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# Ribbon tilings of Ferrers diagrams, flips and the 0-Hecke algebra

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Abstract. In this article we study how the 0-Hecke algebra $H_m(0)$ can be used to give an algebraic structure to ribbon tilings of Ferrers diagrams. The key to this structure is the Stanton-White bijection, which gives a bijection between $n$-ribbon tableaux and $n$-upplets of Young tableaux. Restricting to standard ribbon tableaux, we can define a natural action of $H_m(0)$. Thus we can define local actions on ribbon tableaux, which we call flips or pseudo-flips, and which are generalisations of domino flips. Then with some help from the Yang-Baxter relations we prove some properties about minimal flip chains, properties which remain true for ribbon tilings.

Résumé. Le but de cet article est de montrer comment on peut utiliser la 0-algèbre de Hecke $H_m(0)$ afin de donner une structure algébrique aux pavages par rubans d'un diagramme de Ferrers donné. Cette structure dcoule de la bijection de Stanton-White entre les tableaux de $n$-rubans et les $n$-uplets de tableaux de Young. Si on se limite aux tableaux standards de rubans, cela nous donne une action naturelle de $H_m(0)$, qui nous permet alors de définir des modifications locales sur les tableaux de rubans, que nous appellons flips et pseudo-flips. Ce sont des généralisations du flip classique de dominos. Grâce aux relations de Yang-Baxter on peut alors donner des invariants sur les chaînes minimales de flips, qui se conservent quand on passe aux pavages par rubans.

## 1. Introduction

The goal of this article is to study a class of tilings called ribbon tilings of a Ferrers diagram. The main motivation for this study is to give a general, algebraic generalisation of domino tilings. In order to try and give some order and algebraic structures to these tilings, we will use ribbon tableaux.

Ribbon tableaux originate from rim hook tableaux, introduced to study the representations of the symetric group [**dBR61, GJ81**]. These tableaux have been studied from an algebraic point of view (see [**SW85, CL95, LLT97**]). In particular, [**SW85**] gives us a bijection between Ferrers diagrams and $n$-tuples of Ferrers diagrams, with interesting properties regarding ribbons. This bijection can be extended to a bijection between ribbons tableaux and $n$-tuples of Young tableaux [**Pak90**].

Section 2 gives some definitions and notation, then recalls some existing results :

We first define ribbon tilings and tableaux, recall basic facts about them and define the Stanton-White bijection. If we restrict our view to standard ribbon tableaux, we obtain from this bijection standard $n$-tuples of Young tableaux, or equivalently skew standard Young tableaux, upon which there are classical algebraic structures.

We then consider one such algebraic structure, the Hecke algebra for $q = 0$, $H_m(0)$, which is related to posets (that is partially ordered sets). We recall its classical actions on permutations and Young tableaux.

In Section 3, we extend this action to standard ribbon tableaux, using the Stanton-White bijection. By giving a $H_m(0)$-module structure to standard ribbon tableaux, we prove that it has a lattice structure, which we study.

In Section 4 Having thus given a poset structure to standard ribbon tableaux, we study the covering relation, using elementary generators of $H_m(0)$. We obtain local actions called pseudo-flips and flips, the latter being a generalisation of the flips encountered with domino tiling. After giving a geometric description and classification of these flips, we prove an invariant results concerning minimum flip paths, the key of the proof being the Yang-Baxter relations met by $H_m(0)$.

## 2. Definitions and existing results

In this section we first define ribbon tilings, ribbon tableaux and define some conventions and notation. We then give some basics facts about these objects, before recalling the Stanton-White bijection and the induced bijection for ribbon tableaux.

### 2.1. Presentation of ribbon tilings and notation.

All the geometric objects which we define will be placed in the discrete plane $\mathbb{N}^2$, and we identify a unit square with its lower left corner.

A *n-ribbon* is a polyomino (that is a finite part of the discrete plane) formed by $n$ squares, defining a path composed only of left or up steps. (It is a simply connected polyomino.) Therefore we can define the *head* of a ribbon as its bottom right square. An $n$-ribbon can thus be given be the coordinate of its head in the discrete plane, and by a word in $\{0,1\}^{n-1}$ coding its shape, each 0 representing a left step, and each 1 a up step. Figure 1 gives two examples of 8-ribbons.

