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Finite-Dimensional Crystals for Quantum Affine Algebras of type $D_n^{(1)}$

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ABSTRACT. The combinatorial structure of the crystal basis $B^{(2,2)}$ for the $U_q'(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,2)}$ is given, and a conjecture is presented for the combinatorial structure of the crystal basis $B^{(2,s)}$ for the $U_q'(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,s)}$.

RÉSUMÉ. Nous donnons la structure combinatoire de la base cristalline $B^{(2,2)}$ pour le $U_q'(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,2)}$, et nous conjecturons la structure combinatoire de la base cristalline $B^{(2,s)}$ pour le $U_q'(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,s)}$.

1. Introduction

While studying representations of quantum groups, Kashiwara developed the theory of crystal bases, which allow modules over quantum groups to be studied in terms of a crystal graph, a purely combinatorial object [**5**]. An open question in the area of crystal basis theory is to determine for which irreducible representations of quantum affine algebras a crystal basis exists, and when they exist, what combinorial structure the crystals have. It is conjectured [**3, 4**] that there is a family of irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules $W^{(k,s)}$, called Kirillov-Reshetikhin modules, which have crystal bases $B^{(k,s)}$, where k is a Dynkin node and s is a positive integer; furthermore, it is expected that all irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules which have crystal bases are tensor products of the modules $W^{(k,s)}$. A first step towards understanding these crystals is to determine their combinatorial structure.

For type $A_n^{(1)}$, the existence of the modules $W^{(k,s)}$ has been established [**8**], and the explicit combinatorial structure is also well-known [**14**]. For non-simply laced types, the following well-known algebra embeddings are conjectured to apply to crystals as well [**12**], which would yield the combinatorial structure of the corresponding crystals in terms of the crystal structure for the simply-laced types.:

$$\begin{aligned} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} &\hookrightarrow A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} &\hookrightarrow D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} &\hookrightarrow E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} &\hookrightarrow D_4^{(1)}. \end{aligned}$$

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Therefore, the next step in developing a general theory of affine crystals is to explore crystals of types $D_n^{(1)}$ ($n \geq 4$) and $E_n^{(1)}$ ($n = 6, 7, 8$). In this paper, we concentrate on type $D_n^{(1)}$. For irreducible representations corresponding to multiples of the first fundamental weight (indexed by a one-row Young diagram) or any single fundamental weight (indexed by a one-column Young diagram) the crystals are known to exist and the structure is known [**8, 7**]. It is conjectured in [**3, 4**] that as $U_q(\mathfrak{g}_{I\setminus\{0\}})$-crystals, the crystal $B^{(k,s)}$ decomposes as

$$B^{(k,s)} = \bigoplus_{\Lambda} B(\Lambda),$$

where the direct sum is taken over all partitions which result from removing any number of 2×1 vertical dominoes from a $k \times s$ rectangle, whenever $k \leq n-2$. In the sequel, we consider the case $k = 2$, for which the above direct sum specializes to

$$B^{(2,s)} = \bigoplus_{i=0}^{s} B(i\Lambda_2).$$

First, we will use tableaux of shape (s, s) to define a $U_q(\mathfrak{so}_{2n})$-crystal whose vertices are in bijection with the classical tableaux from the above direct sum decomposition. Because of the way we define our tableaux, we will be able to combinatorially define the unique action of $\tilde{f}_0$ which makes this crystal into a connected perfect crystal of level s. Finally, we present a conjecture for an explicit construction of the representation $W^{(2,s)}$ which is compatible with the crystal basis $B^{(2,s)}$ as constructed. Full details of our results will be forthcoming [**13**].

2. Review of quantum groups and crystal bases

For $n \in \mathbb{Z}$ and a formal parameter q, we use the notations

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \text{ and } \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

These are all elements of $\mathbb{Q}(q)$, called the q-integers, q-factorials, and q-binomial coefficients, respectively.

Let $\mathfrak{g}$ be a Lie algebra with Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$, a Dynkin diagram indexed by I, and let $\{s_i | i \in I\}$ be the entries of the diagonal symmetrizing matrix of A. Let $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$. We may then construct the quantum enveloping algebra $U_q(\mathfrak{g})$ as the associative $\mathbb{Q}(q)$-algebra generated by e_i and f_i for $i \in I$, and q^h for $h \in P^\vee$, with the following relations:

(1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for all $h, h' \in P^\vee$,
(2) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for all $h \in P^\vee$,
(3) $q^h f_i q^{-h} = q^{\alpha_i(h)} f_i$ for all $h \in P^\vee$,
(4) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$,
(5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ for all $i \neq j$,
(6) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ for all $i \neq j$.

