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On Inversions in Standard Young Tableaux

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Abstract. In this work, we present the inversion number of a standard Young tableau, and determine its distribution over certain sets of standard Young tableaux. Specifically, the work determines the distribution of the inversion number over hook-shaped tableaux and over tableaux of shape (n, n). We also study the parity (also referred to as 'sign balance') of the inversion number over hook-shaped tableaux and over $(n-k, k)$-shaped tableaux. The latter results resemble results in the field of pattern-avoiding permutations, achieved by Adin, Roichman and Reifegerste.

1. Preliminaries

Definition 1.1. Let $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers). A *partition* of n is a vector of positive integer numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We write $\lambda \vdash n$. We denote by $\lambda' = (\lambda'_1, \ldots, \lambda'_t)$ the *conjugate* partition, where λ'_i is the number of parts in λ greater or equal to i. We define $\mid \lambda \mid = n$.

Definition 1.2. The set $\{(i,j) \mid i,j \in \mathbb{N},\ i \leq k,\ j \leq \lambda_i\}$ is called the *Young diagram* of shape λ (notice that 'English notation' is used).

Definition 1.3. A *standard Young tableau* of shape λ consists of inserting the integers $1, 2, \ldots, n$ as *entries* in the cells of the Young diagram of λ, allowing no repetitions and having entries increase along rows and columns. λ is normally denoted $Sh(T)$.

Definition 1.4. A *descent* in a standard Young tableau T, is an entry i, such that i+1 is strictly south (and weakly west) of i. Denote the set of all descents in T by $D(T)$. We define two statistics on a standard Young tableau:

1. The *descent number* of T. $des(T) = \sum_{i \in D(T)} 1$.
2. The *major index* of T. $maj(T) = \sum_{i \in D(T)} i$.

Stanley has found a generalization of the hook formula, giving the generating function for $maj(T)$ when T is of shape λ.

Theorem 1.5 (Stanley's q-analogue of the hook formula, see [**ST2**]).

$$\sum_{shape(T)=\lambda} q^{maj(T)} = \frac{\prod_{k=1}^{n} [k]_q}{\prod_{(i,j)\in\lambda} [h_{i,j}]_q} \tag{1.1}$$

where $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$.

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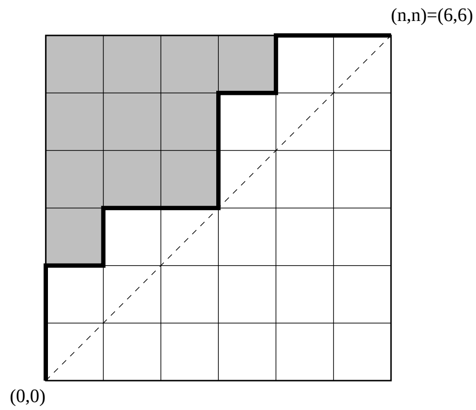

FIGURE 1. A Dyck path. The area of this Dyck path is shaded in gray.

Seeing how natural the generating function of the major index turns out to be, it is surprising that there is no similar result for the descent number. However, Adin and Roichman ina joint study, and Hästö in a parallel study, managed to establish the expected value and variance of $des(T)$ for a random standard Young tableau of a given shape (see [**AR1, H**]).

One can also think of defining the inversion number of a tableau. Not much is known regarding the distribution of the inversion number over tableaux of a fixed shape, and that is in fact the primary goal of this research.

REMARK 1.6. The convention within this paper is that $\binom{n}{k} = 0$ when $k < 0$.

DEFINITION 1.7. A *lattice path* in the plane is defined to be a sequence $L = (v_1, \ldots v_k)$ where $v_i \in \mathbb{N}^2$ and $v_{i+1} - v_i \in \{(1,0), (0,1)\}$. The last condition indicates that when moving from v_i to v_{i+1}, we move either one unit north, or one unit east.

DEFINITION 1.8. A *Dyck path* of order n is a lattice paths starting at $(0,0)$ and ending at (n,n), which always remain above or on the line $x = y$. A Dyck path can be encoded by a sequence $(a_1, \ldots a_{2n})$ where $a_i \in \{1, -1\}$ with $a_i = 1$ indicating a north move at the i-th step, and $a_i = -1$ indicating an east move at the i-th step.

The *area above a dyck path* D (denoted: $area(D)$) is the area between D and the dyck path encoded by $\{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{-1, -1, \ldots, -1}_{n}\}$.