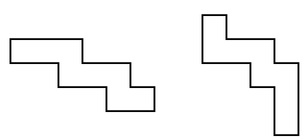

FIGURE 1. Two 8-ribbons

Two particular cases of ribbons are 1-ribbons, which are elementary squares in the discrete planes, and 2-ribbons, which are the classical dominoes, upon which much litterature exists [**CL95**].

Given a partition $\lambda$, that is a decreasing integer sequence of finite length, its *Ferrers* diagram is the shape in $\mathbb{N}^2$ whose length rows are givens by the terms of $\lambda$. (We use the cartesian convention, and thus rows lengths are decreasing upward.) We identify a partition and its Ferrers diagram, and call the et set of all partitions (or equivalently of all Ferrers diagrams) $\Pi$.

$\Pi$ is the set of all partitions, considering two partitions equals if we can obtain one from the other by adding some zeros at its right. For a partition $\lambda$ its *length* $l(\lambda)$, is its length as a finite sequence (the tail zeros do not count in the length). The *weight* of $\lambda$, denoted by $|\lambda|$, is the sum of its terms.

since we are in the discrete plane, we can define the diagonal $d$ as $\Delta_d = \big\{(x,y) \in \mathbb{N}^2, x-y=d\big\}$. The *content* of a cell is the diagonal which it belongs to. The content of a ribbon is the content of its head.

Let us now define a *ribbon tiling* : We fix an integer $n$, and we tile $\lambda$ by removing $n$-ribbons from its rim, in such a way that the remaining polyomino is still the Ferrers diagram associated with a partition, and then we go on, until we cannot remove ribbons anymore. Is is a classical result that the remaining partition, $\lambda_{(n)}$ does not depend on how the ribbons were removed (see [**dBR61, GJ81**]), and is called the $n$-core of $\lambda$. Thus all the tilings we can define this way are tilings of the same part of the discrete plane, $\lambda \smallsetminus \lambda_{(n)}$. Figure 2 gives two ribbon tilings of the same Ferrers diagram.

General ribbon tiling have been studied in [**She**], where it was proved that the set of all tilings of a given Ferrers diagram is in bijection with the set of all acyclics orientation of a particular graph. Reversing an edge in a given orientation, when it is possible then gives an action on ribbon tilings. This action is a

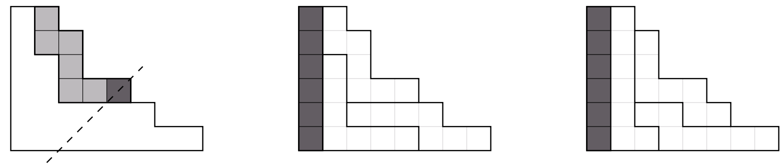

FIGURE 2. An example of rim 7-ribbon with its head and diagonal, and two 7-tiling of $(8, 6, 5, 3, 3, 2)$ with the 7-core shaded.

local one called flip, for which the set of all tilings of a diagram is connex. But it does not give additional structure. We will give an algebraic interpretation for these flips.

Ribbon tilings of Ferrers diagrams have been studied in [**Pak90**], which defines a family of functions on ribbons. It is then proved that these function form a basis for the tiling invariant. That is to say that for a giver Ferrers diagram $\lambda$ these functions are all invariant on the ribbon tilings of $\lambda$, and every such invariant can be obtained from these functions.

Given a Ferrers diagram $\lambda$, a *Young tableau* of shape $\lambda$ is a filling of the cells of $\lambda$ with strictly positive integers, in such a way that in each row the numbers are weakly increasing, and in each column they are strictly increasing upward.

It is natural to extend this notion to ribbons, which gives ribbon tableaux. A *n-ribbon tableau* of shape $\lambda \setminus \lambda_{(n)}$ is a tiling of $\lambda \setminus \lambda_{(n)}$ by $n$-ribbons to which we give integer numbers with the following growth conditions : On each row and each colon, the numbers of the encoutered ribbons must be weakly increasing, and the head of a ribbon cannot be above another ribbon with the same number. A young tableau is actually a 1-ribbon tableau, therefore definitions which apply to both will be given for ribbon tableaux.

