for the density of particles in general scattering media. Perturbations of the theory of ray transforms provide most of the known results on the reconstruction of constitutive parameters in transport equations from phase space boundary measurements [13, 16, 25]. The problem is that phase-space measurements are usually not available. In many application, the measurements are rather either the angularly averaged density or the outgoing current at the physical domain boundary. They are thus similar to the measurements available

*Department of Applied Physics and Applied Mathematics, Columbia University, New York NY, 10027; gb2030@columbia.edu

in the diffusive approximation; e.g. the Dirichlet to Neumann map. There is hardly any available theoretical result for this problem; see [3] for a counter-example. Rather than perturbing ray transforms, one thus would like to "perturb" the theory for diffusion equation; see e.g. [20, 26, 27]. The latter however does not respond very well to perturbations. I will mention a few promising (and largely unexplored) ways to address this issue.

References

[1] E. V. Arbuzov, A. L. Bukhgeim, and S. G. Kazantsev, *Two-dimensional tomography problems and the theory of A− analytic functions*, Sib. Adv. Math., 8 (1998), pp. 1–20.

[2] S. R. Arridge, *Optical tomography in medical imaging*, Inverse Problems, 15 (1999), pp. R41–R93.

[3] H. Babovsky, *An inverse model problem in kinetic theory*, Inverse Problems, 11 (1995), pp. 555–570.

[4] G. Bal, *Reconstructions in impedance and optical tomography with singular interfaces*, Inverse Problems.

[5] ——, *Particle transport through scattering regions with clear layers and inclusions*, J. Comp. Phys., 180(2) (2002), pp. 659–685.

[6] ——, *Transport through diffusive and non-diffusive regions, embedded objects, and clear layers*, SIAM J. Appl. Math., 62(5) (2002), pp. 1677–1697.

[7] ——, *On the attenuated Radon transform with full and partial measurements*, Inverse Problems, 20(2) (2004), pp. 399–419.

[8] ——, *Ray transforms in hyperbolic geometry*, J. Math. Pures Appl., (2005).

[9] G. Bal and P. Moireau, *Fast numerical inversion of the attenuated Radon transform with full and partial measurements*, Inverse Problems, 20(4) (2004), pp. 1137–1164.

[10] G. Bal and K. Ren, *Generalized diffusion model in optical tomography with clear layers*, J. Opt. Soc. Amer. A, 20(12) (2003), pp. 2355–2364.

[11] ——, *Atmospheric concentration profile reconstructions from radiation masurements*, Inverse Problems, 21(1) (2005), pp. 153–168.

[12] ——, *Reconstruction of singular interfaces by shape sensitivity analysis and level set method*, submitted, (2005).

[13] G. Bal and A. Tamasan, *Inverse source problems in transport equations*, submitted, (2005).

[14] A. Y. Bluestone, G. S. Abdoulaev, C. Schmitz, R. L. Barbour, and A. H. Hielscher, *Three-dimensional optical-tomography of hemodynamics in the human head*, Optics Express, 9 (2001), pp. 272–286.

[15] A. A. Bukhgeim and S. G. Kazantsev, *Inversion formula for the Fan-beam attenuated Radon transform in a unit disk*, Sobolev Instit. of Math., Preprint N. 99 (2002).

[16] M. Choulli and P. Stefanov, *Reconstruction of the coefficients of the stationary transport equation from boundary measurements*, Inverse Problems, 12 (1996), pp. L19–L23.

[17] V. Markel and J. Schotland, *Inverse Problem in optical Diffusion tomography. I. Fourier-Laplace inversion formulas*, J. Opt. Soc. Am. A, 18(6) (2001), pp. 1336–1347.

[18] R. G. Mukhometov, *The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry. (Russian)*, Dokl. Akad. Nauk SSSR, 232 (1977), pp. 32–35.

[19] ——, *A problem of integral geometry on the plane. (Russian) Methods of functional analysis in problems of mathematical physics (Russian)*, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 180 (1978), pp. 30–37.

[20] A. Nachman, *Reconstruction from boundary measurements*, Ann. Math., 128 (1988), pp. 531–577.

[21] F. Natterer, *Inversion of the attenuated Radon transform*, Inverse Problems, 17 (2001), pp. 113–119.

[22] F. Natterer and F. Wübbeling, *Mathematical Methods in Image Reconstruction*, SIAM monographs on Mathematical Modeling and Computation, Philadelphia, 2001.

[23] R. G. Novikov, *An inversion formula for the attenuated X-ray transformation*, Ark. Math., 40 (2002), pp. 145–167 (Rapport de Recherche 00/05–3, Université de Nantes, Laboratoire de Mathématiques).

[24] K. Ren, G. Bal, and A. H. Hielscher, *Frequency domain optical tomography based on the equation of radiative transfer*, Submitted, (2005).

[25] P. Stefanov and G. Uhlmann, *Optical tomography in two dimensions*, Methods Appl. Anal., 10 (2003), pp. 1–9.

[26] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., 125(1) (1987), pp. 153–169.

[27] G. Uhlmann, *Developments in inverse problems since Calderón's foundational paper*, Harmonic analysis and partial differential equations, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, (1999), pp. 295–345.