EXAMPLE 1.9. The Dyck path corresponding to the series $\{1, 1, -1, 1, -1, -1, 1, 1, -1, 1, -1, -1\}$ is drawn in figure 1 with its area shaded in gray.

Recall that the number of Dyck paths of order n is called the *n-th Catalan number*, and is denoted C_n.

Recall the following well-known corollary of the q-binomial theorem (see [**GR**, page 7]):

THEOREM 1.10. *(The Cauchy binomial theorem*[**GR**, page 7]*)*

$$\prod_{k=1}^{n} (1 + yq^k) = \sum_{m=0}^{n} y^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q$$

DEFINITION 1.11. The n-th Carlitz-Riordan q-Catalan number is defined as follows: $C_n(q) = \sum_{D \in Dyck(n)} q^{area(D)}$, where $Dyck(n)$ is the set of all Dyck paths of order n.

This q-Catalan number was studied by Carlitz and Riordan (see [**C, CR**]), and further studied by Fürlinger and Hofbauer in 1985 (see [**FH**], which also includes further references within). There is no known generating function for this q-Catalan number, however Fürlinger and Hofbauer expressed it as a term within a generating function, and several determinant formulas were provided, the most recent one by Loehr (see [**L**, Theorem 16]).

LEMMA 1.12. [**FH**, Eq. 2.2] *The Carlitz-Riordan q-Catalan numbers abide to the recursion:*

$$C_{n+1}(q) = \sum_{k=0}^{n} C_k(q)C_{n-k}(q) \cdot q^{(k+1)(n-k)}$$

with starting condition $C_0(q) = 1$.

REMARK 1.13. Some authors define the Carlitz-Riordan q-Catalan as $\tilde{C}_n(q) = q^{\binom{n}{2}}C_n(q)$. These numbers describe the distribution of the area between Dyck paths of order n, and the "diagonal Dyck path" $(1, -1, 1, -1, 1, -1, \dots, 1, -1)$.

We cite the following "common knowledge" result:

LEMMA 1.14. *For any two positive integers $k \le n$,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=-1} \begin{cases} 0 & \begin{matrix} n\, even \\ k\, odd \end{matrix} \\ \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & otherwise \end{cases}$$

COROLLARY 1.15.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q=-1} q^k = (1+q)\sum_{k=0}^{n} \binom{n}{k} q^{2\cdot k}$$

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_{q=-1} q^k = \sum_{k=0}^{n} \binom{n}{k} q^{2\cdot k}$$

2. Inversions in Tableaux and Signs of Tableaux

This chapter presents the most fundamental concept of the work.

As we saw in definition , there is a meaningful way to define the descent set of a tableau. The definitions of the descent number and the major index follow naturally. The following definition of an inversion in a standard Young tableau is natural as an extension of the descent definition. It is a variant of the definition given by Stanley (see [**ST3**, page 15]).

DEFINITION 2.1. An *inversion* in a standard Young tableau is a pair (i, j) such that $i < j$ and the entry for j appears strictly south and strictly west of the entry for i. The *inversion number* of a standard Young tableau T (denoted: $inv(T)$) is the number of inversions in this standard Young tableau.

DEFINITION 2.2. A *weak* inversion in a standard Young tableau T is a pair of integers (i, j) such that $i < j$ and j is weakly south and weakly west of i. The number of weak inversions in T is called the *weak inversion number* of T and denoted $winv(T)$.

There is a simple connection between the inversion and the weak inversion numbers: Let T be a standard Young tableau with $sh(T) = \lambda = (\lambda_1, \dots, \lambda_k)$, and denote $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$ to be the conjugate partition, then $winv(T) = inv(T) + \sum_{i=1}^{\lambda_1} \binom{\lambda'_i}{2}$.

DEFINITION 2.3. Let T be a standard Young tableau. The *sign* of T is defined: $sign(T) = (-1)^{inv(T)}$.

3. Hook Shaped Tableaux

DEFINITION 3.1. A *hook-shaped tableaux* is a tableaux with one row and one column. Alternatively, it is a tableaux T with shape $\lambda = (k, 1, 1, \ldots, 1)$ with $k \geq 1$. The *column length* of T (denoted $col(T)$) is defined as $\lambda_1' - 1$, or equivalently, the number of parts in λ, reducing 1. The *row length* of T is defined as $\lambda_1 - 1$.

DEFINITION 3.2. Write $sh(T) \in hook(n)$ if T is a hook-shaped standard Young tableau of order n. Write $sh(T) \in hook(n, k)$ if T is a hook-shaped standard Young tableau of order n with column length k.