We can define the *weight* of a ribbon tableau (thus of a Young tableau) as the sequence $(w_i)_i$ where $w_i$ is the number of ribbon whose number is $i$, and a *standard ribbon tableau* will be a ribbon tableau of weigth $(1, 1, 1, 1, 0, 0, \ldots)$. A standard ribbon tableau can be seen as a ribbon tiling, together with in which order the ribbons are added from the core. An example of standard ribbon tableau is given in figure 3. From now on, all the ribbon tableaux considered will be standard.

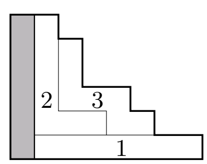


FIGURE 3. A standard ribbon tableau of shape $(8, 6, 5, 3, 3, 2) \setminus (1^6)$

The set of ribbon tableau and the set of standard ribbon tableaux of shape $\lambda \setminus \lambda_{(n)}$ will be denoted respectively by $Tab_n(\lambda \setminus \lambda_{(n)})$, and $STab_n(\lambda \setminus \lambda_{(n)})$.

**2.2. The Stanton-White bijection.**

Let us now recall here the classical Stanton-White bijection, which occurs for a given $n$-core $\mu$ between all partitions whose $n$-core is $\mu$ and all the $n$-tuples of partitions. This bijection can thus be considered as a bijection between the set all partitions and the outer product of the set of all $n$-tuples of partitions and the set of all $n$-cores. Let us define it algorithmically before giving an interpretation :

Given a partition $\lambda \in \Pi$, first add to it the stair $\Lambda_m = (m-1, m-2, \ldots, 2, 1, 0)$ with $m$ such that $m \geqslant l(\lambda)$ and $m = \alpha n$ with $\alpha \in \mathbb{N}^*$. We thus obtain a strict partition $\lambda^\Lambda$ of length $m$, which we decompose modulo $n$ into $n$ partitions $\lambda^\Lambda(0), \ldots, \lambda^\Lambda(n-1)$ in the following way : For each term $x$ of $\lambda$, we make the integer division by $n$, $x = nq + r$, and add the term $q$ (even if $q = 0$) to $\lambda^\Lambda(r)$. (By separating $\lambda^\Lambda$ modulo $n$, we actually separate $\lambda$ along its diagonals modulo $n$.) We set $l_i$ as the length of $\lambda^\Lambda(i)$ (including a possible 0), and then substract $\Lambda_{l_i}$ to $\lambda^\Lambda(i)$, to obtain a partition $\lambda^{(n)}(i)$. $(\lambda^{(n)}(1), \dots, \lambda^{(n)}(n-1))$ is then an $n$-tuple of partitions corresponding to $\lambda$, called the $n$-quotient of $\lambda$, and is denoted by $\lambda^{(n)}$.

This function is surjective, as for any $n$-tuple of partitions $\lambda^{(n)}$, the $\lambda^\Lambda(i)$ are simply obtained by adding stairs, and then $\lambda^\Lambda$ by multiplying the terms $\lambda^\Lambda(i)$ by $n$ and adding $i$ to them, and merging the resulting partition into a strict partition $\lambda^\Lambda$. We then just have to substract a stair to $\lambda^\Lambda$ to obtain a partition whose image is $\lambda^{(n)}$.

Let us illustrate this bijection by taking $\lambda = (5, 5, 3, 2, 1)$ and $n = 3$, we add $(5, 4, 3, 2, 1, 0)$ to $\lambda$ to get $\lambda^\Lambda = (10, 9, 6, 4, 2, 0)$, which we divide modulo 3 to obtain $\lambda^\Lambda(0) = (3, 2, 0)$, $\lambda^\Lambda(1) = (3, 1)$ and $\lambda^\Lambda(2) = (0)$. Substracting the corresponding stairs, we get $\lambda^{(3)}(0) = (1, 1, 0)$, $\lambda^{(3)}(1) = (2, 1)$ and $\lambda^{(3)}(2) = (0)$. This is illustred in figure 4

