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Enumeration of Totally Positive Grassmann Cells

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Abstract. In [**6**], Postnikov gave a combinatorially explicit cell decomposition of the totally nonnegative part of a Grassmannian, denoted $Gr^+_{k,n}$, and showed that this set of cells is isomorphic as a graded poset to many other interesting graded posets, such as the posets of decorated permutations, Ꞁ-diagrams (certain $0-1$ tableau), and positroids. The main result of our work is an explicit generating function which enumerates the cells in $Gr^+_{k,n}$ according to their dimension. Equivalently, we compute rank generating functions for the posets of decorated permutations, Ꞁ-diagrams, and positroids. As a corollary, we give a new proof that the Euler characteristic of $Gr^+_{k,n}$ is 1. Additionally, we use our result to produce a new q-analog of the Eulerian numbers, which interpolates between the Eulerian numbers, the Narayana numbers, and the binomial coefficients.

Résumé. Postnikov a décrit explicitement dans [**6**], en termes combinatoires, la décomposition cellulaire de la partie positive (notée $Gr^+_{k,n}$) d'une variété grassmannienne. Il a montré que cet ensemble de cellules est isomorphe, en tant que treillis gradué, à de nombreux ensembles partiellement ordonnés intéressants, comme les permutations décorées, les Ꞁ-diagrammes (qui sont certains tableaux à coefficients 0, 1) ou les matroïdes positifs. Le résultat principal de notre travail est une fonction génératrice explicite, qui dénombre les cellules de $Gr^+_{k,n}$ selon leur dimension. De façon équivalente, nous calculons la fonction génératrice, pondérée par le rang, pour le treillis des permutations décorées, des Ꞁ-diagrammes et des matroïdes positifs. Nous en déduisons comme corollaire une nouvelle preuve que la caractéristique d'Euler de $Gr^+_{k,n}$ est 1. De plus, nous utilisons notre résultat pour exhiber un nouveau q-analogue des nombres eulériens, qui s'interpole entre les nombres eulériens, les nombres de Narayana et les coefficients binomiaux.

1. Introduction

The classical theory of total positivity concerns matrices in which all minors are nonnegative. While this theory was pioneered by Gantmacher, Krein, and Schoenberg in the 1930s, the past decade has seen a flurry of research in this area initiated by Lusztig [**3, 4, 5**], and continued by works of Fomin and Zelevinsky [**1**], and Rietsch [**7**], among others.

Most recently, Postnikov [**6**] investigated the combinatorics of the totally nonnegative part of a Grassmannian $Gr^+_{k,n}$: he produced a combinatorially explicit cell decomposition of $Gr^+_{k,n}$, giving the set of cells of $Gr^+_{k,n}$ a natural structure of graded poset. Furthermore, he showed that this poset was isomorphic to many other interesting combinatorial posets, such as the posets of decorated permutations, Ꞁ-diagrams, positive oriented matroids, and move-equivalence classes of planar oriented networks. In this paper we continue Postnikov's study of the combinatorics of $Gr^+_{k,n}$: in particular, we enumerate the cells in the cell decomposition

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of $Gr^+_{k,n}$ according to their dimension. Equivalently, we compute the rank generating functions for all of the above posets.

The *totally nonnegative part* of the Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$ is defined as the quotient $Gr^+_{k,n} = \mathrm{GL}^+_k \setminus \mathrm{Mat}^+(k,n)$, where GL^+_k is the group of real $k \times k$ matrices with positive determinant, and $\mathrm{Mat}^+(k,n)$ is the set of real $k \times n$-matrices of rank k with nonnegative maximal minors. If we specify which maximal minors are strictly positive and which are equal to zero, we obtain a cellular decomposition of $Gr^+_{k,n}$, as shown in [**6**]. We refer to the cells in this decomposition as *totally positive cells*. The set of totally positive cells naturally has the structure of a graded poset: we say that one cell covers another if the closure of the first cell contains the second, and the rank function is the dimension of each cell.

