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A Polynomiality Property for Littlewood-Richardson Coefficients

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ABSTRACT. We present a polynomiality property of the Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$. The coefficients are shown to be given by polynomials in λ, μ and ν on the cones of the chamber complex of a vector partition function. We give bounds on the degree of the polynomials depending on the maximum allowed number of parts of the partitions λ, μ and ν. We first express the Littlewood-Richardson coefficients as a vector partition function. We then define a hyperplane arrangement from Steinberg's formula, over whose regions the Littlewood-Richardson coefficients are given by polynomials, and relate this arrangement to the chamber complex of the partition function. As an easy consequence, we get a new proof of the fact that $c_{N\lambda\,N\mu}^{N\nu}$ is given by a polynomial in N, which partially establishes the conjecture of King, Tollu and Toumazet [**KTT03**] that $c_{N\lambda\,N\mu}^{N\nu}$ is a polynomial in N with nonnegative rational coefficients.

RÉSUMÉ. Nous présentons une propriété de polynomialité des coefficients de Littlewood-Richardson $c_{\lambda\mu}^{\nu}$. Nous démontrons que ces coefficients sont donnés par des fonctions polynomiales en λ, μ et ν dans les cônes du complexe d'une fonction de partition vectorielle. Nous donnons des bornes sur les degrés de ces polynômes en termes du nombre de parts des partitions λ, μ and ν. Nous exprimons premièrement les coefficients de Littlewood-Richardson en termes d'une fonction de partition vectorielle. Nous définissons ensuite un arrangement d'hyperplans à partir de la formule de Steinberg, sur les régions duquel les coefficients de Littlewood-Richardson sont donnés par des polynômes, puis faisons le lien entre cet arrangement et le complexe de cônes de la fonction de partition vectorielle. Comme conséquence simple, nous obtenons une preuve élémentaire du fait que $c_{N\lambda\,N\mu}^{N\nu}$ est donné par un polynôme en N, ce qui établit partiellement une conjecture de King, Tollu et Toumazet [**KTT03**], voulant que $c_{N\lambda\,N\mu}^{N\nu}$ soit un polynôme en N avec des coefficients rationnels nonnégatifs.

1. Introduction

Littlewood-Richardson coefficients appear in many fields of mathematics. In combinatorics, they appear in the theory of symmetric functions (see [**Mac95, Sta99**]). The Schur symmetric functions form a linear basis of the ring of symmetric functions, and the Littlewood-Richardson coefficients express the multiplication rule,

$$s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu\,, \tag{1.1}$$

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as well as how to write skew Schur functions in terms of the Schur function basis:

$$s_{\nu/\lambda} = \sum_{\mu} c^{\nu}_{\lambda\mu} s_{\mu} \,. \tag{1.2}$$

In the representation theory of the general and special linear groups, the characters of the irreducible polynomial representations of $\mathrm{GL}_k\mathbb{C}$ are Schur functions in appropriate variables [**FH91, Mac95**]. As such, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ gives the multiplicity with which the irreducible representation V_ν of $\mathrm{GL}_k\mathbb{C}$ appears in the tensor product of the irreducible representations V_λ and V_μ:

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} c^{\nu}_{\lambda\mu} V_\nu \,. \tag{1.3}$$

Littlewood-Richardson coefficients also appear in algebraic geometry: Schubert classes form a linear basis of the cohomology ring of the Grassmannian, and the Littlewood-Richardson coefficients again express the multiplication rule [**Ful97**]:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} \sigma_\nu \,. \tag{1.4}$$

In previous work with Billey and Guillemin [**BGR03**], we studied the Kostka numbers $K_{\lambda\mu}$, which appear when expressing the Schur function s_λ in terms of the monomial symmetric functions: $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$. Kostka numbers also give the weight multiplicities in the weight space decomposition $V_\lambda = \bigoplus_\mu \left(V_\lambda\right)_\mu$ of the irreducible representation V_λ of $\mathfrak{sl}_k\mathbb{C}$:

$$K_{\lambda\mu} = \dim \left(V_\lambda\right)_\mu \,. \tag{1.5}$$

We showed there that the Kostka numbers are given by a vector partition function and that this implies that the function $(\lambda,\mu) \longmapsto K_{\lambda\mu}$ is quasipolynomial in the cones of a chamber complex. We then defined a hyperplane arrangement, the Kostant arrangement, over whose regions this function was given by a polynomial. This allowed us to prove that the quasipolynomials in the cones were actually polynomials. As a corollary, we obtained an alternative proof to that of Kirillov that the function $N \mapsto K_{N\lambda\, N\mu}$ is a polynomial in N for every fixed λ and μ.

In [**KTT03**], King, Tollu and Toumazet conjecture that the Littlewood-Richardson coefficients exhibit a similar "stretching" property:

Conjecture 1.1. **(King, Tollu, Toumazet [KTT03])** *For all partitions λ, μ and ν such that $c^{\nu}_{\lambda\mu} > 0$ there exists a polynomial $P^{\nu}_{\lambda\mu}(N)$ in N with nonnegative rational coefficients such that $P^{\nu}_{\lambda\mu}(0) = 1$ and $P^{\nu}_{\lambda\mu}(N) = c^{N\nu}_{N\lambda\, N\mu}$ for all positive integers N.*

In [**DW02**], Derksen and Weyman prove the polynomiality part of this conjecture using semi-invariants of quivers. They call the functions $P^{\nu}_{\lambda\mu}(N)$ (for fixed λ, μ and ν), *Littlewood-Richardson polynomials*.

Here we extend the results of [**BGR03**] to the case of Littlewood-Richardson coefficients. We first express Littlewood-Richardson coefficients as a vector partition function (Theorem 2.3). This is done using a combinatorial model (the hive model [**Buc00, KT99**]) for computing the Littlewood-Richardson coefficients. This means that these coefficients are quasipolynomial in λ, μ and ν over the conical cells of a chamber complex $\mathcal{LR}_k$.