LEMMA 3.3.

$$F_{n,k}(q) = \sum_{sh(T) \in hook(n+1,k)} q^{inv(T)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{3.1}$$

PROOF. This proposition may be proved using the recursion $F_{n,k}(q) = F_{n-1,k}(q) + q^{n-k} F_{n-1,k-1}(q)$. □

Using Cauchy's binomial theorem (Theorem 1.10) we deduce that

$$\sum_{sh(T) \in hook(n+1)} q^{winv(T)} = \prod_{k=1}^{n} (1 + q^k)$$

for a detailed proof see [**SH**].

4. Tableaux of Two Rows

4.1. Counting Inversions.

DEFINITION 4.1. Let T be a standard Young tableau. If $sh(T) = (n-k, k)$ with $n-k, k \geq 0$, we say T is a *two-rowed tableau*, and write $T \in tworows(n)$. If $n - k = k$ we say T is *equal-rowed*.

LEMMA 4.2. *Let $(x_1, x_2, \ldots, x_{2n})$ be an encoding of a Dyck path (see definition 1.8). In each Dyck path of order n there are exactly n 1's, call them $x_{a_1}, \ldots, x_{a_n}$. Then $area(D) = \sum_{i=1}^{n} (a_i - i)$.*

The proof of this proposition is left to the reader.

THEOREM 4.3. *Recall the definition of $\tilde{C}_n(q)$ in note 1.13.*

$$\sum_{sh(T)=(n,n)} q^{inv(T)} = \tilde{C}_n(q) \tag{4.1}$$

PROOF. There is a well known bijection between Dyck paths of order n, and standard Young tableaux of shape (n, n): Take a standard Young tableaux T of shape (n, n). The corresponding Dyck path encoding is given by $a_i = 1$ if the entry i lies within the first row of T, and $a_i = -1$ if the entry i lies within the second row of T.

Now, observe that the entry values in T are uniquely determined by choosing the entries in the first row, since there is only one unique way to arrange the remainder "unused" entries in the second row. Moreover, the sum of entries in the first row uniquely determines the number of inversions.

To prove this, write:

$$T = \begin{matrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{matrix}$$

Notice that from the definition of an inversion, it must follow that any two entries i, j creating an inversion, must reside in two different rows. Thus, to calculate the number of inversions for our given tableau, it is sufficient to determine the number of inversions involving one element a_i and one element b_j $(j < i)$. Thus, we need to determine the number of elements $b_j < a_i$ with $j < i$. We know there are $a_i - 1$ values smaller than a_i, and $i - 1$ of them are in the first row (all the entries to before the i-th entry), so there are $a_i - i$ entries smaller than a_i in the second row, and they all must appear in smaller column indices than i. That leaves room for exactly $i - 1 - (a_i - i) = 2i - a_i - 1$ entries larger than a_i in the second row, with a smaller

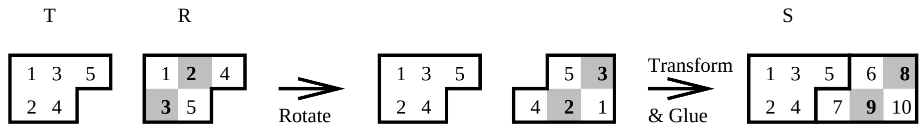

FIGURE 2. Gluing together equal-shaped tableaux. Notice that all inversions in the tableau R are preserved. The entry couple $(2, 3)$ is an inversion. It is highlighted throughout the process. At the end it corresponds with the entry couple $(8, 9)$ which is an inversion in S.

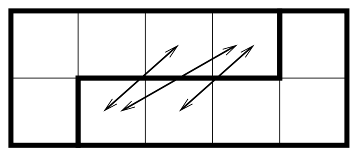

FIGURE 3. In this illustration, there are exactly 3 inversions involving exactly one entry from each of the two merged tableaux. The specific values within the tableaux do not make any difference here.

column index, and hence that is also the number of inversions in which a_i participates. The number of total inversions in the tableau would be $\sum_{i=1}^{n}(2i - a_i - 1) = \binom{n+1}{2} - n + \sum_{i=1}^{n}(i - a_i)$. By proposition 4.2 we get $\sum_{i=1}^{n}(i - a_i) = -area(D)$. Thus, $\sum_{sh(T)=(k,k)} q^{inv(T)} = q^{\binom{n+1}{2}-n} C_n(\frac{1}{q}) = q^{\binom{n}{2}} C_n(\frac{1}{q}) = \tilde{C}_n(q)$. □

COROLLARY 4.4. *Let* $G_{n-k,k}(q) = \sum_{sh(T)=(n-k,k)} q^{inv(T)}$. *Then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{n-2k}{2}} G_{n-k,k}(q)^2 = \tilde{C}_n(q)$$

PROOF. Let T and R be tableaux of shape $(n - k, k)$ $(0 \leq 2k \leq n$ with $n > 0)$. Any two tableaux of the same shape may be "glued" together in a certain fashion we shall describe, to obtain a standard Young tableau of shape (n, n), which we denote S. From there we use 4.1 to conclude the result.