The main result of this paper is an explicit formula for the *rank generating function* $A_{k,n}(q)$ of $Gr^+_{k,n}$. Specifically, $A_{k,n}(q)$ is defined to be the polynomial in q whose q^r coefficient is the number of totally positive cells in $Gr^+_{k,n}$ which have dimension r. As a corollary of our main result, we give a new proof that the Euler characteristic of $Gr^+_{k,n}$ is 1. Additionally, using our result and exploiting the connection between totally positive cells and permutations, we compute generating functions which enumerate (regular) permutations according to two statistics. This leads to a new q-analog of the Eulerian numbers that has many interesting combinatorial properties. For example, when we evaluate this q-analog at $q = 1, 0$, and -1, we obtain the Eulerian numbers, the Narayana numbers, and the binomial coefficients. Finally, the connection with the Narayana numbers suggests a way of incorporating noncrossing partitions into a larger family of "crossing" partitions.

Let us fix some notation. Throughout this paper we use $[i]$ to denote the q-analog of i, that is, $[i] = 1 + q + \cdots + q^{i-1}$. (We will sometimes use $[n]$ to refer to the set $\{1, \ldots, n\}$, but the context should make our meaning clear.) Additionally, $[i]! := \prod_{k=1}^{i} [k]$ and $\begin{bmatrix} i \\ j \end{bmatrix} := \frac{[i]!}{[j]![i-j]!}$ are the q-analogs of $i!$ and $\binom{i}{j}$, respectively.

2. ⅃-Diagrams

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence of nonnegative numbers. For a partition λ, where $\sum \lambda_i = n$, the *Young diagram* Y_λ of shape λ is a left-justified diagram of n boxes, with λ_i boxes in the ith row. Figure 1 shows a Young diagram of shape $(4, 2, 1)$.

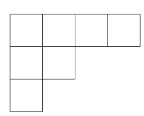

FIGURE 1. A Young diagram of shape $(4, 2, 1)$

Fix k and n. Then a ⅃*-diagram* $(\lambda, D)_{k,n}$ is a partition λ contained in a $k \times (n-k)$ rectangle (which we will denote by $(n-k)^k$), together with a filling $D : Y_\lambda \to \{0, 1\}$ which has the ⅃*-property*: there is no 0 which has a 1 above it and a 1 to its left. (Here, "above" means above and in the same column, and "to its left" means to the left and in the same row.) In Figure 2 we give an example of a ⅃-diagram. [1]

We define the rank of $(\lambda, D)_{k,n}$ to be the number of 1's in the filling D. Postnikov proved that there is a one-to-one correspondence between ⅃-diagrams (λ, D) contained in $(n-k)^k$, and totally positive cells in $Gr^+_{k,n}$, such that the dimension of a totally positive cell is equal to the rank of the corresponding ⅃-diagram. He proved this by providing a modified Gram-Schmidt algorithm A, which has the property that it maps a real $k \times n$ matrix of rank k with nonnegative maximal minors to another matrix whose entries are all positive or 0, which has the ⅃-property. In brief, the bijection between totally positive cells and ⅃-diagrams maps a

[1] The symbol ⅃ is meant to remind the reader of the shape of the forbidden pattern, and should be pronounced as [le], because of its relationship to the letter L. See [**6**] for some interesting numerological remarks on this symbol.

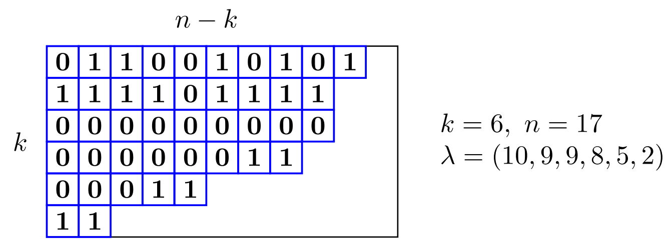

FIGURE 2. A Ⅎ-diagram $(\lambda, D)_{k,n}$

matrix M (representing some totally positive cell) to a Ⅎ-diagram whose 1's represent the positive entries of $A(M)$.