From Steinberg's formula [**Ste61**], giving the multiplicities with which irreducible representations appear in the decomposition into irreducibles of the tensor product of two irreducible representations of a complex semisimple Lie algebra, we then define a hyperplane arrangement, the Steinberg arrangement $\mathcal{SA}_k$. We show that the Littlewood-Richardson coefficients are given by a polynomial over the regions of this arrangement (Proposition 3.3).

Finally, by comparing the chamber complex $\mathcal{LR}_k$ with the Steinberg arrangement $\mathcal{SA}_k$, we are able to show that the quasipolynomials in the cones of $\mathcal{LR}_k$ are actually polynomials in λ, μ and ν, and we provide degree bounds (Theorem 4.1). Because we are working in cones, this provides an alternative proof to that of [**DW02**] of the polynomiality part of the conjecture of King, Tollu and Toumazet; we don't know whether the polynomials $P^{\nu}_{\lambda\mu}$ have nonegative coefficients or not. However, we get global polynomiality results in a chamber complex instead of polynomiality on fixed rays. We understand that Knutson [**Knu03**] also proved polynomiality in cones using symplectic geometry techniques.

1.1. Type A root systems and Littlewood-Richardson coefficients.

The simple Lie algebra $\mathfrak{sl}_k\mathbb{C}$ (of type A_{k-1}) is the subalgebra of $\mathfrak{gl}_k\mathbb{C} \cong \mathrm{End}(\mathbb{C}^k)$ consisting of traceless $k \times k$ matrices over $\mathbb{C}$. We will take as its Cartan subalgebra $\mathfrak{h}$ its subspace of traceless diagonal matrices. The roots and weights live in the dual $\mathfrak{h}^*$ of $\mathfrak{h}$, which can be identified with the subspace $x_1 + \cdots + x_k = 0$ of $\mathbb{R}^k$. The roots are $\{e_i - e_j \ : \ 1 \le i \ne j \le k\}$, and we will choose the positive ones to be $\Delta_+ = \{e_i - e_j \ : \ 1 \le i < j \le k\}$. The simple roots are then $\alpha_i = e_i - e_{i+1}$, for $1 \le i \le k-1$, and for these simple roots, the fundamental weights are

$$\omega_i = \frac{1}{k}(\underbrace{k-i, k-i, \ldots, k-i}_{i \text{ times}}, \underbrace{-i, -i, \ldots, -i}_{k-i \text{ times}}), \qquad 1 \le i \le k-1 . \tag{1.6}$$

The fundamental weights are defined such that $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ is the usual dot product. The integral span of the simple roots and the fundamental weights are the root lattice Λ_R and the weight lattice Λ_W respectively. The root lattice is a finite index sublattice of the weight lattice, with index $k-1$.

For our choice of positive roots, $\delta = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha = \sum_{j=1}^{k-1} \omega_j = \frac{1}{2}(k-1, k-3, \ldots, -(k-3), -(k-1))$. The Weyl group for $\mathfrak{sl}_k\mathbb{C}$ is the symmetric group $\mathfrak{S}_k$ acting on $\{e_1, \ldots, e_k\}$ (i.e. $\sigma(e_i) = e_{\sigma(i)}$), and with the choice of positive roots we made, the fundamental Weyl chamber will be $C_0 = \{(\lambda_1, \ldots, \lambda_k) \ : \ \sum_{i=1}^k \lambda_i = 0 \text{ and } \lambda_1 \ge \cdots \ge \lambda_k\}$. The action of the Weyl group preserves the root and weight lattices. Weights lying in the fundamental Weyl chamber are called *dominant*, and we will call elements of the Weyl orbits of the fundamentals weights *conjugates of fundamental weights.*

The finite dimensional representations of $\mathfrak{sl}_k\mathbb{C}$, or $\mathrm{SL}_k\mathbb{C}$, are indexed by the dominant weights $\Lambda_W \cap C_0$, and for a given dominant weight λ, there is a unique irreducible representation $\rho_\lambda : \mathfrak{sl}_k\mathbb{C} \to \mathfrak{gl}(V_\lambda)$ with highest weight λ, up to isomorphism. The finite dimensional polynomial representations of $\mathfrak{gl}_k\mathbb{C}$, or $\mathrm{GL}_k\mathbb{C}$, are indexed by partitions with at most k parts, that is by sequences $(\lambda_1, \ldots, \lambda_k)$ of integers satisfying $\lambda_1 \ge \cdots \ge \lambda_k \ge 0$. Two irreducible representations V_λ and V_μ of $\mathfrak{gl}_k\mathbb{C}$ restrict to the same irreducible representation of $\mathfrak{sl}_k\mathbb{C}$ if $\lambda_i - \mu_i$ is some constant independent of i for all i. So the irreducible representations of $\mathfrak{sl}_k\mathbb{C}$ correspond to equivalence classes of irreducible representations of $\mathfrak{gl}_k\mathbb{C}$. Consider the map $\lambda \mapsto \bar{\lambda}$ given by

$$(\lambda_1, \ldots, \lambda_k) \quad \longmapsto \quad (\lambda_1, \ldots, \lambda_k) - \frac{\sum \lambda_i}{k} \underbrace{(1, 1, \ldots, 1)}_{k \text{ times}} . \tag{1.7}$$

Then the representations V_λ of $\mathfrak{gl}_k\mathbb{C}$ restricts to the irreducible representation $V_{\bar{\lambda}}$ of $\mathfrak{sl}_k\mathbb{C}$. Details about the construction of the irreducible representations of $\mathrm{SL}_k\mathbb{C}$ and $\mathrm{GL}_k\mathbb{C}$ are well-known and can be found in [**Ful97**] or [**FH91**], for example. We will denote by $|\lambda|$ the sum $\sum \lambda_i$ (so λ is a partition of the integer $|\lambda|$). We will also let $l(\lambda)$ denote the number of nonzero parts of λ.