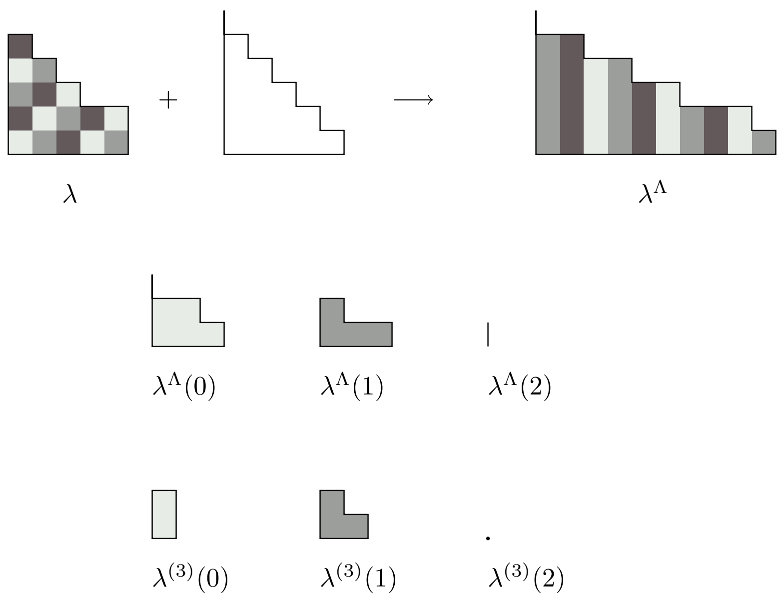

Figure 4

This transformation also gives us a coding of $\lambda_{(n)}$, by setting $w_i = l_i - l(\lambda)$. $w$ is then an $n$-dimensional vector with sum equal to 0, which depends only of $\lambda_{(n)}$.

If we call $\mathbb{Z}_0^n$ the set of all $n$-dimensionals vectors with sum 0. We then can state the following result ([**GJ81, SW85**]) :

Theorem 2.1. *The function* $\lambda \mapsto (\lambda^{(n)}, \lambda_{(n)})$ *is a bijection between* $\Pi$ *and* $\Pi^n \times \mathbb{Z}_0^n$

For $n = 1$, $\lambda^{(1)} = (\lambda)$, and $\lambda_{(1)}$ is empty, so the Stanton-White bijection is the canonical bijection between $\Pi$ and $\Pi \times \{0\}$.

Note that if $\mu$ is obtained by removing an $n$-ribbon from $\lambda$'s rim, then $\mu^\Lambda$ is obtained from $\lambda^\Lambda$ by substracting $n$ to a term and sorting the remaining terms. This term is equal modulo $n$ to the content of the ribbon removed. Furthermore, each term of $\lambda^\Lambda$ to which we can substract $n$ (in such a way that $\mu^\Lambda$ remains a strict partition) corresponds to a ribbon which can be removed from $\lambda$'s rim, as can be seen in figure 5

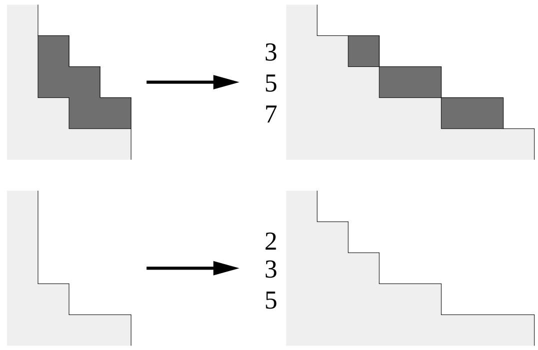


FIGURE 5. Removing a 5-ribbon from $\lambda^{\Lambda}$

We then obtain $\mu^{(n)}$ from $\lambda^{(n)}$ by removing a square of diagonal $d' + c_i$ from $\lambda^i$, where $d = nd' + i$ is the integer division of $d$ by $n$ ( the $c_i$ are offset parameters which depend on $\lambda_{(n)}$). So the squares of $\lambda^{(n)}$ correspond to ribbons, and the number of ribbons in a tiling of $\lambda \setminus \lambda_{(n)}$ is given by $\sum_{i=0}^{n-1} |\lambda^i|$.

This bijection translates to ribbon tableaux. The corresponding objects are then $n$-tuples of Young tableaux, each square corresponding to a precise ribbon. (The growth condition on ribbon tableau gives precisely classical growth condition of Young tableaux.) So, for a partition $\lambda$ we have a bijection between $Tab_n(\lambda \setminus \lambda_{(n)})$ and all $n$-tuples of Young tableaux of shape $\lambda^{(n)}$, and this bijection preserves the weight. Figure 6 gives an example for this bijection.