Because of this correspondence, in order to compute $A_{k,n}(q)$, we need to enumerate Ⅎ-diagrams contained in $(n-k)^k$ according to their number of 1's.

3. Decorated Permutations and the Cyclic Bruhat Order

The poset of decorated permutations (also called the cyclic Bruhat order) was introduced by Postnikov in [**6**]. A *decorated permutation* $\tilde{\pi} = (\pi, d)$ is a permutation π in the symmetric group S_n together with a coloring (decoration) d of its fixed points $\pi(i) = i$ by two colors. Usually we refer to these two colors as "clockwise" and "counterclockwise," for reasons which the next paragraph will make clear.

We represent a decorated permutation $\tilde{\pi} = (\pi, D)$, where $\pi \in S_n$, by its *chord diagram*, constructed as follows. Put n equally spaced points around a circle, and label these points from 1 to n in clockwise order. If $\pi(i) = j$ then this is represented as a directed arrow, or chord, from i to j. If $\pi(i) = i$ then we draw a chord from i to i (i.e. a loop), and orient it either clockwise or counterclockwise, according to d. We refer to the chord which begins at position i as Chord(i), and we use ij to denote the directed chord from i to j. Also, if $i, j \in \{1, \dots, n\}$, we use $\mathrm{Arc}(i, j)$ to denote the set of points that we would encounter if we were to travel clockwise from i to j, including i and j.

For example, the decorated permutation $(3, 1, 5, 4, 8, 6, 7, 2)$ (written in list notation) with the fixed points 4, 6, and 7 colored in counterclockwise, clockwise, and counterclockwise, respectively, is represented by the chord diagram in Figure 3.

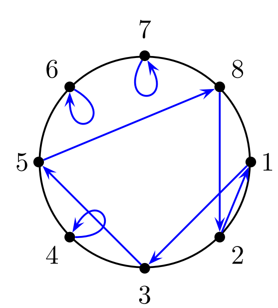

FIGURE 3. A chord diagram for a decorated permutation

The symmetric group S_n acts on the permutations in S_n by conjugation. This action naturally extends to an action of S_n on decorated permutations, if we specify that the action of S_n sends a clockwise (respectively, counterclockwise) fixed point to a clockwise (respectively, counterclockwise) fixed point.

We say that a pair of chords in a chord diagram forms a *crossing* if they intersect inside the circle or on its boundary.

Every crossing looks like Figure 4, where the point A may coincide with the point B, and the point C

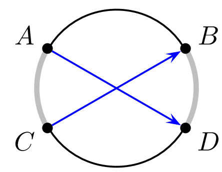

FIGURE 4. A crossing

may coincide with the point D. A crossing is called a *simple crossing* if there are no other chords that go from $\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$. Say that two chords are *crossing* if they form a crossing.

Let us also say that a pair of chords in a chord diagram forms an *alignment* if they are not crossing and they are relatively located as in Figure 5. Here, again, the point A may coincide with the point B, and

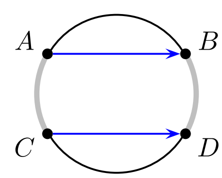

FIGURE 5. An alignment

the point C may coincide with the point D. If A coincides with B then the chord from A to B should be a counterclockwise loop in order to be considered an alignment with $\mathrm{Chord}(C)$. (Imagine what would happen if we had a piece of string pointing from A to B, and then we moved the point B to A). And if C coincides with D then the chord from C to D should be a clockwise loop in order to be considered an alignment with $\mathrm{Chord}(A)$. As before, an alignment is a *simple alignment* if there are no other chords that go from $\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$. We say that two chords are *aligned* if they form an alignment.

We now define a partial order on the set of decorated permutations. For two decorated permutations π_1 and π_2 of the same size n, we say that π_1 *covers* π_2, and write $\pi_1 \to \pi_2$, if the chord diagram of π_1 contains a pair of chords that forms a simple crossing and the chord diagram of π_2 is obtained by changing them to the pair of chords that forms a simple alignment: If the points A and B happen to coincide then the chord from A to B in the chord diagram of π_2 degenerates to a counterclockwise loop. And if the points C and D coincide then the chord from C to D in the chord diagram of π_2 becomes a clockwise loop. These degenerate situations are illustrated in Figure 7.