We can view $U_q(\mathfrak{g})$ as a q-deformation of $U(\mathfrak{g})$. Similarly, a $U_q(\mathfrak{g})$-module V may be seen as a q-deformation of a $U(\mathfrak{g})$-module. The representation theory of $U_q(\mathfrak{g})$ does not depend on q, provided $q \neq 0$ and $q^k \neq 1$ for all $k \in \mathbb{Z}$. Furthermore, through appropriate tensoring and factoring, we may "take the limit as q goes to zero" in $U_q(\mathfrak{g})$ and V. This process makes V very simple, so that we may study it using a colored directed graph whose vertices correspond to a canonical basis of V. In the solvable lattice models which provided the original motivation for quantum groups, q parameterized temperature, so letting q approach 0 in the quantum group corresponds to the temperature approaching absolute zero in the physical models.

Thus, the graph described above is called a crystal graph, and its vertices are a crystal basis B for V [**5**]. The edges are colored by the index set I, which indicates the action of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on B. The Kashiwara operators are a "crystal version" of the Chevalley generators of $\mathfrak{g}$.

We are particularly interested in a class of crystals called perfect crystals, since they allow us to construct infinite-dimensional highest weight modules over $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is of affine type [**9**]. To define them, we need a few preliminary definitions.

Let P be the weight lattice of an affine Lie algebra $\mathfrak{g}$. Define $P_{cl} = P/\mathbb{Z}\delta$, $P_{cl}^+ = \{\lambda \in P_{cl} | \langle h_i, \lambda\rangle \geq 0 \text{ for all } i \in I\}$, and $U_q'(\mathfrak{g})$ to be the quantum enveloping algebra with the Cartan datum $(A, \Pi, \Pi^\vee, P_{cl}, P_{cl}^\vee)$.

A crystal pseudobase for a module V is a set B such that there is a crystal base B' for V such that $B = B' \cup -B'$.

Denote by c the canonical central element of $\mathfrak{g}$. In the sequel, we only consider $\mathfrak{g}$ of type $D_n^{(1)}$, in which case

$$c = \Lambda_0 + \Lambda_1 + 2\Lambda_2 + \cdots + 2\Lambda_{n-2} + \Lambda_{n-1} + \Lambda_n.$$

Define the set of level ℓ weights to be $(P_{cl}^+)_\ell = \{\lambda \in P_{cl}^+ | \langle c, \lambda\rangle = \ell\}$. For a crystal basis element $b \in B$, define $\varepsilon_i(b) = \max\{n \geq 0 | \tilde{e}_i^n(b) \in B\}$, and $\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i$, and similarly, $\varphi_i(b) = \max\{n \geq 0 | \tilde{f}_i^n(b) \in B\}$, and $\varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i$. Finally, for a crystal basis B, we define B_{min} to be the set of crystal basis elements b such that $\langle c, \varepsilon(b)\rangle$ is minimal over $b \in B$.

A crystal B is a perfect crystal of level ℓ if:

1. $B \otimes B$ is connected;
2. there exists $\lambda \in P_{cl}$ such that $wt(B) \subset \lambda + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\alpha_i$ and $\#(B_\lambda) = 1$;
3. there is a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module V with a crystal pseudobase of which B is an associated crystal;
4. for any $b \in B$, we have $\langle c, \varepsilon(b)\rangle \geq \ell$;
5. the maps ε and φ from B_{min} to $(P_{cl}^+)_\ell$ are bijective.

We may now state the main result of this paper.

THEOREM 2.1. *Suppose that the $U_q'(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,2)}$ has a crystal basis $B^{(2,2)}$ as conjectured in* [**3**]. *Then $B^{(2,2)} \cong \tilde{B}^{(2,2)}$, where $\tilde{B}^{(2,2)}$ is the affine crystal given explicitly by the construction below. Furthermore, we conjecture that the construction of $\tilde{B}^{(2,s)}$ below explicitly gives the crystal graph associated to the $U_q'(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,s)}$.*

Specifically, we will construct a $U_q'(\widehat{\mathfrak{so}}_{2n})$-crystal $\tilde{B}^{(2,s)}$ with the conjectured classical decomposition, and then show that it is the only perfect crystal which can admit such a decomposition. This is the first step in confirming Conjecture 2.1 of [**4**], which states that as modules over the embedded classical quantum group, $W^{(2,s)}$ decomposes as $\bigoplus_{i=0}^{s} V(i\Lambda_2)$, where $V(\Lambda)$ is the classical module with highest weight Λ, $W^{(2,s)}$ has a crystal basis, and this is a perfect crystal of level s.

