Operads and the interaction between algebraic and geometric topology: preliminary outline

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1. Introduction

Throughout much of their history algebraic and geometric topology were unified. Eventually two distinct subcultures emerged, the former pushing around categories and functors and the latter pushing around surgery diagrams and geometric structures. In a number of recent developments algebraic topology has once again been brought to bear on geometric questions. But instead of the algebraic topology of manifolds themselves one studies that of "moduli" spaces associated to manifolds, for example spaces of maps, embeddings, or diffeomorphisms.

In this lecture series we will touch on some recent developments in studying algebraic topology of moduli spaces coming from geometric topology – the stable mapping class group, spaces of knots, and string topology. Given this breadth of examples, we will sacrifice studying some of the deeper mathematics which goes into each one (for example, Harer's stability theorem and singularity theory for the stable mapping class group, or Goodwillie calculus and the JSJ decomposition of three-manifolds for spaces of knots). But we will highlight the role of operad actions in each setting, a common thread which binds them. We will organize our lectures around this thread and develop the theory of operads including recent general work on operads and (co)simplicial technology.

2. Topics

The following list of topics, which also gives a rough lecture plan, is *tentative*.

i) Introduction to operads
   (a) Definitions and classical examples.
   (b) Little cube actions and resulting operations in homology (F. Cohen's thesis)
   (c) McClure-Smith's technology for cubes actions on cosimplicial spaces.

ii) String topology
   (a) The Chas-Sullivan bracket.
   (b) Operad actions of Kauffman, Penner (and presumably others inappropriately left out).
   (c) Approach through cosimplicial technology.

iii) Spaces of knots
   (a) Budney's little cubes action.
   (b) Budney's results in dimension three.
   (c) Goodwillie-Weiss and Sinha's work in higher dimensions; connection with cosimplicial technology.
   (d) Turchin's calculations.

iv) The stable mapping class group.

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(a) Tillman's operad action.
(b) The Mumford conjecture.
(c) Madsen-Tillman-Weiss's equivalence to $\mathbb{C}P^{\infty}_{-1}$.

3. Reading

For each of these topics, we give literature of which we are aware (omissions are inevitable). Our lectures are really meant as a survey, so that such papers will be slightly more approachable after this introduction. The one paper it would be particularly helpful for you to read is McClure-Smith's "Operads and cosimplicial objects: an introduction," as these ideas and technology are used a few times in our study. We adorn this paper with three stars *** below. Survey papers which would help build perspective are adorned by **. Technical papers whose first sections would serve as useful introductions are adorned by *.

i) Introduction to operads
 (a) J. Stasheff. H-Spaces from a Homotopy Point of View. Lecture Notes in Mathematics, Vol. 161, 1970.
 (b) J.M. Boardman and R. Vogt. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347, 1973.
 (c) J. P. May. The geometry of iterated loop spaces. Lecture Notes in Mathematics, Vol. 271, 1972.
 (d) M. Markl, S. Shnider, and J. Stasheff. Operads in algebra, topology and physics. Math. Surveys and Monographs, 96. AMS, 2002.
 (e) *** J. McClure and J Smith. Operads and cosimplicial objects: an introduction. math.QA/0402117.

ii) String topology
 (a) ** R. Cohen and A. Voronov. Notes on string topology. math.GT/0503625.
 (b) M. Chas and D. Sullivan. String topology. math.GT/9911159.
 (c) A number of papers from different viewpoints on the Arxiv.

iii) Spaces of knots
 (a) * R. Budney. Little cubes and long knots. math.GT/0309427.
 (b) * D. Sinha. The topology of spaces of knots. math.AT/0202287 and Operads and knot space. math.AT/040703.
 (c) Turchin's On the other side of the bialgebra of chord diagrams. math.QA/0411436 and Dyer-Lashof-Cohen operations in Hochschild cohomology. math.RA/0504017

iv) The stable mapping class group.
 (a) ** U. Tillmann. Strings and the stable cohomology of mapping class groups. math.GT/0304300.
 (b) U. Tillmann. Higher genus surface operad detects infinite loop spaces. Math. Ann. 317 (2000), 613–628.
 (c) Madsen-Tillmann. The stable mapping class group and $Q(\mathbb{C}P^{\infty})$. Invent. Math. 145 (2001), 509–544.
 (d) Madsen-Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. math.AT/021232.

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