Rook Numbers and the Normal Ordering Problem

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ABSTRACT. For an element w in the Weyl algebra generated by D and U with relation $DU = UD + 1$, the normally ordered form is $w = \sum c_{i,j} U^i D^j$. We demonstrate that the normal order coefficients $c_{i,j}$ of a word w are rook numbers on a Ferrers board. We use this interpretation to give a new proof of the rook factorization theorem, which we use to provide an explicit formula for the coefficients $c_{i,j}$. We calculate the Weyl binomial coefficients: normal order coefficients of the element $(D + U)^n$ in the Weyl algebra. We extend some of these results to the q-analogue of the Weyl algebra.

RÉSUMÉ. Pour un élément dans l'algèbre du Weyl généré par D et U avec de relation $DU = UD + 1$, la forme d'ordre normal est $w = \sum c_{i,j} U^i D^j$. Nous démontrons que les coefficients d'ordre normal $c_{i,j}$ sont des nombres de tours sur d'amier de Ferrers. Nous employons cette interpétation pour fournir une nouvelle preuve de théorème de factorisation de tours, à la laquelle nous mène une formule explicite pour les coefficients $c_{i,j}$. Nous calculons les coefficients binomiaux du Weyl: les coefficients d'ordre normal d'élément $(D + U)^n$ de l'algèbre du Weyl. Nous prolongeons quelque de ces résultats à q-analogue d'algèbre du Weyl.

1. Introduction

For an element w in the Weyl algebra generated by D and U with relation $DU = UD + 1$, the normally ordered form is $w = \sum c_{i,j} U^i D^j$. For example, in the algebra of differential operators where $D = \frac{d}{dx}$ and U acts as multiplication by x, the operator w, applied to the polynomial $f(x)$, is expressed in the normally ordered form as

$$w\big(f(x)\big) = \sum c_{i,j} \; x^i \, \frac{d^j f}{dx^j}(x) \; .$$

The problem of finding explicit formulae for the normal order coefficients $c_{i,j}$ appears more frequently in the context where the Weyl algebra is the algebra of boson operators [**BPS, Ka, Ma, Mi1, Mi2, Sc**], generated by the creation and annihilation operators typically denoted as $a^\dagger$ and a. A boson is a type of particle like the light particle, the photon. According to the theory of quantum mechanics, the possible amount of energy that a particle can have is not continuous but quantized, so there is the smallest amount of energy—the zeroth state, frequently referred to as the ground state; there is the first state, which is the next smallest amount of energy allowed, and so on. The boson operators change the energy state of the particle like the differential operators change the power of x. While the machanics of it are fascinating, for our purposes the assurance of the commutation relation $aa^\dagger - a^\dagger a = 1$ is sufficient.

As a hint of combinatorial interest in the problem of normal ordering, it has long been known that the normal order coefficients of $(UD)^n$ are the Stirling numbers $S(n, k)$ of the second kind. These numbers can

be defined algebraically by the formula

$$x^n = \sum_{k=0}^{n} S(n,k) \, x(x-1)\cdots(x-k+1),$$

and they also have a combinatorial interpretation of counting the number of ways to partition a set of n elements into k subsets.

Mikhailov and Katriel [**Ka, Mi1, Mi2**] have extended the definition of the Stirling numbers, finding explicit formulas for the normal ordered form of operators such as $(D + U^r)^n$, and $(UD + U^r)^n$. Recently, Blasiak, Penson, and Solomon [**BPS**] have generalized the Stirling numbers even further to address the normal ordering problem for operators of the form $(U^r D^s)^n$.

The interpretation of the normal order coefficients of a word as rook numbers on a Ferrers board was given by Navon [**Na**] in 1973, but it requires the power of the rook factorization theorem, presented two years later by Goldman, Joichi, and White [**GJW**] to give an explicit formula. Interestingly enough, the interpretation provides a proof of the rook factorization theorem.