3. Decomposition of $\tilde{B}^{(2,s)}$

Let $B(k\Lambda_2)$ denote the crystal basis of the irreducible representation of $U_q(\mathfrak{so}_{2n})$ with highest weight $k\Lambda_2$ for $k \in \mathbb{Z}_{\geq 0}$. We may associate with each crystal element a tableau of shape $\lambda = (k, k)$ on the partially ordered alphabet

$$1 < 2 < \cdots < n-1 < \begin{matrix} n \\ \bar{n} \end{matrix} < \overline{n-1} < \cdots \bar{2} < \bar{1}$$

such that [**2**, page 202]

1. if ab is in the filling, then $a \leq b$;
2. if $\begin{smallmatrix} a \\ b \end{smallmatrix}$ is in the filling, then $b \not\leq a$;
3. no configuration of the form $\begin{smallmatrix} a & a \\ & \bar{a} \end{smallmatrix}$ or $\begin{smallmatrix} a & \\ \bar{a} & \bar{a} \end{smallmatrix}$ appears;
4. no configuration of the form $\begin{smallmatrix} n-1 & \\ n & \frac{n}{n-1} \end{smallmatrix}$ or $\begin{smallmatrix} n-1 & \\ \bar{n} & \frac{\bar{n}}{n-1} \end{smallmatrix}$ appears;

(5) no configuration of the form $\begin{smallmatrix}1\\1\end{smallmatrix}$ appears.

Note that for $k \geq 2$, condition 5 follows from conditions 1 and 3.

Consider the set $\mathcal{T}$ of tableaux of shape (s,s) which violate only condition 3. We will construct a bijection between $\mathcal{T}$ and the vertices of $\bigoplus_{i=0}^{s-1} B(i\Lambda_2)$, so that $\mathcal{T} \cup B(s\Lambda_2)$ may be viewed as a $U_q(\mathfrak{so}_{2n})$-crystal with the conjectured classical decomposition of $B^{(2,s)}$. In section 4 we will define $\tilde{f}_0$ on $\mathcal{T} \cup B(s\Lambda_2)$ to give it the structure of a perfect $U_q'(\widehat{\mathfrak{so}}_{2n})$-crystal. This will be the crystal $\tilde{B}^{(2,s)}$ mentioned in Theorem 2.1. For proofs of all claims, see [**13**].

Let $T \in \mathcal{T}$, and define $\bar{\bar{i}} = i$ for $1 \leq i \leq n$. Then there is a unique $a \in \{1, \ldots, n, \bar{n}\}$, $m \in \mathbb{Z}_{>0}$ such that T has exactly one configuration of one of the following forms:

$$\begin{matrix} a & a & & a & c_1 \\ b_1 & \bar{a} & \cdots & \bar{a} & d_1 \end{matrix}, \quad \text{(}m\text{ columns } \begin{smallmatrix}a\\ \bar a\end{smallmatrix}\text{)} \qquad \text{where } b_1 \neq \bar{a}, \text{ and } c_1 \neq a \text{ or } d_1 \neq \bar{a};$$

$$\begin{matrix} b_2 & a & & a & d_2 \\ c_2 & \bar{a} & \cdots & \bar{a} & \bar{a} \end{matrix}, \quad \text{(}m\text{ columns } \begin{smallmatrix}a\\ \bar a\end{smallmatrix}\text{)} \qquad \text{where } d_2 \neq a, \text{ and } b_2 \neq a \text{ or } c_2 \neq \bar{a};$$

$$\begin{matrix} b_3 & a & & a & d_3 \\ c_3 & \bar{a} & \cdots & \bar{a} & e_3 \end{matrix}, \quad \text{(}m+1\text{ columns } \begin{smallmatrix}a\\ \bar a\end{smallmatrix}\text{)} \qquad \text{where } b_3 \neq a \text{ and } e_3 \neq \bar{a}.$$

To find the corresponding element of $\bigoplus_{i=0}^{s-1} B(i\Lambda_2)$, remove $\underbrace{\begin{smallmatrix} a & \cdots & a \\ \bar{a} & & \bar{a} \end{smallmatrix}}_{m}$ from T. The result will be a tableau in $B((s-m)\Lambda_2)$. Denote the image of T under this map by $D_2(T)$. We call D_2 the height-two drop map. For example, we have

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}, \quad D_2(T) = \begin{array}{|c|c|c|} \hline 1 & 3 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}.$$

Let $t \in B(i\Lambda_2)$. The map F_2 (the height-two fill map) which inverts D_2 is given by finding a configuration $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}$ in t such that either $c \leq \bar{a} \leq d$ or $a \leq \bar{d} \leq b$, and filling with $\underbrace{\begin{smallmatrix} a & \cdots & a \\ \bar{a} & & \bar{a} \end{smallmatrix}}_{s-i}$ or $\underbrace{\begin{smallmatrix} \bar{d} & \cdots & \bar{d} \\ d & & d \end{smallmatrix}}_{s-i}$, respectively. If more than one such configuration exists, or if both pairs of inequalities are satisfied, then $F_2(t)$ is independent of any of these choices. For example,

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \bar{4} & \bar{2} & \bar{1} \\ \hline \end{array}, \quad F_2(t) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline \bar{4} & \bar{2} & \bar{2} & \bar{1} \\ \hline \end{array}.$$

While we could choose either column two or column three as the filling location, either choice results in the same tableau.