Let us define two statistics A and K on decorated permutations. For a decorated permutation π, the numbers $A(\pi)$ and $K(\pi)$ are given by

$$A(\pi) = \#\{\text{pairs of chords forming an alignment}\},$$

$$K(\pi) = \#\{i \mid \pi(i) > i\} + \#\{\text{counterclockwise loops}\}.$$

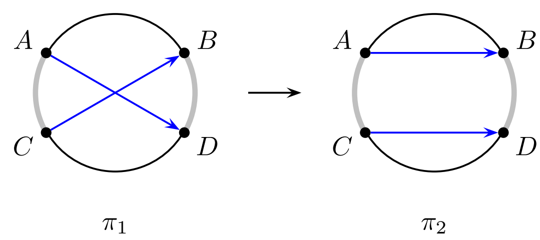

FIGURE 6. Covering relation

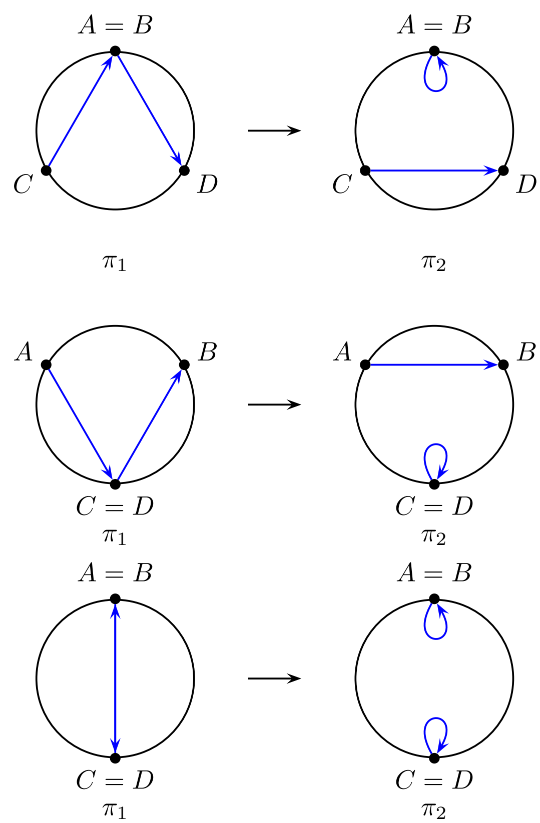

FIGURE 7. Degenerate covering relations

In our previous example $\pi = (3, 1, 5, 4, 8, 6, 7, 2)$ we have $A = 11$ and $K = 5$. The 11 alignments in π are $(13, 66)$, $(21, 35)$, $(21, 58)$, $(21, 44)$, $(21, 77)$, $(35, 44)$, $(35, 66)$, $(44, 66)$, $(58, 77)$, $(66, 77)$, $(66, 82)$.

LEMMA 3.1. [**6**] *If* π_1 *covers* π_2 *then* $A(\pi_1) = A(\pi_2) - 1$ *and* $K(\pi_1) = K(\pi_2)$.

Note that if π_1 covers π_2 then the number of crossings in π_1 is greater then the number of crossings in π_2. But the difference of these numbers is not always 1.

Lemma 3.1 implies that the transitive closure of the covering relation "$\to$" has the structure of a partially ordered set and this partially ordered set decomposes into $n+1$ incomparable components. For $0 \le k \le n$, we define the *cyclic Bruhat order* $\mathcal{CB}_{kn}$ as the set of all decorated permutations π of size n such that $K(\pi) = k$ with the partial order relation obtained by the transitive closure of the covering relation "$\to$". By Lemma 3.1 the function A is the corank function for the cyclic Bruhat order $\mathcal{CB}_{kn}$.