Given two irreducible representations V_λ and V_μ of $\mathrm{GL}_k\mathbb{C}$, their tensor product $V_\lambda \otimes V_\mu$ is again a representation of $\mathrm{GL}_k\mathbb{C}$, and we can decompose it in terms of irreducibles of $\mathrm{GL}_k\mathbb{C}$:

$$V_\lambda \otimes V_\mu = \bigoplus_\nu c_{\lambda\mu}^\nu V_\nu \,, \tag{1.8}$$

where $c_{\lambda\mu}^\nu V_\nu = V_\nu^{\oplus c_{\lambda\mu}^\nu}$, for some nonnegative integer numbers $c_{\lambda\mu}^\nu$, called the *Littlewood-Richardson coefficients*. The direct sum ranges over all partitions ν, but $c_{\lambda\mu}^\nu = 0$ unless $|\lambda| + |\mu| = |\nu|$ and λ and μ are contained in ν. We have a similar decomposition for the tensor product of two irreducible representations of $\mathfrak{sl}_k\mathbb{C}$:

$$V_{\bar\lambda} \otimes V_{\bar\mu} = \bigoplus_{\bar\nu} m_{\bar\lambda\bar\mu}^{\bar\nu} V_{\bar\nu} \,, \tag{1.9}$$

for nonnegative integers $m_{\bar\lambda\bar\mu}^{\bar\nu}$, where the sum ranges over all dominant weights $\bar\nu \in C_0$.

There is a general formula due to Steinberg [**Hum72, Ste61**] giving the multiplicity with which an irreducible representation V_ν occurs in the tensor product of two irreducible representations V_λ and V_μ of a complex semisimple Lie algebra. This will give us a way of computing the $m_{\bar\lambda\bar\mu}^{\bar\nu}$, and also the $c_{\lambda\mu}^\nu$, but first we have to define the Kostant partitition function.

DEFINITION 1.2. The *Kostant partition function* for a root system Δ, given a choice of positive roots Δ_+, is the function

$$K(v) = \left|\left\{(k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{N}^{|\Delta_+|} \;:\; \sum_{\alpha\in\Delta_+} k_\alpha \alpha = v\right\}\right| , \tag{1.10}$$

i.e. $K(v)$ is the number of ways that v can be written as a sum of positive roots.

THEOREM 1.3. **(Steinberg [Ste61])**

$$m_{\bar\lambda\bar\mu}^{\bar\nu} = \sum_{\sigma\in\mathfrak{S}_k}\sum_{\tau\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma\tau)} K(\sigma(\bar\lambda+\delta) + \tau(\bar\mu+\delta) - (\bar\nu+2\delta)) \,, \tag{1.11}$$

where $\mathrm{inv}(\psi)$ *is the number of inversions of the permutation* ψ*.*

Restricting equation (1.8) to $\mathrm{SL}_k\mathbb{C}$, we get

$$V_{\bar\lambda} \otimes V_{\bar\mu} = \sum_\nu c_{\lambda\mu}^\nu V_{\bar\nu} \,, \tag{1.12}$$

and comparing with (1.9) gives

$$c_{\lambda\mu}^\nu = m_{\bar\lambda\bar\mu}^{\bar\nu} \,. \tag{1.13}$$

Hence Steinberg's formula also computes the Littlewood-Richardson coefficients, and we can further simplify things by noticing that if we let $\mathbf{1}_k$ denote the vector $(1,1,\ldots,1) \in \mathbb{R}^k$, then

$$\begin{aligned}
\sigma(\bar\lambda+\delta) + \tau(\bar\mu+\delta) - (\bar\nu+2\delta) &= \sigma(\bar\lambda) + \tau(\bar\mu) - \bar\nu + \sigma(\delta) + \tau(\delta) - 2\delta \\
&= \sigma(\lambda - \frac{|\lambda|}{k}\mathbf{1}_k) + \tau(\mu - \frac{|\mu|}{k}\mathbf{1}_k) - (\nu - \frac{|\nu|}{k}\mathbf{1}_k) + \sigma(\delta) + \tau(\delta) - 2\delta \\
&= \sigma(\lambda) - \frac{|\lambda|}{k}\mathbf{1}_k + \tau(\mu) - \frac{|\mu|}{k}\mathbf{1}_k - \nu + \frac{|\nu|}{k}\mathbf{1}_k + \sigma(\delta) + \tau(\delta) - 2\delta \\
&= \sigma(\lambda+\delta) + \tau(\mu+\delta) - (\nu+2\delta) + \frac{1}{k}(|\nu| - |\lambda| - |\mu|)\mathbf{1}_k \\
&= \sigma(\lambda+\delta) + \tau(\mu+\delta) - (\nu+2\delta) \,.
\end{aligned}$$

In view of (1.11) and (1.13), this gives

$$c^{\nu}_{\lambda\mu} = \sum_{\sigma\in\mathfrak{S}_k}\sum_{\tau\in\mathfrak{S}_k}(-1)^{\mathrm{inv}(\sigma\tau)}K(\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta))\,. \tag{1.14}$$

In Section 3, we will use this formula to define a hyperplane arrangement over whose regions the Littlewood-Richardson coefficients are given by polynomials in λ, μ and ν.

1.2. Partition functions and chamber complexes. Partition functions arise in the representation theory of the semisimple Lie algebras in the form of Kostant's partition function, which sends a vector in the root lattice to the number of ways it can be written down as a linear combination with nonnegative integer coefficients of the positive roots. The Kostant partition function is a simple example of a more general class of functions, called *vector partition functions*.

Definition 1.4. Let M be a $d\times n$ matrix over the integers, such that $\ker M\cap\mathbb{R}^n_{\geq 0}=0$. The *vector partition function* (or simply *partition function*) associated to M is the function

$$\begin{array}{cccc}\phi_M: & \mathbb{Z}^d & \longrightarrow & \mathbb{N}\\ & b & \mapsto & |\{x\in\mathbb{N}^n \;:\; Mx=b\}|\end{array}$$

The condition $\ker M\cap\mathbb{R}^n_{\geq 0}=0$ forces the set $\{x\in\mathbb{N}^n \;:\; Mx=b\}$ to have finite size, or equivalently, the set $\{x\in\mathbb{R}^n_{\geq 0} \;:\; Mx=b\}$ to be compact, in which case it is a polytope P_b, and the partition function is the number of integral points (lattice points) inside it.