The action of the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ for $i \in \{1, \ldots, n\}$ on $\mathcal{T} \cup B(s\Lambda_2)$ can be defined by direct combinatorial construction, but for the sake of simplicity, we describe them in terms of the above bijection. Let $T \in \mathcal{T} \cup B(s\Lambda_2)$. We define

$$\begin{aligned} \tilde{e}_i(T) &= F_2(\tilde{e}_i(D_2(T))) \\ \tilde{f}_i(T) &= F_2(\tilde{f}_i(D_2(T))), \end{aligned}$$

where the $\tilde{e}_i$ and $\tilde{f}_i$ on the right are the standard Kashiwara operators on $U_q(\mathfrak{so}_{2n})$-crystals [**10**].

4. Affine Kashiwara operators

We know that once $B^{(2,s)}$ is determined, there will be a map $\sigma : B^{(2,s)} \to B^{(2,s)}$ such that $\tilde{e}_0 = \sigma \tilde{e}_1 \sigma$ and $\tilde{f}_0 = \sigma \tilde{f}_1 \sigma$, corresponding to the automorphism of $U_q'(\widehat{\mathfrak{so}}_{2n})$ which interchanges nodes 0 and 1 of the Dynkin diagram. With this in mind, suppose we have defined $\tilde{f}_0$ on $\mathcal{T} \cup B(s\Lambda_2)$ to produce $\tilde{B}^{(2,s)}$, and consider the following operations on $\tilde{B}^{(2,s)}$; let $J \subset I$, and denote by B_J the graph which results from removing all j-colored edges from $\tilde{B}^{(2,s)}$ for $j \in J$. Then as directed graphs, we expect $B_{\{0\}}$ to be isomorphic to $B_{\{1\}}$, otherwise, $\tilde{B}^{(2,s)}$ and $B^{(2,s)}$ will not be isomorphic. We can gain some information about σ by considering $B_{\{0,1\}}$.

It is easy to see that the connected components of $B_{\{0,1\}}$ will be $U_q(\mathfrak{so}_{2n-2})$-crystals, indexed by partitions as described below. This decomposition produces a "branching component graph" for $\tilde{B}^{(2,s)}$, which we denote $\mathcal{BC}(\tilde{B}^{(2,s)})$. It suffices to describe the decomposition of the component of $\tilde{B}^{(2,s)}$ with classical highest weight $k\Lambda_2$ into $U_q(\mathfrak{so}_{2n-2})$-crystals. Denote this branching component subgraph by $\mathcal{BC}(k\Lambda_2)$. Each vertex v_λ of this graph will be labeled by a partition indicating the classical highest weight λ of the corresponding $U_q(\mathfrak{so}_{2n-2})$-crystal. Let $B(v_\lambda)$ denote the set of tableaux in $B(k\Lambda_2)$ contained in the $U_q(\mathfrak{so}_{2n-2})$-crystal indexed by v_λ. Then $\mathcal{BC}(k\Lambda_2)$ has a 1-colored edge from v_λ to v_μ if there is a tableau $b \in B(v_\lambda)$ such that $\tilde{f}_1(b) \in B(v_\mu)$.

We can give an explicit combinatorial description of $\mathcal{BC}(k\Lambda_2)$ as follows. The "highest weight" component of $\mathcal{BC}(k\Lambda_2)$ is a $1 \times k$ rectangle; call this vertex v_k. The function

$$r_k(v) = d(v, v_k) = \min_{P(v,v_k)} (\text{number of edges in } P(v, v_k))$$

is a rank function on $\mathcal{BC}(k\Lambda_2)$, where $P(v, v_k)$ is the set of all paths from v to v_k in $\mathcal{BC}(k\Lambda_2)$. For any partition λ, in each rank no more than one vertex may be indexed by λ. Let $v_\lambda \in \mathcal{BC}(k\Lambda_2)$ have rank less than k; then there is a 1-edge from v_λ to v_μ, where $r_k(v_\mu) = r_k(v_\lambda) + 1$ and there is an edge between λ and μ in the Young lattice. Also note that if $v_\lambda \in \mathcal{BC}(k\Lambda_2)$, then $\lambda \subset (k, k)$.

If $v_\lambda \in \mathcal{BC}(k\Lambda_2)$ has rank p, there is a vertex v'_λ, called the complementary vertex of v_λ, with rank $2k-p$. Let v_λ have a 1-edge to v_μ. Then there is also a 1-edge from v'_μ to v'_λ. Combined with the above description of the first $k+1$ ranks, this completely characterizes $\mathcal{BC}(k\Lambda_2)$.

Observe that $\mathcal{BC}(\tilde{B}^{(2,s)}) = \bigcup_{i=0}^{s} \mathcal{BC}(i\Lambda_2)$. Let $v_\lambda \in \mathcal{BC}(i\Lambda_2) \subset \mathcal{BC}(\tilde{B}^{(2,s)})$. Define $R(v_\lambda) = r_i(v_\lambda) + s - i$. This puts a rank on all of $\mathcal{BC}(B^{(2,s)})$. Note that $\mathcal{BC}(i\Lambda_2) \subset \mathcal{BC}((i+1)\Lambda_2)$, and this inclusion is compatable with R. Also note that if $R(v_\lambda) = p$, then v'_λ, the complementary vertex to v_λ, is now defined to be the vertex of rank $2s - p$ with the same shape and in the same component as v_λ.