Here is an outline of the article. In section 2, we cover the basic definitions concerning Ferrers boards and rook numbers. We demonstrate that the normal order coefficents of a word are rook numbers on a Ferrers board in section 3, and give an explicit formula for them in section 4, together with a new proof of the rook factorization theorem. We discuss the Weyl binomial coefficients in section 5. Finally, we extend the interpretation of the normal order coefficients to the q-analogue in section 6.

2. Definitions concerning rook numbers and Ferrers boards

For n a positive integer, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. A *board* is a subset of $[n] \times [m]$, where n and m are positive integers. Intuitively, we think of a board as an array of squares arranged in rows and columns. An element $(i, j) \in B$ is then represented by a square in the ith column and jth row. It will be convenient to consider columns numbered from left to right, and rows numbered from top to bottom, so that the square $(1, 1)$ appears in the top left corner.

A board B is a *Ferrers* board if there is a non-increasing sequence of positive integers $h(B) = (h_1, h_2, \ldots, h_n)$ such that $B = \{(i, j) \mid i \le n \text{ and } j \le h_i\}$. Intuitively, a Ferrers board is a board made up of adjacent solid columns with a common upper edge, such that the heights of the columns from left to right form a non-increasing sequence.

EXAMPLE 2.1. A Ferrers board with height sequence $(4, 4, 3, 1, 1)$ can be visually represented as
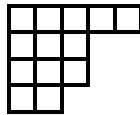
The connection between Ferrers boards and words composed of two letters is as follows. We represent the letter D as a step to the right, and the letter U as a step up. The resulting path outlines a Ferrers board.

EXAMPLE 2.2. The word $w = DDUDUUDDU$ outlines the Ferrers board in example 2.1. This is easy to see from the path representing w:
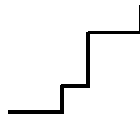
Note that a word $w' = U^i w D^j$ outlines the same Ferrers board for any nonnegative integers i and j. If the Ferrers board B is contained in a rectangle with n columns and m rows, then there is a unique word with n D's and m U's that outlines B.

We denote the Ferrers board outlined by the word w by B_w.

For a board B, let $r_k(B)$ denote the number of ways of marking k squares of the board B, no two in the same row or column. In chess terminology, we are placing k rook pieces on the board B in non-attacking positions. The number $r_k(B)$ is called the kth *rook number* of B.

3. Normal order coefficients of a word

Recall that the Weyl algebra is the algebra generated by D and U, with the commutation relation $DU = UD + 1$.

DEFINITION 3.1. For w an element in the Weyl algebra, the *normally ordered form* of w is the sum

$$w = \sum_{i,j} c_{i,j} U^i D^j,$$

where in each term the D's appear to the right of the U's. The numbers $c_{i,j}$ are the *normal order coefficients* of w.

We call w a *word* if w has a representation $w = w_1 w_2 \ldots w_n$, where $w_i \in \{D, U\}$. We demonstrate that the normal order coefficients of a word w are rook numbers on the Ferrers board outlined by w. This combinatorial interpretation was originally given by Navon [**Na**].

THEOREM 3.2. **Normally Ordered Word**
Let the element w in the Weyl algebra be a word composed of n D's and m U's. Then

$$w = \sum_{k=0}^{n} r_k(B_w) U^{m-k} D^{n-k}.$$

PROOF. It is easy to see that the terms in the normally ordered form of w are $U^{m-k} D^{n-k}$, where $k = 0, 1, \ldots, \min(m, n)$. Every time we replace DU with $UD + 1$ and expand the result, one of the new terms retains the same number of D's and U's as before, and the other term has one fewer of each.

By definition, the normal order coefficient $c_{m-k,n-k}$ is the number of terms $U^{m-k} D^{n-k}$ in the normally ordered form of w, which is obtained by successively replacing DU with $UD + 1$ and expanding. For the sake of consistency, we always choose to replace the rightmost DU.

We can regard the terms as words. Then the normal order coefficient $c_{m-k,n-k}$ is the number of ways to get the word $U^{m-k} D^{n-k}$ from the word w, by successively replacing the rightmost DU with either UD or 1 (that is, deleting it), choosing to do the latter k times.