Also, if we let $M_1,\dots,M_n$ denote the columns of M (as column-vectors), and $x=(x_1,\dots,x_n)\in\mathbb{R}^n_{\geq 0}$, then $Mx=x_1M_1+x_2M_2+\cdots+x_nM_n$ and for this to be equal to b, b has to lie in the cone $\mathrm{pos}(M)$ spanned by the vectors M_i. So ϕ_M vanishes outside of $\mathrm{pos}(M)$.

It is well-known that partition functions are piecewise quasipolynomial, and that the domains of quasipolynomiality form a complex of convex polyhedral cones, called the *chamber complex*. Sturmfels gives a very clear explanation in [**Stu95**] of this phenomenon. The explicit description of the chamber complex is due to Alekseevskaya, Gel'fand and Zelevinskiĭ [**AGZ98**]. There is a special class of matrices for which partition functions take a much simpler form. Call an integer $d\times n$ matrix M of full rank d *unimodular* if every non-singular $d\times d$ submatrix has determinant ±1. For unimodular matrices, the chamber complex determines domains of polynomiality instead of quasipolynomiality [**Stu95**].

It is useful for what follows to describe how to obtain the chamber complex of a partition function. Let M be a $d\times n$ integer matrix of full rank d and ϕ_M its associated partition function. For any subset $\sigma\subseteq\{1,\dots,n\}$, denote by M_σ the submatrix of M with column set σ, and let $\tau_\sigma=\mathrm{pos}(M_\sigma)$, the cone spanned by the columns of M_σ. Define the set $\mathcal{B}$ of *bases* of M to be

$$\mathcal{B}=\{\sigma\subseteq\{1,\dots,n\} \;:\; |\sigma|=d \;\text{ and }\; \mathrm{rank}(M_\sigma)=d\}\,.$$

$\mathcal{B}$ indexes the invertible $d\times d$ submatrices of M. The *chamber complex* of ϕ_M is the common refinement of all the cones τ_σ, as σ ranges over $\mathcal{B}$ (see [**AGZ98**]). A theorem of Sturmfels [**Stu95**] describes exactly how partition functions are quasipolynomial over the chambers of that complex.

If we let M_{A_n} be the matrix whose columns are the positive roots $\Delta_+^{(A_n)}$ of A_n, written in the basis of simple roots, then we can write Kostant's partition function in the matrix form defined above as

$$K_{A_n}(v)=\phi_{M_{A_n}}(v)\,.$$

The following lemma is a well-known fact about M_{A_n} and can be deduced from general results on matrices with columns of 0's and 1's where the 1's come in a consecutive block (see [**Sch86**]).

Lemma 1.5. *The matrix M_{A_n} is unimodular for all n.*

M_{A_n} unimodular means that the Kostant partition functions for A_n is polynomial instead of quasipolynomial on the cells of the chamber complex. In general, for M unimodular, the polynomial pieces have degree at most the number of columns of the matrix minus its rank (see [**Stu95**]). In our case, M_{A_n} has rank n and as many columns as A_n has positive roots, $\binom{n+1}{2}$. Hence the Kostant partition function for A_n is piecewise polynomial of degree at most $\binom{n+1}{2} - n = \binom{n}{2}$.

Remark 1.6. In view of Steinberg's formula (1.11), this means that the Littlewood-Richardson coefficients are given by a piecewise polynomial function of degree at most $\binom{n}{2}$ in the three sets of variables λ, μ and ν, if these partitions have at most $n+1$ parts. This will be made precise in Sections 3 and 4

2. A vector partition function for the Littlewood-Richardson coefficients

There are many combinatorial ways to compute the Littlewood-Richardson coefficients, in particular the Littlewood-Richardson rule [**Sta99**], honeycombs [**KT99**] and Berenstein-Zelevinsky triangles [**BZ92**]. The model that is most convenient for us is the hive model [**Buc00, KT99**].

Definition 2.1. A *k-hive* is an array of numbers a_{ij} with $0 \le i, j \le k$ and $i + j \le k$. We will represent hives in matrix form. For example, a 4-hive is

$$\begin{array}{ccccc} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & \\ a_{20} & a_{21} & a_{22} & & \\ a_{30} & a_{31} & & & \\ a_{40} & & & & \end{array} \tag{2.1}$$

We will call a hive *integral* if all its entries are nonnegative integers

Following the terminology of [**KTT03**], we will call *hive conditions* (HC) the conditions

$$\begin{array}{ccc} & j & j+1 \\ i & \bullet & \boxed{\bullet} \\ i+1 & \boxed{\bullet} & \bullet \end{array} \qquad \begin{array}{ccc} & j & j+1 \\ i & & \bullet \\ i+1 & \boxed{\bullet} & \boxed{\bullet} \\ i+2 & \bullet & \end{array} \qquad \begin{array}{cccc} & j & j+1 & j+2 \\ i & & \boxed{\bullet} & \bullet \\ i+1 & \bullet & \boxed{\bullet} & \end{array} \tag{2.2}$$

where in each diagram, the sum of the boxed entries is at least as large as the sum of the other two entries. In terms of the a_{ij}, (HC) is

$$a_{i+1\,j} + a_{i\,j+1} \ge a_{ij} + a_{i+1\,j+1} \tag{2.3}$$

$$a_{i+1\,j} + a_{i+1\,j+1} \ge a_{i+2\,j} + a_{i\,j+1} \tag{2.4}$$

$$a_{i\,j+1} + a_{i+1\,j+1} \ge a_{i+1\,j} + a_{i\,j+2} \tag{2.5}$$

for $i+j \le k-2$.