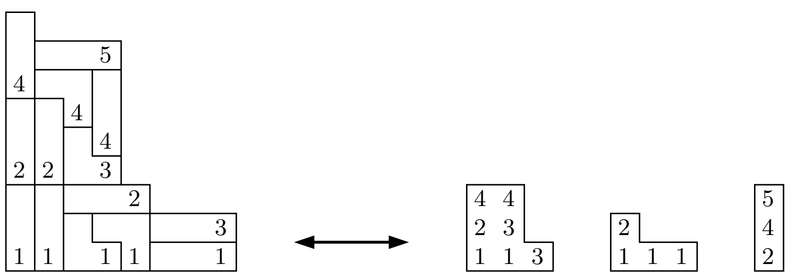

FIGURE 6

Moreover, we can arrange an $n$-tuple of Young tableau in such a way as to obtain a skew Young tableau. Figure 7 shows how it is done for the example we took in figure 6. (There are $2^n n!$ skew different Young

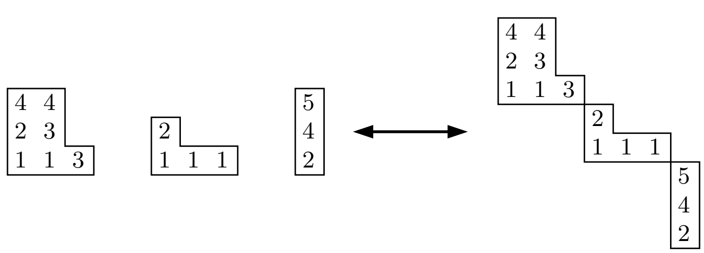

FIGURE 7

tableaux we can obtain this way, by taking the $n$ Young tableaux in different orders and by transposing some of them.)

So for a given shape $\lambda$, we can obtain a skew Ferrers diagram $\mu \setminus \nu$ such that we have a bijection between $Tab_n(\lambda \setminus \lambda_{(n)})$ and skew Young tableaux of shape $\mu \setminus \nu$, which leaves the weight invariant. If we restrict ourselves to standard ribbon tableaux, we then have a bijection between standard ribbon tableaux and standard Young tableaux, upon which we can define algebraic structures, which we are now going to do.

### 2.3. The 0-Hecke algebra $H_m(0)$ and its actions.

In this section we define a classical combinatorics algebra, the 0-Hecke algebra ([**KT97**]) and explain how it can be used to give en ordering structure on permutations and Young tableaux.

We define $H_m(0)$, the *Hecke algebra* for $q = 0$ (over the field of complex numvers), as the $\mathbb{C}$-algebra spanned by $n-1$ generators $T_1, \dots, T_{m-1}$ with the following relations[1] :

$$\begin{cases} {T_i}^2 = T_i & \forall 1 \leqslant i \leqslant m-1 \quad (1) \\ T_iT_j = T_jT_i & \text{if } |i-j| > 1 \quad (2) \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & \forall 1 \leqslant i \leqslant m-2 \quad (3) \end{cases}$$

Actually, $H_m(0)$ (and more generally $H_m(q)$) is a deformation of the symetric group, and such algebras can be defined for all Coxeter groups. If we take $q = 1$, $H_m(1)$ is $\mathbb{K}\mathfrak{S}_m$, the symetric group algebra.

Let us first remark that (1) can can be rewritten as $T_i(T_i - 1) = 0$, and thus 0 and 1 are possible eigenvalues for $T_i$ seen as an operator. The fact that 0 is a possible eigenvalue is an important aspect of $H_m(0)$, as we will see later.

Given a permutation $\sigma \in \mathfrak{S}_m$, let us define $T_\sigma$ in the following way : starting from a elementary decomposition of $\sigma$, $\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k}$, where $\sigma_i = (i, i+1)$, we set $T_\sigma = T_{i_1}T_{i_2}\dots T_{i_k}$. $T_\sigma$ does not depend on the elementary decomposition chosen, as all such decomposition are congruent by the braid relation in the symetric group, and thus the corresponding products of $T_i$ are congruent by (3).