We now can consider the substitutions in terms of the outlined Ferrers boards. The rightmost DU outlines the rightmost inner corner square of the board. Replacing DU with UD amounts to deleting that square, whereas deleting the DU amounts to deleting the square together with its row and column. Therefore the normal order coefficient $c_{m-k,n-k}$ is the number of ways to reduce the Ferrers board B_w outlined by w to the trivial board by successively deleting the rightmost inner corner square either alone, or together with its row and column, choosing to do the latter k times.

The k squares that are deleted together with their rows and columns cannot share either a row or a column. So the normal order coefficient $c_{m-k,n-k}$ is the number of ways to mark k squares on the Ferrers board B_w outlined by the word w, no two in the same row or column. This is exactly the kth rook number $r_k(B_w)$.

Finally, since $r_k(B_w) = 0$ for $k > \min(m, n)$, we let the sum range from 1 to n. □

REMARK 3.3. As mentioned in the introduction, the Stirling numbers $S(n, k)$ of the second kind are the normal order coefficients of the word $(UD)^n$. Mikhailov [**Mi1**] defined, in a purely algebraic way, a more generalized version of the Stirling numbers to find the normal ordered form of operators of the form $(U^r + D)^n$. In a recent paper unrelated to the normal ordering problem, Lang [**La**] studied a similar generalization of the Stirling numbers, finding combinatorial interpretations for certain particular cases. Recently Blasiak, Penson,

and Solomon [**BPS**] introduced the generalized Stirling numbers of the second kind, denoted $S_{r,s}(n,k)$ for $r \geq s \geq 0$, defined by the relation

$$(U^r D^s)^n = U^{n(r-s)} \sum_{k=s}^{ns} S_{r,s}(n,k) U^k D^k.$$

The standard Stirling numbers of the second kind are $S_{1,1}(n,k)$, and the generalized Stirling numbers of Mikhailov are $S_{r,1}(n,k)$.

We define the *staircase board* $J_{r,s,n}$ to be the Ferrers board outlined by the word $(U^r D^s)^n$.

COROLLARY 3.4.
$$S_{r,s}(n,k) = r_{ns-k}(J_{r,s,n}).$$

REMARK 3.5. We can easily adapt the proof of Theorem 3.2 to work in the case where the algebra generated by D and U has the commutation relation $DU = UD + c$. For a word w, we get the normal ordering form of w by successively replacing DU by $UD + c$, and expanding. Just as before, we consider w as a word in the letters D, U, and the substitution as a choice of either replacing the rightmost DU by UD or deleting it, but the choice of deleting is weighted by c. In terms of the associated Ferrers board, we weight each placement of a rook by c. So we get

$$w = \sum_{k=0}^{n} c^k r_k(B_w) U^{m-k} D^{n-k}.$$

We should note that this algebra is isomorphic to the Weyl algebra, because if $DU - UD = 1$, then $D(cU) - (cU)D = c$.

We can also assign a weight to the choice of replacing DU by UD, and thus extend the result to algebra with the relation $DU = qUD + 1$. The algebra with this relation is know as the q-Weyl algebra, and is of interest both to combinatorialists and to physicists. To the latter, because such algebras are models for q-degenerate bosonic operators [**Sc**]. To the former, because it involves the q-analogue of rook numbers [**GR**]. We discuss this case in detail in section 6.

First, we show how Theorem 3.2 allows for a new proof of the Rook Factorization Theorem [**GJW**], which in turn leads to an explicit formula for computing the normal order coefficients of a word.

4. Computing the normal order coefficients

For a general board B, rook numbers can be computed recursively [**Ri**]. Choose a square of B, and let B_1 be the board obtained from B by deleting that square, and let B_2 be the board obtained from B by deleting the square together with its row and column. Then $r_k(B) = r_k(B_1) + r_{k-1}(B_2)$, reflecting the fact we may or may not mark the square in question.