PROPOSITION 2.2. **(Knutson-Tao [KT99], Fulton [Buc00])** *For λ, μ and ν partitions with at most k parts and $|\lambda|+|\mu|=|\nu|$, the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ is the number of integral k-hives satisfying (HC) and the boundary conditions*

$$\begin{aligned} a_{00} &= 0\,, \\ a_{0j} &= \lambda_1+\cdots+\lambda_j && 1 \le j \le k \\ a_{i0} &= \nu_1+\cdots+\nu_i && 1 \le i \le k \\ a_{m,k-m} &= |\lambda|+\mu_1+\cdots+\mu_m && 1 \le m \le k\,. \end{aligned} \tag{2.6}$$

Once the boundary conditions are imposed, we are left with a system of inequalitites in the nonnegative integral variables a_{ij} for $1 \le i,j \le k-1$ and $i+j \le k-1$. There are $n(k) = 3\binom{k}{2}$ inequalities. If we let these a_{ij} take real values, the inequalitites define a rational polytope $Q_{\lambda\mu}^{\nu}$, and the Littlewood-Richardson coefficient corresponding to the boundary conditions is the number of integral (lattice) points inside $Q_{\lambda\mu}^{\nu}$.

Given a d-dimensional rational polytope Q in $\mathbb{R}^n$, we will denote by mQ the polytope Q blown up by a factor of m. The function $m \in \mathbb{N} \mapsto |mQ \cap \mathbb{Z}^n|$ is called the *Ehrhart function* of Q, and is known [**Ehr77, Sta99**] to be a quasipolynomial of degree d in m. Furthermore, if Q is integral, the Ehrhart function is a degree d polynomial in m. This means that the function

$$N \longmapsto c_{N\lambda\,N\mu}^{N\nu} \tag{2.7}$$

is the Ehrhart quasipolynomial of the polytope $Q_{\lambda\mu}^{\nu}$. It is known that $Q_{\lambda\mu}^{\nu}$ is not integral in general (see examples in [**KTT03**]).

This describes the behavior of the Littlewood-Richardson coefficients on a ray in (λ,μ,ν)-space, but we will get more general results by showing that we can find a vector partition function that gives these coefficients. We will then be able to work with conical chambers in (λ,μ,ν)-space instead of simple rays. This is accomplished in a way very similar to the one introduced for the weight multiplicities in [**BGR03**], and this case is even simpler because the variables a_{ij} are already constrained to be nonnegative.

THEOREM 2.3. *There are integer matrices E_k and B_k such that the function $(\lambda,\mu,\nu) \mapsto c_{\lambda\mu}^{\nu}$ for λ, μ, ν partitions with at most k parts such that $|\lambda|+|\mu|=|\nu|$ and $\lambda,\mu \subseteq \nu$ is given by*

$$c_{\lambda\mu}^{\nu} = \phi_{E_k}\left(B_k\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix}\right). \tag{2.8}$$

The chamber complex defined by E_k is much too big for our purposes. For one thing, its cones have dimension $n(k) = 3\binom{k}{2}$, whereas (λ,μ,ν)-space is $3k$-dimensional. To simplify things, we can first restrict ourselves with the intersection of the complex of E_k with the subspace

$$\mathcal{B}^{(k)} = \left\{\left(B_k\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix}\right) \;:\; \lambda,\mu,\nu \in \mathbb{R}^k\right\} \tag{2.9}$$

of $\mathbb{R}^{n(k)}$ to get a complex $\mathcal{C}_k$. Then we can pull back the cones along the transformation B_k to (λ,μ,ν)-space. Cones in $\mathcal{B}^{(k)}$ are given by inequalitites of the form

$$\left\langle v_i, B_k\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix}\right\rangle \ge 0$$

for some directions $v_i \in \mathbb{R}^{n(k)}$. But

$$\left\langle v_i, B_k \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} \right\rangle \geq 0 \quad \Leftrightarrow \quad \left\langle B_k^T v_i, \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} \right\rangle \geq 0\,,$$

where B_k^T is the transpose of B_k. So we can pull back the cones to get a complex $B_k^* \mathcal{C}_k$ in (λ, μ, ν)-space. As a final simplification, we can note that $c_{\lambda\mu}^{\nu} = 0$ unless $\lambda, \mu \subseteq \nu$ and $|\lambda| + |\mu| = |\nu|$ and that these conditions define a cone $C_k^{(1)}$ since the containment equations can be written $\lambda_i, \mu_i \leq \nu_i$ for $1 \leq i \leq k$. The conditions $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$, $\mu_1 \geq \cdots \geq \mu_k \geq 0$ and $\nu_1 \geq \cdots \geq \nu_k \geq 0$ also define a cone $C_k^{(2)}$.

Definition 2.4. We will call the intersection of the cones $C_k^{(1)}$ and $C_k^{(2)}$ with the rectified complex $B_k^* \mathcal{C}_k$ the *Littlewood-Richardson complex*, and denote it $\mathcal{LR}_k$. This complex lives on the subspace $|\lambda| + |\mu| = |\nu|$ of $\mathbb{R}^{3k}$.

As a result of the general theory of vector partition functions, we get the following corollary.

Corollary 2.5. *Under the conditions of the theorem above, the function $(\lambda, \mu, \nu) \mapsto c_{\lambda\mu}^{\nu}$ is quasipolynomial of degree at most $3\binom{k}{2} + n(k) - \operatorname{rank} E_k = 3\binom{k}{2}$ over the chambers of the complex $\mathcal{LR}_k$.*

We will explain in Section 4 that we actually get polynomials in the chambers.

It rapidly becomes computationally hard to work out the chamber complex and the associated polynomials; we present an example of how the computations are done on the simplest nontrivial example, $k = 3$, in Section 5.