Let T, R be standard Young tableaux of shape $(n - k, k)$. Take the second row of R. Reverse the order of elements in it, and then replace each element a by $2n - a - 1$. Add the result to the end of T's first row. This is the first row of S. The second row is acquired from applying the same transformation on R's first row, and adding it to T's second row. It is required to verify that S is indeed a standard Young tableau, which is left as an exercise for the reader. Notice that all elements originating from R are bigger than all elements in T. See Figure 2 for an illustration of the process.

Now we look at the relation between inversions of T and R, and those of S. First notice that for any two entries $i < j$ in R, with j strictly southwest of i the corresponding entries in S would be $2n-i-1 > 2n-j-1$ and $2n - i - 1$would be strictly southwest of $2n - j - 1$. Thus, these entries would form an inversion in S. All inversions in T are obviously preserved in S. Furthermore: any inversion in S not derived from R or T would have to consist of one element from T and one element from R (else the inversion would have to appear either in R or S). The only inversions in S consisting of one element from T and one element from R could be found in the "middle region" of S, i.e. in column indices ranging from $k + 1$ to $n - k$. All elements in R are bigger than those in T, thus we can calculate exactly this number of inversions: it is the number of matchings of elements from the first row within this "middle region", and elements of the second row, also in this "middle region". This gives us exactly $\binom{n-2k}{2}$ extra inversions. See Figure 3 for an example.

This transformation is a bijection from standard Young tableaux of shapes $(n-k,k)$ with $0 \le k \le \lfloor \frac{n}{2} \rfloor$ to (n,n)-shaped tableaux. Thus, the inversions over $q^{\binom{n-2k}{2}} G_{n-k,k}(q)^2$ distribute exactly as they do over (n,n)-shaped tableaux. □

4.2. Sign Balance. When addressing standard Young tableaux of shape $(n-k,k)$, we can give an explicit formula for the sign distribution.

DEFINITION 4.5. $row_2(T)$ will denote the length of the second row of T. i.e. if T is of shape $(n-k,k)$ then $row_2(T) = k$.

THEOREM 4.6. *Recall that we defined* $\binom{n}{k} = 0$ *whenever* k *is negative. Then:*

$$\sum_{T \in tworows(2n+1)} sign(T) q^{row_2(T)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n}{k} - \binom{n}{k-1} \right] q^{2k}$$

$$\sum_{T \in tworows(2n)} sign(T) q^{row_2(T)} = (1+q) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-1}{k} - \binom{n-1}{k-1} \right] q^{2k}$$

PROOF. Denote $Sum(n,k) = \sum_{sh(T)=(n-k,k)} sign(T)$. Then it is sufficient to prove that for $0 \le 2k \le n$:

(1) $Sum(2n+1, 2k) = (-1)^k \left[\binom{n}{k} - \binom{n}{k-1} \right]$.
(2) $Sum(2n+1, 2k+1) = 0$. $(2k \ne n)$
(3) $Sum(2n, 2k) = (-1)^k \left[\binom{n-1}{k} - \binom{n-1}{k-1} \right]$.
(4) $Sum(2n, 2k+1) = (-1)^k \left[\binom{n-1}{k} - \binom{n-1}{k-1} \right]$. $(2k \ne n)$

The proof is done by induction on n. It is clear that $Sum(1,0) = 1$. We give the induction step for the first case. The other three cases are very similar.