There are better methods for calculating rook numbers on Ferrers boards, owing to the fact that the generating function of rook numbers on a Ferrers board, expressed in terms of falling factorials, completely factors.

We define the kth *falling factorial* of x by
$$x^{\underline{k}} = x(x-1)\cdots(x-k+1).$$

Goldman et al. [**GJW**] show that the factorial rook polynomial $r(B,x) := \sum_{k=0}^{n} r_k(B) x^{\underline{n-k}}$ of a Ferrers board is a product of linear factors.

THEOREM 4.1. **Rook Factorization Theorem**
For a Ferrers board B with column heights $h(B) = (h_1, \ldots, h_n)$,

$$\sum_{k=0}^{n} r_k(B) x^{\underline{n-k}} = \prod_{i=1}^{n} (x + h_i - n + i)$$

We provide a new proof the Rook Factorization Theorem, using Theorem 3.2.

PROOF. Let w be the word with n D's and h_1 U's that outlines the Ferrers board B. By Theorem 3.2,

$$w = \sum_{k=0}^{n} r_k(B_w) U^{h_1-k} D^{n-k}.$$

as an element in the Weyl algebra.

We consider a particular manifestation of the Weyl algebra as the algebra of operators generated by $D = \frac{d}{dt}$ and $U=$ multiplication by t, acting on functions in the variable t. So

$$w = \sum_{k=0}^{n} r_k(B_w) t^{h_1-k} \left(\frac{d}{dt}\right)^{n-k}.$$

We apply both sides of the equation to t^x, where x is a real number.

Since $\left(\frac{d}{dt}\right)^{n-k}(t^x) = x(x-1)\cdots(x-(n-k)+1)t^{x-n+k} = x^{\underline{n-k}}\, t^{x-n+k}$, the right hand side is

$$\sum_{k=0}^{n} r_k(B)\, x^{\underline{n-k}}\, t^{x-n+h_1}.$$

On the left-hand side we get the product of the following factors. The jth application of D gives the factor of $x + a_U - a_D$, where a_U the number of times U was previously applied, and a_D the number of times D was previously applied. There are $j-1$ D's to the right of the jth D, so $a_D = j-1$. The jth D from the right is the $(n-j+1)$st D from the left, so $a_U = h_{n-j+1}$. Therefore the left-hand side is

$$t^{x-n+h_1} \prod_{j=1}^{n} (x + h_{n-j+1} - j + 1).$$

If we let $i = n - j + 1$, then the left-hand side is

$$t^{x-n+h_1} \prod_{i=1}^{n} (x + h_i - n + i).$$

Now we set $t = 1$ to get the desired result. $\qquad \square$

EXAMPLE 4.2. For $w = DDUUUDDUD$, by Theorem 3.2,

$$\frac{d}{dt}\frac{d}{dt} t \cdot t \cdot t \cdot \frac{d}{dt}\frac{d}{dt} t \cdot \frac{d}{dt} (t^x) = \sum_{k=0}^{5} r_k(B_w)\, t^{4-k} \left(\frac{d}{dt}\right)^{5-k} (t^x)$$

$$x \cdot (x+1) \cdot (x-1) \cdot x \cdot x \cdot (t^{x-1}) = \sum_{k=0}^{5} r_k(B_w)\, x^{\underline{5-k}} \cdot (t^{x-1})$$

$$x \cdot (x+1) \cdot (x-1) \cdot x \cdot x = \sum_{k=0}^{5} r_k(B_w)\, x^{\underline{5-k}}.$$

The left hand side is the complete factorization of the factorial rook polynomial of B_w.