3. The Steinberg arrangement

In this section, we will construct a hyperplane arrangement whose regions are domains of polynomiality for the Littlewood-Richardson coefficients. We will deduce the form of this arrangement from a closer look at Steinberg's formula (1.11) and the chamber complex of the Kostant partition function defined in Section 1.2.

The following lemma describes the set of normals to the hyperplanes supporting the cells of the chamber complex for the Kostant partition function.

Lemma 3.1. *The set of normals to the facets of the maximal cones of the chamber complex of the Kostant partition function of A_n consists of all the conjugates of the fundamental weights.*

To compute the Littlewood-Richardson coefficients using Steinberg's formula (1.11), we look at the points $\sigma(\lambda + \delta) + \tau(\mu + \delta) - (\nu + 2\delta)$, as σ and τ range over the Weyl group $\mathfrak{S}_k$ (we assume here that λ, μ and ν have at most k parts and index irreducible representations of $\mathrm{GL}_k\mathbb{C}$). Some of these points will lie inside the chamber complex for the Kostant partition function and we compute the Littlewood-Richardson coefficients by finding which cells contain them and evaluating the corresponding polynomials at those points. We will call (λ, μ, ν) *generic* if none of the points $\sigma(\lambda + \delta) + \tau(\mu + \delta) - (\nu + 2\delta)$ lies on a wall of the chamber complex of the Kostant partition function. If we change a generic (λ, μ, ν) to (λ', μ', ν') on the hyperplane $|\lambda| + |\mu| = |\nu|$ in such a way that none of the $\sigma(\lambda + \delta) + \tau(\mu + \delta) - (\nu + 2\delta)$ crosses a wall, we will obtain $c_{\lambda'\mu'}^{\nu'}$ by evaluating the same polynomials. So there is a neighborhood of (λ, μ, ν) on which the Littlewood-Richardson coefficients are given by the same polynomial in the variables λ, μ and ν.

Lemma 3.1 describes the walls of the chamber complex for the Kostant partition function in terms of the normals to the hyperplanes (through the origin) supporting the facets of the maximal cells. Now a point

$\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta)$ will be on one of those walls (hyperplane though the origin) when its scalar product with the hyperplane's normal, say $\theta(\omega_j)$, vanishes, that is when

$$\langle \sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta), \theta(\omega_j)\rangle = 0 \tag{3.1}$$

Consider the arrangement on the subspace $|\lambda|+|\mu|=|\nu|$ of $\mathbb{R}^{3k}$ consisting of all such hyperplanes, for $1 \le j \le k$ and σ, τ, $\theta \in \mathfrak{S}_k$. For (λ,μ,ν) and (λ',μ',ν') in the same region of this arrangement and any fixed $\sigma,\tau \in \mathfrak{S}_k$, the points $\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta)$ and $\sigma(\lambda'+\delta)+\tau(\mu'+\delta)-(\nu'+2\delta)$ lie on the same side of every wall of the chamber complex for the Kostant partition function. We will call this arrangement the *Steinberg arrangement*, and denote it $\mathcal{SA}_k$.

Definition 3.2. Fix a labelling on the chambers of the complex for the Kostant partition function, and let p_1, p_2, … be the polynomials associated to the chambers. For generic λ, μ and ν, let $v_{\sigma\tau}(\lambda,\mu,\nu)$ be the label of the region containing the point $\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta)$ (this label is unique for generic λ, μ and ν). Define the *type* of λ, μ and ν to be the matrix

$$\mathrm{Type}(\lambda,\mu,\nu) = \big(v_{\sigma\tau}(\lambda,\mu,\nu)\big)_{\sigma,\tau\in\mathfrak{S}_k},$$

for some fixed total order on $\mathfrak{S}_k$. Furthermore, define

$$P(\lambda,\mu,\nu) = \sum_{\sigma\in\mathfrak{S}_k}\sum_{\tau\in\mathfrak{S}_k}(-1)^{\mathrm{inv}(\sigma\tau)} p_{v_{\sigma\tau}}\big(\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta)\big). \tag{3.2}$$

Proposition 3.3. *$P(\lambda,\mu,\nu)$ is a polynomial function in λ, μ and ν on the interior of the regions of $\mathcal{SA}_k$ and gives the Littlewood-Richardson coefficients there.*

The reason why Proposition 3.3 is restricted to the interior of the regions is that while polynomials for adjacent regions of the chamber complex for the Kostant partition function have to coincide on the intersection of their closures, there is a discontinuous jump in the value of the Kostant partition function (as a piecewise polynomial function) when going from a region on the boundary of the complex to region 0 (outside the complex).

To summarize, the hyperplanes of the Steinberg arrangement are defined by the equations

$$\langle \sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta), \theta(\omega_j)\rangle = 0 \tag{3.3}$$

or

$$\langle \sigma(\lambda)+\tau(\mu)-\nu, \theta(\omega_j)\rangle = \langle 2\delta-\sigma(\delta)-\tau(\delta), \theta(\omega_j)\rangle. \tag{3.4}$$

Note that the right hand side of (3.4) doesn't depend on λ, μ and ν, and we will call it the *δ-shift*:

$$s(\sigma,\tau,\theta,j) = \langle 2\delta-\sigma(\delta)-\tau(\delta), \theta(\omega_j)\rangle. \tag{3.5}$$

4. Polynomiality in the chamber complex

We have now expressed the Littlewood-Richardson coefficients in two ways: as a quasipolynomial function over the cones of the chamber complex $\mathcal{LR}_k$, and as a polynomial function over the interior of the regions of the hyperplane arrangement $\mathcal{SA}_k$. In this section, we relate the chamber complex to the hyperplane arrangement to show that the quasipolynomials are actually polynomials.

Theorem 4.1. *The quasipolynomials giving the Littlewood-Richardson coefficients in the cones of the chamber complex $\mathcal{LR}_k$ are polynomials of total degree at most $\binom{k-1}{2}$ in the three sets of variables $\lambda=(\lambda_1,\dots,\lambda_k)$, $\mu=(\mu_1,\dots,\mu_k)$ and $\nu=(\nu_1,\dots,\nu_k)$.*

From this, we can deduce a "stretching" property for Littlewood-Richardson coefficients.