Take a tableau of shape $(2n+1-2k, 2k)$ with $n \ge 2k > 0$. Each such tableaux is uniquely achieved either by a $(2n+1-2k, 2k-1)$-shaped tableaux with the entry $2n+1$ added to its second row, or by a $(2n-2k, 2k)$-shaped tableaux with the entry $2n+1$ added to its first row. If the entry $2n+1$ resides in the first row, it would participate in no inversions, and thus its removal would not alter the sign. If it resides in the second row, it would participate in $2n+1-4k$ inversions (the number of elements in the first row with greater column index), and its removal would flip the sign. Thus,

$$\begin{aligned}
Sum(2n+1, 2k) &= Sum(2n, 2k) - Sum(2n, 2(k-1)+1) = \\
&= (-1)^k \binom{n-1}{k} - \binom{n-1}{k-1} + (-1)^k \binom{n-1}{k-1} - \binom{n-1}{k-2} = \\
&= (-1)^k \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] - (-1)^k \left[\binom{n-1}{k-1} + \binom{n-1}{k-2} \right] = \\
&= (-1)^k \left[\binom{n}{k} - \binom{n}{k-1} \right]
\end{aligned}$$

Notice that this result would be true also for $k = 0$ since then we would look only at $Sum(2n, 2 \cdot 0) = (-1)^0 \left[\binom{n}{0} - \binom{n}{0-1} \right] = \binom{n}{0}$, which is also what we would get by substituting $k = 0$ in the formula for $Sum(2n+1, 0)$. □

5. The "$\frac{n}{2}$ Phenomenon"

The following results are corollaries of previous theorems in this work.

THEOREM 5.1. *Let* $sh(T)' = (\lambda'_1, \ldots, \lambda'_l)$*. Denote* $col(T) = \lambda'_1 - 1$*. Then*

$$\sum_{T \in hook(2n-1)} sign(T) q^{col(T)} = \sum_{T \in hook(n)} q^{2 \cdot col(T)}$$

$$\sum_{T \in hook(2n)} sign(T) q^{col(T)} = (1+q) \sum_{T \in hook(n)} q^{2 \cdot col(T)}$$

THEOREM 5.2.

$$\sum_{T \in tworows(2n+1)} sign(T) q^{row_2(T)} = \sum_{T \in tworows(n)} (-q^2)^{row_2(T)}$$

$$\sum_{T \in tworows(2n+2)} sign(T) q^{row_2(T)} = (1+q) \sum_{T \in tworows(n)} (-q^2)^{row_2(T)}$$

REMARK 5.3. As a special case of Theorem 5.2, we see that for the Carlitz-Riordan q-Catalan numbers:

$$\sum_{n=1}^{\infty} q^n \cdot \tilde{C}_n(-1) = \sum_{n=1}^{\infty} q^{2n+1} \cdot \tilde{C}_n$$

These results resemble recent results of Adin and Roichman (see [**AR2**]) and Reifegerste (see [**R**]) regarding 321-avoiding permutations, which are brought hereby.

DEFINITION 5.4. Let $T_n := \{\pi \in S_n \mid \not\exists i < j < k \text{ such that } \pi(i) > \pi(j) > \pi(k)\}$ be the set of all 321-avoiding permutations. Define $ldes(\pi) := \max\{1 \leq i \leq n-1 \mid \pi(i) > \pi(i+1)\}$ and define $ldes(id) = 0$.

THEOREM 5.5. [**AR2**]

$$\sum_{\pi \in T_{2n+1}} sign(\pi) \cdot q^{ldes(\pi)} = \sum_{\pi \in T_n} q^{2 \cdot ldes(\pi)}$$

$$\sum_{\pi \in T_{2n}} sign(\pi) \cdot q^{ldes(\pi)} = (1-q) \sum_{\pi \in T_n} q^{2 \cdot ldes(\pi)}$$

DEFINITION 5.6. Define $lis(\pi)$ as the longest increasing subsequence in π.

THEOREM 5.7. [**R**]

$$\sum_{\pi \in T_{2n+1}} sign(\pi) \cdot q^{lis(\pi)} = \sum_{\pi \in T_n} q^{2 \cdot lis(\pi)+1}$$

$$\sum_{\pi \in T_{2n+2}} sign(\pi) \cdot q^{lis(\pi)} = (q-1) \sum_{\pi \in T_n} q^{2 \cdot lis(\pi)+1}$$

THEOREM 5.8. [**R**]

$$\sum_{\pi \in T^*_{2n+1}} sign(\pi) \cdot q^{lis(\pi)} t^{ldes(\pi)} = \sum_{\pi \in T_n} q^{2 \cdot lis(\pi)+1} t^{2 \cdot ldes(\pi)}$$

$$(1+q) \sum_{\pi \in T^*_{2n}} sign(\pi) q^{lis(\pi)} t^{ldes(\pi)} = \sum_{\pi \in T_n} q^{2 \cdot lis(\pi)+1} t^{2 \cdot ldes(\pi)}$$

A fuller understanding of such results would require additional research.

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