The definitions of the covering relation and of the statistic A will not change if we rotate a chord diagram. The definition of K depends on the order of the boundary points $1, \dots, n$, but it is not hard to see that the statistic K is invariant under the cyclic shift conj_σ for the long cycle $\sigma = (1, 2, \dots, n)$. Thus the order $\mathcal{CB}_{kn}$ is invariant under the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on decorated permutations.

In [**6**], Postnikov proved that the number of totally positive cells in $Gr^+_{k,n}$ of dimension r is equal to the number of decorated permutations in $\mathcal{CB}_{kn}$ of rank r. Thus, $A_{k,n}(1)$ is the cardinality of $\mathcal{CB}_{kn}$, and the coefficient of $q^{k(n-k)-\ell}$ in $A_{k,n}(q)$ is the number of decorated permutations in $\mathcal{CB}_{kn}$ with ℓ alignments.

4. The Rank Generating Function of $Gr^+_{k,n}$

Recall that the coefficient of q^r in $A_{k,n}(q)$ is the number of cells of dimension r in the cellular decomposition of $Gr^+_{k,n}$. In this section we give an explicit expression for $A_{k,n}(q)$, as well as expressions for the generating functions $A_k(q,x) := \sum_n A_{k,n}(q)x^n$ and $A(q,x,y) := \sum_{k\ge 1}\sum_n A_{k,n}(q)x^n y^k$. Our main theorem is the following:

Theorem 4.1.

$$A(q,x,y) = \frac{-y}{q(1-x)} + \sum_{i\ge 1} \frac{y^i(q^{2i+1}-y)}{q^{i^2+i+1}(q^i - q^i[i+1]x + [i]xy)}$$

$$A_k(q,x) = \sum_{i=0}^{k-1} (-1)^{i+k} \frac{x^{k-i-1}[i]^{k-i-1}}{q^{ki+i+1}(1-[i+1]x)^{k-i}} + \sum_{i=0}^{k} (-1)^{i+k} \frac{x^{k-i}[i]^{k-i}}{q^{ki}(1-[i+1]x)^{k-i+1}}$$

$$A_{k,n}(q) = q^{-k^2} \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (q^{ki}[k-i]^i[k-i+1]^{n-i} - q^{(k+1)i}[k-i-1]^i[k-i]^{n-i})$$

$$= \sum_{i=0}^{k-1} \binom{n}{i} q^{-(k-i)^2} ([i-k]^i[k-i+1]^{n-i} - [i-k+1]^i[k-i]^{n-i}).$$

Corollary 4.2. The Euler characteristic of the totally non-negative part of the Grassmannian $Gr^+_{k,n}$ is 1.

Recall that the Euler characteristic of a cell complex is defined to be $\sum_i (-1)^i f_i$, where f_i is the number of cells of dimension i. So to prove Corollary 4.2, simply set $q = -1$ in Theorem 4.1 and simplify.

One interesting ingredient in the proof of Theorem 4.1 is the following lemma. We prove this lemma by interpreting the two equations as statements about partitions, and overpartitions, respectively. Alternatively, Christian Krattenthaler has pointed out to us that this lemma follows from the ${}_1\phi_1$ summation described in Appendix II.5 of [**2**].

Lemma 4.3.

$$\sum_{i\ge 0} (-1)^i y^i q^{\binom{i+1}{2}} \prod_{r=1}^{i+1} \frac{1}{1-q^r y} = 1. \tag{4.1}$$

$$(-1)^j q^{-\binom{j+1}{2}} y^{-j} \sum_{i \geq j} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} i \\ j \end{bmatrix} y^i \prod_{r=1}^{i+1} \frac{1}{1-q^{r+j}y} = 1. \tag{4.2}$$

In Table 1, we have listed some of the values of $A_{k,n}(q)$ for small k and n. It is easy to see from the definition of Ⅎ-diagrams that $A_{k,n}(q) = A_{n-k,n}(q)$: one can reflect a Ⅎ-diagram $(\lambda, D)_{k,n}$ of rank r over the main diagonal to get another Ⅎ-diagram $(\lambda', D')_{n-k,n}$ of rank r. Alternatively, one should be able to prove the claim directly from the expression in Theorem 4.1, using some q-analog of Abel's identity.