To illustrate, $\mathcal{BC}(\tilde{B}^{(2,2)})$ is given in Figure 1, with rank 0 in the first line, rank 1 in the second, etc.

Since we know that $B_{\{0\}}$ and $B_{\{1\}}$ are isomorphic as directed graphs, it is clear that we can put 0-colored edges in the branching component graph in such a way that interchanging the 1-edges and the 0-edges and applying some shape-preserving bijection $\hat{\sigma}$ (defined below) to the vertices will produce an isomorphic colored directed graph. Such a bijection can be naturally extended to $\sigma : \tilde{B}^{(2,s)} \to \tilde{B}^{(2,s)}$ as follows. Let $b \in B(v_\lambda) \subset \tilde{B}^{(2,s)}$ for some supercrystal vertex v_λ, and let u_λ denote the $U_q(\mathfrak{so}_{2n-2})$ highest weight vector of $B(v_\lambda)$. Then for some finite sequence $i_1, \ldots, i_k$ of integers in $\{2, \ldots, n\}$, we know that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda = b$. Let $v^*_\lambda = \hat{\sigma}(v_\lambda)$, and let u^*_λ be the highest weight vector of $B(v^*_\lambda)$. We may define $\sigma(b) = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u^*_\lambda$. This involution of $\tilde{B}^{(2,s)}$ will satisfy $\tilde{f}_0 = \sigma \tilde{f}_1 \sigma$.

We will define $\hat{\sigma}(v_\lambda)$ for $R(v_\lambda) \le s$, and observe that $\hat{\sigma}(v'_\lambda) = \hat{\sigma}(v_\lambda)'$, where v' denotes the complementary vertex of v. Let $v_\lambda \in \mathcal{BC}(k\Lambda_2)$, and $R(v_\lambda) = p$. Then by the inclusion $\mathcal{BC}(i\Lambda_2) \subset \mathcal{BC}((i+1)\Lambda_2)$, there are $p+1$ vertices of the same shape as v_λ of rank p in $\mathcal{BC}(\tilde{B}^{(2,s)})$, one in each $\mathcal{BC}(j\Lambda_2)$ for $j = \{s-p, \ldots, s\}$. We define $\hat{\sigma}(v_\lambda)$ to be the vertex of the same shape as v_λ of rank $2s-p$ in $\mathcal{BC}((2s-p-k)\Lambda_2)$.

The action of $\hat{\sigma}$ on $\mathcal{BC}(\tilde{B}^{(2,2)})$ is given in Figure 2.

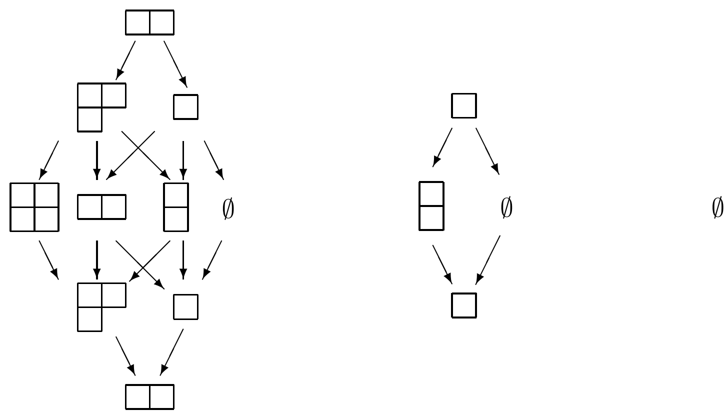

FIGURE 1. The branching component graph $\mathcal{BC}(\tilde{B}^{(2,2)})$

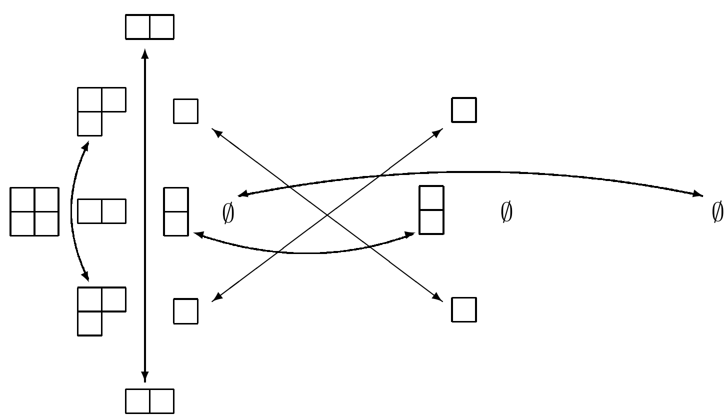

FIGURE 2. Definition of $\hat{\sigma}$ on $\mathcal{BC}(\tilde{B}^{(2,2)})$

The observant reader will note that there are other permutations of the set of vertices of $\mathcal{BC}(\tilde{B}^{(2,s)})$ which respect the shape of the associated partitions. First, note that if a tableau T is in a vertex of rank p, we expect $\tilde{f}_0(T) = \sigma\tilde{f}_1\sigma(T)$ to be in a vertex with rank $p-1$; otherwise there will be some T for which $\tilde{f}_0(T) = \tilde{f}_1(T)$, which must not be the case. Even this does not completely specify $\hat{\sigma}$, since (for instance) we might permute the three empty partitions in any manner and still satisfy all the above requirements. Note, however, that ε_0 depends entirely on the definition of σ, and the perfectness of a crystal depends on the function ε_0. (Recall the definitions from section 2.) For a detailed proof of the following theorem, see [**13**].