Let us now define two natural actions of $H_m(0)$ on the symetric group algebra $\mathbb{C}\mathfrak{S}_m$, the left action $L$ and the right action $R$. $L$ is also called the *action on values* and $R$ the *action on places.*

$$\begin{cases} R(T_i)\sigma = \sigma & \text{if } \sigma(i) < \sigma(i+1) \\ R(T_i)\sigma = \sigma\sigma_i & \text{otherwise} \end{cases}$$

$$\begin{cases} L(T_i)\sigma = \sigma & \text{if } \sigma^{-1}(i) < \sigma^{-1}(i+1) \\ L(T_i)\sigma = \sigma_i\sigma & \text{otherwise} \end{cases}$$

These action are in duality, by $L(T_i)\sigma = \left(R(T_i)\sigma^{-1}\right)^{-1}$. From now on, we will only consider the right action of $H_m(0)$, which is shown in figure 8 for $n = 3$.

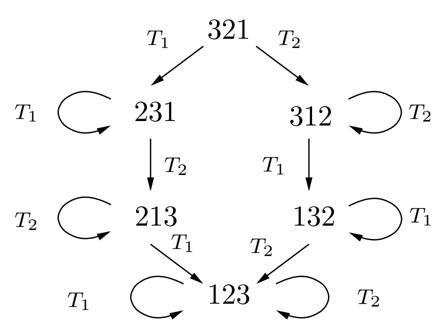


FIGURE 8. The right action of $H_3(0)$ on $\mathfrak{S}_3$

[1]Note that this is not the usual presentation of $H_m(0)$. The Hecke algebra for $q$ given is usually defined by the quadratic relation $(T_i - 1)(T_i + q) = 0$, or equivalently ${T_i}^2 = (1-q)T_1 + q$, hence for $q = 0$ ${T_i}^2 = -T_i$. What we define here is a renormalisation obtained by replacing $T_i$ with $-T_i$, which is more convenient in the present case.

Actually, $H_m(0)$ is linked with bubble sort : The action of a $T_i$ is exactly an elementary step of bubble sort, and if we take the maximal permutation $\omega_0 = (m, m-1, \ldots, 2, 1) = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 \ldots$
$\sigma_{m-1}\sigma_{m-2} \ldots \sigma_2\sigma_1$, then $T_{\omega_0}$ corresponds to the whole bubble sort.

Moreover, if we define a partial ordering on $\mathfrak{S}_n$ by setting $\sigma \preccurlyeq \sigma'$ when there exists a permutation $\omega$ such that $\sigma = T_\omega\sigma'$ or $\sigma = \sigma'$, then we find the classical weak order of $\mathfrak{S}_m$. The graph whose vertices are the permutations of $\mathfrak{S}_n$ and the edges are the action of the $T_i$ minus the loops is the right $n$-permutaedron. The poset whose Hasse diagram is the permutaedron is a graded lattice ([**GR63, Bjö84**]), its maximum element is the trivial permutation 1 and its minimum element the permutation $\omega_0$. (If we take the left action instead of the right action, we obtain another poset structure on $\mathfrak{S}_n$, isomorph to the one defined by the right action.) Note that this lattice is not distributive for $n \geqslant 3$.

(We recall that a *lattice* is a poset in which every pair of elements admits a supremum and infimum, noted by $\vee$ and $\wedge$. A lattice is *distributive* if $\vee$ and $\wedge$ are each distributive over the other, and a lattice $P$ is *graded* if it admits a graduation, that is a function $f : P \to \mathbb{Z}$ such that if $a$ covers $b$ in $P$, then $f(a) = f(b) + 1$. See [**DP90**])

This action can easily be extended to space spanned by all the standard Young tableau of a given shape $\lambda$ with $|\lambda| = m$ : Given a standard Young tableau $t$, we call $\sigma_t$ the permutation given by its line reading, and inversely $t_\sigma$ is the tableau of shape $\lambda$ whose line reading is $\sigma$.

$$\begin{cases} T_i t = t_{L(T_i)\sigma_t} & \text{if } i \text{ and } i+1 \text{ are not adjacent in } t \\ T_i t = 0 & \text{otherwise} \end{cases}$$

If $i$ and $i+1$ are not adjacent, they cannot be on the same line as $t$ is standard, and so the action of $T_i$ is to reorder them so that $i+1$ is higher than $i$. Thus $T_{\omega_0}$ reorders $t$ into the line filling of $\lambda$. This is illustrated in figure 9 for the tableaux of shape $(3, 2)$.