$A_{1,1}(q)$	1
$A_{1,2}(q)$	$q+2$
$A_{1,3}(q)$	q^2+3q+3
$A_{1,4}(q)$	q^3+4q^2+6q+4
$A_{2,4}(q)$	$q^4+4q^3+10q^2+12q+6$
$A_{2,5}(q)$	$q^6+5q^5+15q^4+30q^3+40q^2+30q+10$
$A_{2,6}(q)$	$q^8+6q^7+21q^6+50q^5+90q^4+120q^3+110q^2+60q+15$
$A_{3,6}(q)$	$q^9+6q^8+21q^7+56q^6+114q^5+180q^4+215q^3+180q^2+90q+20$
$A_{3,7}(q)$	$q^{12}+7q^{11}+28q^{10}+84q^9+203q^8+406q^7+679q^6+938q^5+1050q^4+$ $910q^3+560q^2+210q+35$

TABLE 1. $A_{k,n}(q)$

Note that it is possible to see directly from the definition that $Gr^+_{1,n}$ is just some deformation of a simplex with n vertices. This explains the simple form of $A_{1,n}(q)$.

5. A New q-Analog of the Eulerian Numbers

If $\pi \in S_n$, we say that π has a *weak excedence* at position i if $\pi(i) \geq i$. The *Eulerian number* $E_{k,n}$ is the number of permutations in S_n which have k weak excedences. (One can define the Eulerian numbers in terms of other statistics, such as descent, but this will not concern us here.)

Using the rank generating function for the poset of decorated permutations, we can enumerate (regular) permutations according to two statistics: weak excedences and alignments. This gives us a new q-analog of the Eulerian numbers.

Recall that the statistic K on decorated permutations was defined as

$$K(\pi) = \#\{i \mid \pi(i) > i\} + \#\{\text{counterclockwise loops}\}.$$

Note that K is related to the notion of weak excedence in permutations. In fact, we can extend the definition of weak excedence to decorated permutations by saying that a decorated permutation has a weak excedence in position i, if $\pi(i) > i$, or if $\pi(i) = i$ and $d(i)$ is counterclockwise. This makes sense, since the limit of a chord from 1 to 2 as 1 approaches 2, is a counterclockwise loop. Then $K(\pi)$ is the number of weak excedences in π.

We will call a decorated permutation *regular* if all of its fixed points are oriented counterclockwise. Thus, a fixed point of a regular permutation will always be a weak excedence, as it should be. Recall that the Eulerian number $E_{k,n}$ is the number of permutations of $[n]$ with k weak excedences. Earlier, we saw that the coefficient of $q^{k(n-k)-\ell}$ in $A_{k,n}(q)$ is the number of decorated permutations in $\mathcal{CB}_{kn}$ with ℓ alignments. By analogy, let $E_{k,n}(q)$ be the polynomial in q whose coefficient of $q^{k(n-k)-\ell}$ is the number of (regular) permutations with k weak excedences and ℓ alignments. Thus, the family $E_{k,n}(q)$ will be a q-analog of the Eulerian numbers.

We can relate decorated permutations to regular permutations via the following lemma.

LEMMA 5.1.

$$E_{k,n}(q) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} A_{k,n-i}(q).$$

Putting this together with Theorem 4.1, we get the following.

COROLLARY 5.2.

$$E_{k,n}(q) = q^{n-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]^n q^{ki-k} (\binom{n}{i} q^{k-i} + \binom{n}{i-1}).$$

Notice that by substituting $q = 1$ into the formula, we get

$$E_{k,n} = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} (k-i)^n,$$

the well-known exact formula for the Eulerian numbers.

Now we will investigate the properties of $E_{k,n}(q)$. Actually, since $E_{k,n}(q)$ is a multiple of q^{n-k}, we first define $\hat{E}_{k,n}(q)$ to be $q^{k-n} E_{k,n}(q)$, and then work with this renormalized polynomial. Table 2 lists $\hat{E}_{k,n}(q)$ for $n = 4, 5, 6, 7$.