THEOREM 4.1. *The above definition of* σ*, interpreted as a permutation of the vertices of* $\bigoplus_{i=0}^{s} B(i\Lambda_2)$*, is the only map such that defining* $\tilde{f}_0 = \sigma\tilde{f}_1\sigma$ *produces a perfect crystal of level* s *for* $s = 2$*. We conjecture that this is true for all* s*.*

5. Perfectness of $\tilde{B}^{(2,s)}$, Part 1

We must show that $\tilde{B}^{(2,s)}$ satisfies conditions 1-5 from Section 2 with $\ell = s$. Condition 1 is verified by showing that each vertex of $\tilde{B}^{(2,s)} \otimes \tilde{B}^{(2,s)}$ is connected to $u_\emptyset \otimes u_\emptyset$, where $u_\emptyset \in \tilde{B}^{(2,s)}$ is the unique vector of the $U_q(\mathfrak{so}_{2n})$-crystal $B(0)$ [**13**]. Condition 2 is satisfied by $\lambda = s\Lambda_2 - 2s\Lambda_0$. We discuss a conjecture which satisfies Condition 3 in section 7. Conditions 4 and 5 can be dealt with simultaneously, and have been proved for $s = 2$ as described below. For proofs of all claims, see [**13**].

Given a weight $\lambda \in (P_{cl}^+)_s$, we can construct a tableau $T_\lambda \in \tilde{B}^{(2,s)}$ such that $\varepsilon(T_\lambda) = \varphi(T_\lambda) = \lambda$. First, observe the following. Let $T \in B(k\Lambda_2) \subset \tilde{B}^{(2,s)}$, and let $T^* = \iota_k^s(T)$, where $\iota_i^j : B(i\Lambda_2) \hookrightarrow B(j\Lambda_2)$ is the natural inclusion map which is compatible with the inclusion $\mathcal{BC}(i\Lambda_2) \hookrightarrow \mathcal{BC}(j\Lambda_2)$. Assume T to be such that $T^* \in \iota_{s-1}^s(B((s-1)\Lambda_2))$. Let $T_m = (\iota_m^s)^{-1}(T^*)$ for $m = s, s-1, \ldots, k$, where k is the smallest number such that $T_k \notin \iota_{k-1}^k(B((k-1)\Lambda_2))$. Then we have

$$\langle \varepsilon(T_s), \Lambda_0 + \Lambda_1 \rangle = \langle \varepsilon(T_{s-1}), \Lambda_0 + \Lambda_1 \rangle = \quad \cdots \quad = \langle \varepsilon(T), \Lambda_0 + \Lambda_1 \rangle \neq 0,$$

and for $i = 2, \ldots, n$,

$$\langle \varepsilon(T_s), \Lambda_i \rangle = \langle \varepsilon(T_{s-1}), \Lambda_i \rangle = \quad \cdots \quad = \langle \varepsilon(T_k), \Lambda_i \rangle.$$

This allows us to temporarily restrict our attention to those level s weights λ which satisfy $\langle \lambda, \Lambda_0 \rangle = \langle \lambda, \Lambda_1 \rangle = 0$; i.e., which can be expressed as $\lambda = \sum_{i=2}^n a_i \Lambda_i$. These weights correspond to tableaux $T_\lambda \in B_{min} \cap B(s\Lambda_2) \setminus \iota_{s-1}^s(B((s-1)\Lambda_2))$. We will later recursively construct the tableaux corresponding to all other level s weights.

First, let $\lambda = k\Lambda_{n-1} + (s-k)\Lambda_n$. If s is even and $k \geq s/2$, the corresponding tableau is

$$T_\lambda = \underbrace{\begin{matrix} n-2 \\ n-1 \end{matrix} \cdots \begin{matrix} n-2 \\ n-1 \end{matrix}}_{s-k} \underbrace{\begin{matrix} n-1 \\ \bar{n} \end{matrix} \cdots \begin{matrix} n-1 \\ \bar{n} \end{matrix}}_{k-s/2} \underbrace{\begin{matrix} n \\ \overline{n-1} \end{matrix} \cdots \begin{matrix} n \\ \overline{n-1} \end{matrix}}_{k-s/2} \underbrace{\begin{matrix} \overline{n-2} \\ \overline{n-1} \end{matrix} \cdots \begin{matrix} \overline{n-2} \\ \overline{n-1} \end{matrix}}_{s-k}$$