The falling factorials $1, x, x(x-1), \ldots$ form a basis of polynomials in x. If $P(x) = \sum_{k=0}^{n} p_k\, x^{\underline{k}}$, it is well known that the coefficients are $p_k = \frac{1}{k!}\Delta^k P(x)\Big|_{x=0}$, where Δ is the difference operator defined by $\Delta P(x) = P(x+1) - P(x)$. Explicitly [St], the coefficients are

$$p_k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} P(i).$$

COROLLARY 4.3. **Computing the Normal Order Coefficients of a Word**
For w a word in the Weyl algebra composed of n D's and m U's, let $P(x) = \prod_{i=1}^{n}(x + h_i - n + i)$, where $h_1, h_2, \ldots, h_n$ are the column heights of the Ferrers board outlined by w. Then

$$w = \sum_{k=0}^{n} r_k U^{m-k} D^{n-k},$$

where

$$r_k = \frac{1}{(n-k)!} \sum_{i=0}^{n-k} (-1)^{n-k-i} \binom{n-k}{i} P(i).$$

5. Weyl binomial coefficients

The Weyl binomial coefficient $\binom{n}{m}_k$ is the coefficient of the term $U^{n-m-k}D^{m-k}$ in $(D+U)^n$, where the commutation relation is $DU = UD + 1$. The binomial product $(D+U)^n$ is the sum of all words in letters D and U of length n, and the normally ordered term $U^{n-m-k}D^{m-k}$ comes from all words with $n-m$ U's and m D's, where k paris of D and U are deleted during the normal ordering. Each of these words outlines a unique Ferrers board with at most m columns of height at most $(n-m)$. Therefore the Weyl binomial coefficient $\binom{n}{m}_k$ can be expressed as the sum of the kth rook numbers over all Ferrers boards contained in the m-by-$(n-m)$ rectangle.

The classical binomial coefficient $\binom{n}{m}$, and its q-analogue $\begin{bmatrix} n \\ m \end{bmatrix}$, can be similarly expressed in terms of Ferrers boards. The binomial coefficient $\binom{n}{m}$ is the coefficient of the term $U^{n-m}D^m$ in $(D+U)^n$, where the commutation relation is $DU = UD$. The term $U^{n-m}D^m$ is the normally ordered form of any word w with $n-m$ U's and m D's. Since all the letters commute, there is only one way to normally order a word, so the binomial coefficient $\binom{n}{m}$ counts the words with $n-m$ U's and m D's. Since there is a bijection between the set of such words and the set of Ferrers boards contained in m-by-$(n-m)$ rectangle, $\binom{n}{m}$ counts such Ferrers boards. Therefore

$$\binom{n}{m} = \sum_{B \subseteq [m] \times [n-m]} 1.$$

Similarly, the q-binomial coefficient $\begin{bmatrix} n \\ m \end{bmatrix}$ is the coefficient of the term $U^{n-m}D^m$ in $(D+U)^n$, where the commutation relation is $DU = qUD$. Again, the term $U^{n-m}D^m$ comes from any word w with $n-m$ U's and m D's, which outlines the Ferrers board B_w contained in the rectangle $[m] \times [n-m]$. The relation $DU = qUD$ specifies that each square of B_w has weight q, so $w = q^{|B_w|}U^{n-m}D^m$. In other words,

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{B \subseteq [m] \times [n-m]} q^{|B|}.$$

THEOREM 5.1. *Let $k \leq m$ be an integer. Then*

$$\binom{n}{m}_k = \sum_{B \subseteq [m] \times [n-m]} r_k(B) = \frac{n!}{2^k\, k!\, (m-k)!\, (n-m-k)!}$$

PROOF. Any Ferrers board B in $[m] \times [n-m]$ is B_w for a particular word w with $n-m$ U's and m D's. From the proof of Theorem 3.2, the number $r_k(B_w)$ is the number of ways to get the word $U^{n-m-k}D^{m-k}$ from the word w, by successively either replacing DU with UD or deleting it, choosing to do the latter k times. Equivalently, $r_k(B_w)$ is the number of ways to mark k pairs of the letters D and U in the word w, such that in each pair the D appears to the left of the U. The marked pairs are deleted, and the rest of the letters commute into the normally ordered form.