$\hat{E}_{1,4}(q)$	1
$\hat{E}_{2,4}(q)$	$6 + 4q + q^2$
$\hat{E}_{3,4}(q)$	$6 + 4q + q^2$
$\hat{E}_{4,4}(q)$	1
$\hat{E}_{1,5}(q)$	1
$\hat{E}_{2,5}(q)$	$10 + 10q + 5q^2 + q^3$
$\hat{E}_{3,5}(q)$	$20 + 25q + 15q^2 + 5q^3 + q^4$
$\hat{E}_{4,5}(q)$	$10 + 10q + 5q^2 + q^3$
$\hat{E}_{5,5}(q)$	1
$\hat{E}_{1,6}(q)$	1
$\hat{E}_{2,6}(q)$	$15 + 20q + 15q^2 + 6q^3 + q^4$
$\hat{E}_{3,6}(q)$	$50 + 90q + 84q^2 + 50q^3 + 21q^4 + 6q^5 + q^6$
$\hat{E}_{4,6}(q)$	$50 + 90q + 84q^2 + 50q^3 + 21q^4 + 6q^5 + q^6$
$\hat{E}_{5,6}(q)$	$15 + 20q + 15q^2 + 6q^3 + q^4$
$\hat{E}_{6,6}(q)$	1
$\hat{E}_{1,7}(q)$	1
$\hat{E}_{2,7}(q)$	$21 + 35q + 35q^2 + 21q^3 + 7q^4 + q^5$
$\hat{E}_{3,7}(q)$	$105 + 245q + 308q^2 + 259q^3 + 161q^4 + 77q^5 + 28q^6 + 7q^7 + q^8$
$\hat{E}_{4,7}(q)$	$175 + 441q + 588q^2 + 532q^3 + 364q^4 + 196q^5 + 84q^6 + 28q^7 + 7q^8 + q^9$
$\hat{E}_{5,7}(q)$	$105 + 245q + 308q^2 + 259q^3 + 161q^4 + 77q^5 + 28q^6 + 7q^7 + q^8$
$\hat{E}_{6,7}(q)$	$21 + 35q + 35q^2 + 21q^3 + 7q^4 + q^5$
$\hat{E}_{7,7}(q)$	1

TABLE 2. $\hat{E}_{k,n}(q)$

We can make a number of observations about these polynomials. For example, we can generalize the well-known result that $E_{k,n} = E_{n+1-k,n}$, where $E_{k,n}$ is the Eulerian number corresponding to the number of permutations of S_n with k weak excedences.

PROPOSITION 5.3. $\hat{E}_{k,n}(q) = \hat{E}_{n+1-k,n}(q)$.

PROPOSITION 5.4. [**6**] The coefficient of the highest degree term of $\hat{E}_{k,n}(q)$ is 1.

PROPOSITION 5.5. $\hat{E}_{k,n}(-1) = \pm\binom{n-1}{k-1}$.

PROPOSITION 5.6. $\hat{E}_{k,n}(q)$ is a polynomial of degree $(k-1)(n-k)$, and $\hat{E}_{k,n}(0)$ is the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

COROLLARY 5.7. $\hat{E}_{k,n}(q)$ interpolates between the Eulerian numbers, the Narayana numbers, and the binomial coefficients, at $q = 1, 0$, and -1, respectively.

REMARK 5.8. The coefficients of $\hat{E}_{k,n}(q)$ appear to be unimodal. However, these polynomials do not in general have real zeroes.

6. Connection with Narayana Numbers

A *noncrossing partition* of the set $[n]$ is a partition π of the set $[n]$ with the property that if $a < b < c < d$ and some block B of π contains both a and c, while some block B' of π contains both b and d, then $B = B'$. Graphically, we can represent a noncrossing partition on a circle which has n labeled points equally spaced around it. We represent each block B as the polygon whose vertices are the elements of B. Then the condition that π is noncrossing just means that no two blocks (polygons) intersect each other.