If s is odd and $k \geq s/2$, we have

$$T_\lambda = \underbrace{\begin{matrix} n-2 \\ n-1 \end{matrix} \cdots \begin{matrix} n-2 \\ n-1 \end{matrix}}_{s-k} \underbrace{\begin{matrix} n-1 \\ \bar{n} \end{matrix} \cdots \begin{matrix} n-1 \\ \bar{n} \end{matrix}}_{k-s/2} \begin{matrix} n \\ \bar{n} \end{matrix} \underbrace{\begin{matrix} n \\ \overline{n-1} \end{matrix} \cdots \begin{matrix} n \\ \overline{n-1} \end{matrix}}_{k-s/2} \underbrace{\begin{matrix} \overline{n-2} \\ \overline{n-1} \end{matrix} \cdots \begin{matrix} \overline{n-2} \\ \overline{n-1} \end{matrix}}_{s-k}$$

In either case, if $k < s/2$, interchange n and $\bar{n}$ in T_λ.

Next, we describe how to construct T_λ recursively when $\lambda = \sum_{i=2}^n a_i \Lambda_i$ and $\langle \lambda, \Lambda_{n-1} + \Lambda_n \rangle < s$. Let j be the minimal index for which $\langle \lambda, \Lambda_j \rangle = k \neq 0$, let $\lambda' = \lambda - k\Lambda_j$, and let $T_{\lambda'}$ be the tableaux such that $\varepsilon(T_{\lambda'}) = \lambda'$. We then set

$$T_\lambda = \underbrace{\begin{matrix} j-1 \\ j \end{matrix} \cdots \begin{matrix} j-1 \\ j \end{matrix}}_{k} \boxed{T_{\lambda'}} \underbrace{\begin{matrix} \bar{j} \\ \overline{j-1} \end{matrix} \cdots \begin{matrix} \bar{j} \\ \overline{j-1} \end{matrix}}_{k},$$

which is simply the result of inserting $T_{\lambda'}$ between the two $2 \times k$ tableaux on either side.

We now consider level s weights λ such that $\langle \lambda, \Lambda_1 \rangle = k_1 \neq 0$ or $\langle \lambda, \Lambda_0 \rangle = k_0 \neq 0$ (or both). Let $\lambda' = \lambda - k_1\Lambda_1 - k_0\Lambda_0$, let $k_{\lambda'} = \langle c, \lambda' \rangle$, and once again, let $T_{\lambda'}$ be such that $\varepsilon(T_{\lambda'}) = \lambda'$. It follows that $T_{\lambda'}$ is a tableau of shape $(k_{\lambda'}, k_{\lambda'})$. If $k_1 \leq k_{\lambda'}$, then change $T_{\lambda'}$ into a skew tableau $S_{\lambda'}$ of shape $(k_{\lambda'} + k_1, k_{\lambda'})/(k_1)$ by Lecouvey D-equivalence [**11**], then fill the northwest boxes with 1's and the southeast boxes with $\bar{1}$'s to get a tableau of shape $(k_{\lambda'} + k_1, k_{\lambda'} + k_1)$. If $k_1 > k_{\lambda'}$, change $T_{\lambda'}$ into a skew tableau $S_{\lambda'}$ of shape $(2k_{\lambda'}, k_{\lambda'})/(k_{\lambda'})$ by Lecouvey D-equivalence, fill the northwest and southwest boxes as above, and insert a tableau of the form

$$\underbrace{\begin{matrix} 1 \\ \bar{2} \end{matrix} \cdots \begin{matrix} 1 \\ \bar{2} \end{matrix} \begin{matrix} 2 \\ \bar{1} \end{matrix} \cdots \begin{matrix} 2 \\ \bar{1} \end{matrix}}_{k_1 - k_{\lambda'}} \qquad \text{if } k_1 - k_{\lambda'} \text{ is even;}$$

$$\underbrace{\begin{matrix} 1 \\ \bar{2} \end{matrix} \cdots \begin{matrix} 1 \\ \bar{2} \end{matrix} \begin{matrix} 2 \\ \bar{2} \end{matrix} \begin{matrix} 2 \\ \bar{1} \end{matrix} \cdots \begin{matrix} 2 \\ \bar{1} \end{matrix}}_{k_1 - k_{\lambda'}} \qquad \text{if } k_1 - k_{\lambda'} \text{ is odd;}$$

between the first $k_{\lambda'}$ columns and the last $k_{\lambda'}$ columns to get a tableau $T_{\lambda''}$ of shape $(k_{\lambda'}+k_1, k_{\lambda'}+k_1)$. Observe that $\varepsilon(T_{\lambda''}) = \lambda'' = \lambda - k_0\Lambda_0$.

Finally, use the filling map of section 3 to add k_0 columns to $T_{\lambda''}$, yielding T_λ with $\varepsilon(T_\lambda) = \lambda$.