We therefore count the number of ways to construct words with $n - m$ U's and m D's, with k marked pairs of the letters D and U, the former to the left of the latter. We begin with n spaces for the letters in w. Choose $n - m - k$ of these to be U, and $m - k$ to be D. There are

$$\binom{n}{n-m-k,\ m-k} = \frac{n!}{(n-m-k)!\,(m-k)!\,(2k)!}$$

ways to do so. For the $2k$ remaining spaces, we pair them, forming k pairs. There are $(2k-1)\cdot(2k-3)\cdots 5\cdot 3\cdot 1$ ways to do this. For each pair, let the space on the left be D, and the space on the right be U.

By this construction, the sum of the kth rook numbers over all Ferrers boards contained in the m-by-$(n - m)$ rectangle is

$$\frac{(2k-1)\cdot(2k-3)\cdots 5\cdot 3\cdot 1}{(2k)!}\frac{n!}{(n-m-k)!\,(m-k)!} = \frac{n!}{2^k\,k!\,(m-k)!\,(n-m-k)!}.$$

$\square$

Since $(D + U)^n$ is the sum of all words composed of letters D and U of length n, we have a formula for normal ordering of $(D + U)^n$, as shown by Mikhailov in [**Mi1**].

COROLLARY 5.2.

$$(D + U)^n = \sum_{m=0}^{n} \sum_{k=0}^{\min(m,n-m)} \frac{n!}{2^k\,k!\,(m-k)!\,(n-m-k)!} U^{m-k} D^{n-m-k}$$

REMARK 5.3. The Weyl binomial coefficients obey the recursive formula

$$\binom{n}{m}_k = \binom{n-1}{m}_k + \binom{n-1}{m-1}_k + m\binom{n-2}{m-1}_{k-1},$$

with boundary conditions $\binom{1}{0}_0 = \binom{1}{1}_0 = 1$, $\binom{n}{m}_{k-1} = 0$. Consider the pairs (B, C), where B is a Ferrers board in $[m] \times [n - m]$ and C is a placement of k rooks on B. The Weyl binomial coefficient counts the number of such pairs. The set of such pairs is a disjoint union of three sets: one where the height h_1 of the first column B is strictly less than $n - m$, one where $h_1 = m$ and C doesn't place a rook in the first column, and one where $h_1 = m$ and C places a rook in the first column. The recursive formula follows.

The first two terms in the recursive formula are the same as for the classical binomial coefficients. In fact, the Weyl binomial coefficients can be expressed in terms of classical coefficients as follows.

COROLLARY 5.4. Let $C(y) = \sum_{k \geq 0} \binom{n}{k} \frac{y^k}{k!}$ be the exponential generating function of the binomial coefficients. Then the ordinary generating function of the Weyl binomial coefficients is

$$\sum_{k \geq 0} \binom{n}{m}_k x^k = \left(\frac{d}{dy}\right)^{n-m} C(y)\Bigg|_{y=\frac{x}{2}}.$$

PROOF.

$$C(y) = \sum_{k \geq 0} \binom{n}{k} \frac{y^k}{k!}$$

$$= \sum_{k \geq 0} \binom{n}{n-k} \frac{y^k}{k!},$$

so

$$\left(\frac{d}{dy}\right)^{n-m} C(y) = \sum_{k \geq 0} \binom{n}{m-k} \frac{y^k}{k!} ,$$

therefore

$$\left(\frac{d}{dy}\right)^{n-m} C(y) \Bigg|_{y=\frac{x}{2}} = \sum_{k \geq 0} \binom{n}{m-k} \frac{1}{k!} \frac{x^k}{2^k}$$

$$= \sum_{k \geq 0} \frac{n!}{2^k \, k! \, (m-k)! \, (n-m-k)!} \, x^k .$$

$\square$

6. The q-analogue

We extend the combinatorial interpretation of the normal order coefficients to the q-Weyl algebra: the algebra with two generators D and U, and the relation $DU = qUD + 1$.

The commutation relation twisted by q comes up in physics as the relation obeyed by the creation and annihilation operators of q-deformed bosons [**Sc**]. The problem of normally ordering these operators has been studied by Katriel [**Ka**], and recently by Schork [**Sc**].