It is known that the number of noncrossing partitions of $[n]$ which have k blocks is equal to the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ (see Exercise 68e in [**8**]).

To prove the following proposition we will find a bijection between permutations of S_n with k excedences and the maximal number of alignments, and noncrossing partitions on $[n]$.

PROPOSITION 6.1. Fix k and n. Then $(k-1)(n-k)$ is the maximal number of alignments that a permutation in S_n with k weak excedences can have. The number of permutations in S_n with k weak excedences that achieve the maximal number of alignments is the Narayana number $N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

To figure out what the maximal-alignment permutations look like, imagine starting from any given permutation and applying the covering relations in the cyclic Bruhat order as many times as possible, such that the result is a regular permutation. Note that of the four cases of the covering relation (illustrated in section 3), we can use only the first and second cases. We cannot use the third and fourth operations because these add clockwise fixed points, which are not allowed in regular permutations. It is easy to see that after applying the first two operations as many times as possible, the resulting permutation will have no crossings among its chords and all cycles will be directed counterclockwise.

The map from maximal-alignment permutations to noncrossing partitions is now obvious. We simply take our permutation and then erase the directions on the edges. Since the covering relations in the cyclic Bruhat order preserve the number of weak excedences, and since each counterclockwise cycle in a permutation contributes one weak excedence, the resulting noncrossing partitions will all have k blocks. In Figure 8 we show the permutations in S_4 which have 2 weak excedences and 2 alignments, along with the corresponding noncrossing partitions.

Conversely, if we start with a noncrossing partition on $[n]$ which has k blocks, and then orient each cycle counterclockwise, then this gives us a maximal-alignment permutation with k weak excedences.

COROLLARY 6.2. The number of permutations in S_n which have the maximal number of alignments, given their weak excedences, is $C_n = \frac{1}{n}\binom{2n}{n+1}$, the nth Catalan number.

PROOF. It is known that $\sum_k N_{k,n} = C_n$. □

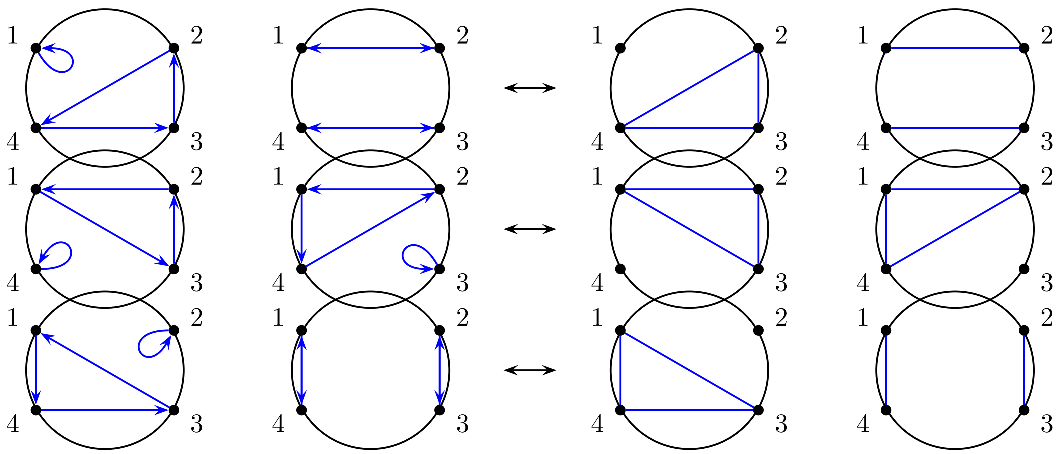

FIGURE 8. The bijection between maximal-alignment permutations and noncrossing partitions

REMARK 6.3. The bijection between maximal-alignment permutations and noncrossing partitions is especially interesting because the connection gives a way of incorporating noncrossing partitions into a larger family of "crossing" partitions; this family of crossing partitions is a ranked poset, graded by alignments.

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