6. Perfectness of $\tilde{B}^{(2,s)}$, Part II

We must now show that the tableaux constructed in section 5 are in bijection with $(P_{cl}^+)_s$. Once again, for proofs of the following Lemmas, see [**13**].

LEMMA 6.1. *Let ι be the crystal endomorphism of $\tilde{B}^{(2,s)}$ defined by $\iota = \bigoplus_{i=0}^{s-1} \iota_i^{i+1}$, and let $T \in \tilde{B}^{(2,s)}$ be a tableau in the range of ι. Then $\varepsilon(\iota(T)) = \varepsilon(T) + \Lambda_1 - \Lambda_0$.*

This means that given a weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \Lambda'$, where $\langle \Lambda', \Lambda_0\rangle = \langle \Lambda', \Lambda_1\rangle = 0$, it suffices to search for tableaux which correspond to the weight Λ'. Furthermore, such a tableau will appear in the "new" part of $B(s\Lambda_2)$, where s is the level of Λ'. We may thus restrict our attention to tableaux $T \in B(s\Lambda_2) \setminus \iota_{s-1}^s(B((s-1)\Lambda_2))$.

LEMMA 6.2. *Let $v_\lambda \in \mathcal{BC}(\tilde{B}^{(2,s)})$ with complimentary vertex v'_λ. (Recall the definitions of the complimentary vertex of v_λ and $B(v_\lambda)$ from section 4.) If $B(v_\lambda)$ contains no minimal tableaux, then neither does $B(v'_\lambda)$.*

Therefore, we need only consider tableaux in the upper half (including the middle row) of the branching component graph.

LEMMA 6.3. *Let $k \geq s/2$. If T has k or more 1's in the first row and no $\bar{1}$'s, then T is not minimal.*

This eliminates many tableaux. In particular, in $\tilde{B}^{(2,2)}$, we only need to check the middle vertices of the branching component graph with shape $(2,2)$ and (2). Exhaustion shows the conjectured tableaux to be the only tableaux of level 2 in those sets.

7. Construction of $W^{(2,s)}$

In [**9**], Kang et al. discuss the relationship between an arbitrary finite-dimensional $U'_q(\mathfrak{g})$-module M (where $\mathfrak{g}$ is of affine type) and $\mathrm{Aff}(M)$, the infinite-dimensional $U_q(\mathfrak{g})$-module constructed by "affinizing" M. Loosely speaking, $\mathrm{Aff}(M) \simeq \bigoplus_{n\in\mathbb{Z}} T^n M$, where e_0 and f_0 respectively raise and lower the degree of T in addition to their ordinary action on M. To make the weight spaces of $\mathrm{Aff}(M)$ finite-dimensional, we add $n\delta$ to the weight of a vector in T^nM, where δ is the null root of $\mathfrak{g}$. Kang et al. also construct $\mathrm{Aff}(B)$ for any $U'_q(\mathfrak{g})$-crystal B, and state that if (L,B) is a crystal base of M, then $(\mathrm{Aff}(L), \mathrm{Aff}(B))$ is a crystal base of $\mathrm{Aff}(M)$.

The inverse of this process for level zero extremal weight modules generated by a basic weight vector is given in [**6**] as follows: given a fundamental infinite-dimensional $U_q(\mathfrak{g})$-module $V(\varpi_i)$, there is a $U'_q(\mathfrak{g})$-linear automorphism z_i of $V(\varpi_i)$ of weight $d_i\delta$, where d_i is an integer constant determined by the root system of $\mathfrak{g}$. The finite-dimensional $U'_q(\mathfrak{g})$-module $W(\varpi_i)$ is given by $W(\varpi_i) = V(\varpi_i)/(z_i-1)V(\varpi_i)$, and $V(\varpi_i)$ can be naturally embedded in $\mathrm{Aff}(W(\varpi_i))$.

Later, Kashiwara also conjecturally gives an embedding for $V(\lambda) \subset \bigotimes V(\varpi_i)^{\otimes m_i}$, where $\lambda = \sum m_i\varpi_i$ is a level zero extremal weight. This conjecture is verified in [**1**] for symmetric untwisted affine Lie algebras, using Schur functions in the operators $z_{i,\nu}$, which correspond to z_i as above acting on the i,ν-th component of the tensor product.

We conjecture that the $U'_q(\widehat{\mathfrak{so}}_{2n})$-module $W^{(2,s)} = W(s\varpi_2)$ can be constructed as the quotient $V(s\varpi_2)/(z_{2s}-1)V(s\varpi_2)$, where z_{2s} is the $U'_q(\widehat{\mathfrak{so}}_{2n})$-linear automorphism of $V(s\varpi_2)$ of weight $2s\delta$. Such a construction would be compatible with $B^{(2,s)}$ as constructed here, and would give an embedding of $V(s\varpi_2)$ in $\mathrm{Aff}(W(s\varpi_2))$ similar to the embedding in [**6**] for fundamental representations.

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