COROLLARY 4.2. *The Littlewood-Richardson coefficients $c_{N\lambda\, N\mu}^{N\nu}$ are given by a polynomial in N with rational coefficients. This polynomial has degree at most $\binom{k-1}{2}$ in N.*

REMARK 4.3. King, Tollu and Toumazet conjectured in [**KTT03**] that the $c_{N\lambda\, N\mu}^{N\nu}$ are polynomial in N with nonnegative rational coefficients (Conjecture 1.1 above). Corollary 4.2 establishes this conjecture, except for the nonnegativity of the coefficients. Derksen and Weyman [**DW02**] have a proof of this part of the conjecture using semi-invariants of quivers, and Knutson [**DW02, Knu03**] a proof using symplectic geometry techniques.

In fact, we can prove something stronger: we can perturb (λ, μ, ν) a bit and get a more global stretching property.

COROLLARY 4.4. *Let Υ be the set*

$$\Upsilon = \{(\lambda, \mu, \nu)\ :\ \max\{l(\lambda), l(\mu), l(\nu)\} \leq k, |\lambda| + |\mu| = |\nu|, \lambda, \mu \subseteq \nu\}. \tag{4.1}$$

For any generic $(\lambda, \mu, \nu) \in \Upsilon$ we can find a neighborhood U of that point over which the function

$$(\lambda, \mu, \nu, t) \in (U \cap \Upsilon) \times \mathbb{N} \longmapsto c_{t\lambda\, t\mu}^{t\nu} \tag{4.2}$$

is polynomial of degree at most $\binom{k-1}{2}$ in t and $\binom{k-1}{2}$ in the λ, μ and ν coordinates.

5. An example: partitions with at most 3 parts

We want to find a vector partition function counting the number of integral 3-hives of the form

$$\begin{array}{llll} 0 & \lambda_1 & \lambda_1 + \lambda_2 & |\lambda| \\ \nu_1 & a_{11} & |\lambda| + \mu_1 & \\ \nu_1 + \nu_2 & |\nu| - \mu_3 & & \\ |\nu| & & & \end{array} \tag{5.1}$$

The hives conditions are given by

$$\begin{array}{lll} a_{11} \leq \nu_1 + \lambda_1 & -a_{11} \leq -\lambda_2 - \nu_1 & -a_{11} \leq -\lambda_1 - \nu_2 \\ -a_{11} \leq -\lambda_1 - \lambda_3 - \mu_1 & a_{11} \leq \lambda_1 + \lambda_2 + \mu_1 & -a_{11} \leq -\lambda_1 - \lambda_2 - \mu_2 \\ -a_{11} \leq -\lambda_1 - \lambda_2 - \lambda_3 - \mu_1 - \mu_2 + \nu_2 & -a_{11} \leq \mu_2 - \nu_1 - \nu_2 & a_{11} \leq \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 - \nu_3 \end{array} \tag{5.2}$$

This corresponds to the matrix system

$$E_3 \cdot \begin{pmatrix} a_{11} \\ s_1 \\ s_2 \\ \vdots \\ s_9 \end{pmatrix} \quad = \quad B_3 \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{5.3}$$

with

$$E_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.4}$$

and

$$B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.5}$$

Note that μ_3 doesn't not appear in this system. This is because it is determined by $|\lambda| + |\mu| = |\nu|$; we could have chosen another variable to disappear.

To get the chamber complex for the vector partition function associated to E_3, we have to find the sets of columns determining maximal nonsingular square matrices in E_3. These determine the bases cones whose common refinement gives the chamber complex. In our case, all subsets of 9 columns determine a nonsingular matrix, so we get 10 base cones. We can find their common refinement using a symbolic calculator like Maple or Mathematica; here we used Maple (version 8) and the package `convex` by Matthias Franz [**Fra01**]. We find the chamber complex $\mathcal{LR}_3$ by rectifying the cones to (λ, μ, ν)-space using B_3^T and intersecting them with the cones $C_3^{(1)}$ and $C_3^{(2)}$. The list of rays of the cones of $\mathcal{LR}_3$

$$a_1 = (\,1\ 1\ 1 \mid 0\ 0\ 0 \mid 1\ 1\ 1\,) \qquad a_2 = (\,0\ 0\ 0 \mid 1\ 1\ 1 \mid 1\ 1\ 1\,)$$
$$b = (\,2\ 1\ 0 \mid 2\ 1\ 0 \mid 3\ 2\ 1\,)$$
$$c = (\,1\ 1\ 0 \mid 1\ 1\ 0 \mid 2\ 1\ 1\,)$$
$$d_1 = (\,1\ 1\ 0 \mid 1\ 0\ 0 \mid 1\ 1\ 1\,) \qquad d_2 = (\,1\ 0\ 0 \mid 1\ 1\ 0 \mid 1\ 1\ 1\,)$$
$$e_1 = (\,1\ 1\ 0 \mid 0\ 0\ 0 \mid 1\ 1\ 0\,) \qquad e_2 = (\,0\ 0\ 0 \mid 1\ 1\ 0 \mid 1\ 1\ 0\,)$$
$$f = (\,1\ 0\ 0 \mid 1\ 0\ 0 \mid 1\ 1\ 0\,)$$
$$g_1 = (\,1\ 0\ 0 \mid 0\ 0\ 0 \mid 1\ 0\ 0\,) \qquad g_2 = (\,0\ 0\ 0 \mid 1\ 0\ 0 \mid 1\ 0\ 0\,)$$

where the bars separate the entries corresponding to the sets of variables λ, μ and ν.

The following table gives the maximal (8-dimensional) cones of $\mathcal{LR}_3$, as well as the polynomial associated to each (computed by polynomial interpolation).