The basic idea of the q-analogue of numbers is that the polynomial $q^0 + q^1 + q^2 + \cdots + q^{n-1}$ plays the role of the positive integer n. We denote the q-analogue of n by $[n]$. Since $1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, we can extend the definition of the q-analogue to all numbers t by defining

$$[t] := \frac{1-q^t}{1-q}.$$

The q-analogue of the derivative $\frac{d}{dx}$ acting on the ring of polynomials in x is defined as

$$D_q \, f(x) := \frac{f(qx) - f(x)}{(q-1)x} ,$$

and it is easy to check that $D_q(x^n) = [n]x^{n-1}$. For a good exposition of the q-analogue of the derivative, we refer the reader to "Quantum Calculus" by Kac and Cheung [**KC**].

If we let $D = D_q$, and let U be the operator acting by multiplication by x, then the algebra generated by D and U has the relation $DU = qUD + 1$, and is therefore the q-Weyl algebra.

Let the element w in the q-Weyl algebra be a word composed of the letters D and U. We adapt the proof of Theorem 3.2 to find the normal order coefficients of w.

In terms of algebraic operations, we get the normal ordering form of w by successively replacing DU by $qUD + 1$, and expanding. As before, we can consider w as a formal word in the letters D and U, and the substitution as a choice of either replacing the rightmost DU by UD, weighting this choice by q, or deleting the rightmost DU. In terms of the Ferrers board B_w outlined by w, we assign the weight q to each square that doesn't have a rook either on it, below it in the same column, or to the right of it in the same row. If we consider the weight of a rook placement to be the product of the weights of all squares of the board, such weights of rook placements describe exactly the q-rook numbers of Garsia and Remmel [**GR**].

DEFINITION 6.1. Let B be a board, and denote by $C_k(B)$ the collection of all placements of k marked squares (rooks) on B, no two in the same row or column. We define the kth q-rook number of B to be

$$R_k(B,q) = \sum_{C \in C_k(B)} q^{\mathrm{inv}(C)},$$

where $\text{inv}(C)$ is the number of squares in the placement C that do not have a rook either on them, below them in the same column, or to the right of them in the same row.

REMARK 6.2. To clarify the statistic $\text{inv}(C)$, we demonstrate by an example. Suppose B is a Ferrers board with column heights $h(B) = (4, 4, 3, 1)$, and we have the following placement C of two rooks
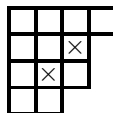
If we mark with a dot the squares above or to the left of a rook, and with a circle the rest of the squares, we get
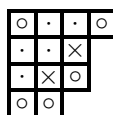
Then $\text{inv}(C)$ is the number of squares marked with a circle, which in this example is 5.

The statistic $\text{inv}(C)$ is a generalization of the inversion statistic on permutations. Given a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$, we get a placement C of n rooks on an n-by-n board where the rook in column i is placed in row σ_i. Then each square marked with a circle has a rook to the left of it and a rook above it, so the square corresponds to an inversion pair $i < j$ such that $\sigma_i > \sigma_j$. So in this case, $\text{inv}(C)$ is the number of inversions of σ.

Analogous to Theorem 3.2, the normal order coefficients of a word w are the q-rook numbers of the Ferrers board outlined by w.

THEOREM 6.3. *Let the element w in the q-Weyl algebra be a word composed of n D's and m U's. Then*

$$w = \sum_{k=0}^{n} R_k(B_w, q) U^{m-k} D^{n-k}.$$

The Factorization Theorem for q-rook numbers [**GR**] can be proved as a corollary.

THEOREM 6.4. **Factorization Theorem for q-Rook Numbers**
For a Ferrers board B with column heights $h(B) = (h_1, \ldots, h_n)$,

$$\sum_{k=0}^{n} R_k(B, q)[x][x-1]\cdots[x-(n-k)+1] = \prod_{i=1}^{n}[x + h_i - n + i]$$

The proof is exactly the same as for Theorem 4.1, replacing the real numbers, the falling factorials, and the derivative with their q-analogue.

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