Cone	Positive hull description	Polynomial
κ_1	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, e_1, e_2)$	$1 - \lambda_2 - \mu_2 + \nu_1$
κ_2	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, g_1, g_2)$	$1 + \nu_2 - \nu_3$
κ_3	$\mathrm{pos}(a_1, a_2, b, c, e_1, e_2, g_1, g_2)$	$1 + \lambda_1 + \mu_1 - \nu_1$
κ_4	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, e_1, e_2, f)$	$1 + \nu_1 - \nu_2$
κ_5	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, f, g_1, g_2)$	$1 + \lambda_2 + \mu_2 - \nu_3$
κ_6	$\mathrm{pos}(a_1, a_2, b, e_1, e_2, f, g_1, g_2)$	$1 - \lambda_3 - \mu_3 + \nu_3$
κ_7	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, e_1, g_1)$	$1 + \lambda_3 + \mu_1 - \nu_3$
κ_8	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, e_2, g_2)$	$1 + \lambda_1 + \mu_3 - \nu_3$
κ_9	$\mathrm{pos}(a_1, a_2, b, c, d_1, e_1, e_2, g_2)$	$1 + \lambda_1 - \lambda_2$
κ_{10}	$\mathrm{pos}(a_1, a_2, b, c, d_2, e_1, e_2, g_1)$	$1 + \mu_1 - \mu_2$
κ_{11}	$\mathrm{pos}(a_1, a_2, b, c, d_1, e_1, g_1, g_2)$	$1 - \lambda_2 - \mu_3 + \nu_2$
κ_{12}	$\mathrm{pos}(a_1, a_2, b, c, d_2, e_2, g_1, g_2)$	$1 - \lambda_3 - \mu_2 + \nu_2$
κ_{13}	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, e_1, f, g_1)$	$1 - \lambda_1 - \mu_3 + \nu_3$
κ_{14}	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, e_2, f, g_2)$	$1 - \lambda_3 - \mu_1 + \nu_3$
κ_{15}	$\mathrm{pos}(a_1, a_2, b, d_1, e_1, f, g_1, g_2)$	$1 + \mu_2 - \mu_3$
κ_{16}	$\mathrm{pos}(a_1, a_2, b, d_2, e_2, f, g_1, g_2)$	$1 + \lambda_2 - \lambda_3$
κ_{17}	$\mathrm{pos}(a_1, a_2, b, d_1, e_1, e_2, f, g_2)$	$1 + \lambda_1 + \mu_2 - \nu_2$
κ_{18}	$\mathrm{pos}(a_1, a_2, b, d_2, e_1, e_2, f, g_1)$	$1 + \lambda_2 + \mu_1 - \nu_2$

Remark 5.1. The symmetry $c^{\nu}_{\lambda\mu} = c^{\nu}_{\mu\lambda}$ implies that we can interchange the λ and μ coordinates. This corresponds to a symmetry of the chamber complex $\mathcal{LR}_3$ under this transformation. This is why some of the rays and cones have been grouped in pairs.

Remark 5.2. We observe from the form of the polynomials in the table above that the equation

$$c^{N\nu}_{N\lambda\, N\mu} = 1 + N(c^{\nu}_{\lambda\mu} - 1) \tag{5.6}$$

holds for $l(\lambda), l(\mu), l(\nu) \leq 3$. This was previously observed in [**KTT03**].

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References

[AGZ98] T. V. Alekseevskaya, I. M. Gel'fand, and A. V. Zelevinskiĭ, *An arrangement of real hyperplanes and the partition function connected with it*, Soviet Math. Dokl. **36** (1998), no. 3, 589–593.

[BGR03] Sara Billey, Victor Guillemin, and Etienne Rassart, *A vector partition function for the multiplicities of* $\mathfrak{sl}_k\mathbb{C}$, To appear in the *Journal of Algebra*, arXiv:math.CO/0307227, 2003.

[Buc00] Anders S. Buch, *The saturation conjecture (after A. Knutson and T. Tao*, Enseign. Math. (2) **46** (2000), 43–60.

[BZ92] A. D. Berenstein and A. V. Zelevinsky, *Triple multiplicities for* $\mathfrak{sl}(r+1)$ *and the spectrum of the exterior algebra of the adjoint representation*, J. Alg. Comb. **1** (1992), 7–22.

[DW02] Harm Derksen and Jerzy Weyman, *On the Littlewood-Richardson polynomials*, J. Algebra **255** (2002), no. 2, 247–257.

[Ehr77] E. Ehrhart, *Polynômes arithmétiques et méthode des polyèdres en combinatoire*, International Series of Numerical Mathematics, vol. 35, Birkhäuser, 1977.

[FH91] William Fulton and Joe Harris, *Representation theory*, GTM, vol. 129, Springer, 1991.

[Fra01] Matthias Franz, *Convex – A Maple package for convex geometry*, Maple package, 2001, Version 0.91, Konstanz, available via internet from `http://www.mathe.uni-konstanz.de/~franz/convex/`.

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, 1997.

[Hum72] James E. Humphreys, *Introduction to Lie algebras and representation theory*, GTM, vol. 9, Springer, 1972.

[Knu03] Allen Knutson, Personal communication, 2003.

[KT99] A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products I: Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), 1055–1090.

[KTT03] R. C. King, C. Tollu, and F. Toumazet, *Stretched Littlewood-Richardson and Kostka coefficients*, To appear in CRM Proceedings and Lecture Notes, volume 34, 2003.

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials, second edition*, Oxford Mathematical Monographs, Clarendon Press, Oxford Science Publications, 1995.

[Sch86] Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience series in discrete mathematics, Wiley, 1986.

[Sta99] Richard P. Stanley, *Enumerative combinatorics, volume 2*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.

[Ste61] R. Steinberg, *A general Clebsch-Gordan theorem*, Bull. Amer. Math. Soc. **67** (1961), 406–407.

[Stu95] Bernd Sturmfels, *On vector partition functions*, Journal of Combinatorial Theory, Series A **72** (1995), 302–309.

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