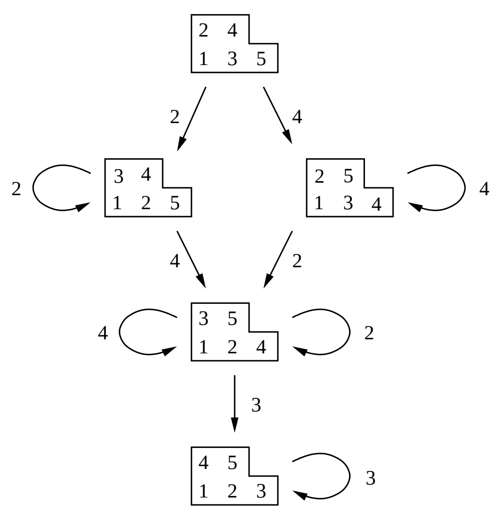


FIGURE 9. The action of $H_5(0)$ on standard Young tableaux of shape $(3, 2)$

This defines what is called a Specht module structure on the vector space spanned by standard Young tableaux of shape $\lambda$. It also gives an ordered structure on standard Young tableaux : Given two standard Young tableaux of same shape, $t$ and $t'$, we set $t \preccurlyeq t'$ if there exists a permutation $\omega$ such that $t = T_\omega t'$ (including the cas $t = t'$). The same kind of results hold as for the permutation group :

Theorem 2.2. *Take $\lambda \in \Pi$, and $m = |\lambda|$. The poset defined by the action of $H_m(0)$ on the set of all standard Young tableaux of shape $\lambda$ is a lattice. The maximum element of the lattice is the row filling of $\lambda$ and the minimum element is the column filing of $\lambda$.*

This lattice is usually not distributive. (For example the lattice of all standard Young tableaux of shape $(3, 2, 1)$ admits the 3-permutaedron as a sub-lattice.)

## 3. The lattice structure on standard ribbon tableaux

Having recalled all these results, we can now extend them to standard ribbon tableaux. This action of $H_m(0)$ on standard Young tableaux naturally extends in the same way to skew Young tableaux, and it keeps all its properties, including the lattice structure it defines.

As the Stanton White bijection induces a bijection between standard ribbon tableaux and standard skew Young tableaux, we naturally have an action of $H_m(0)$ on the vector space spanned by $STab_n(\lambda \setminus \lambda_{(n)})$ (where $m$ is the number of $n$-ribbons in a tiling of $\lambda \setminus \lambda_{(n)}$, that is $\sum |\lambda^{(n)}(i)|$), and the following result :

Theorem 3.1. *$H_m(0)$ induces a graded structure lattice on $STab_n(\lambda \setminus \lambda_{(n)})$*

Figure 10 gives this structure for $STab_3\left((4^3)\right)$, and figure 11 gives the isomorphic structure for the corresponding triplet of Ferrers diagrams, which is $((2), (1), (1))$. Note that in this case different standard ribbons tableaux give different ribbons tilings, but this is not always the case.

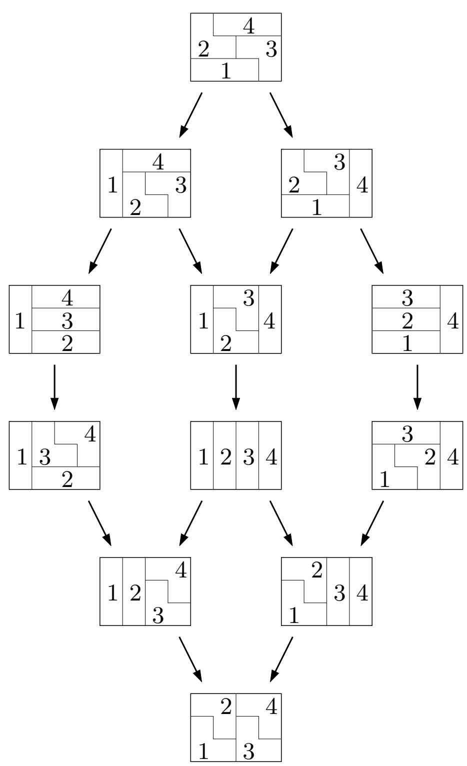

Figure 10

Depending on the way we build the skew Ferrers diagram on which $H_m(0)$ acts, we can obtain up to $2^n n!$ different action of $H_m(0)$, which will define as many different lattice structures (some of them will often coincide), but the non-oriented graph underlying the Hasse diagram will remain the same, as the edge

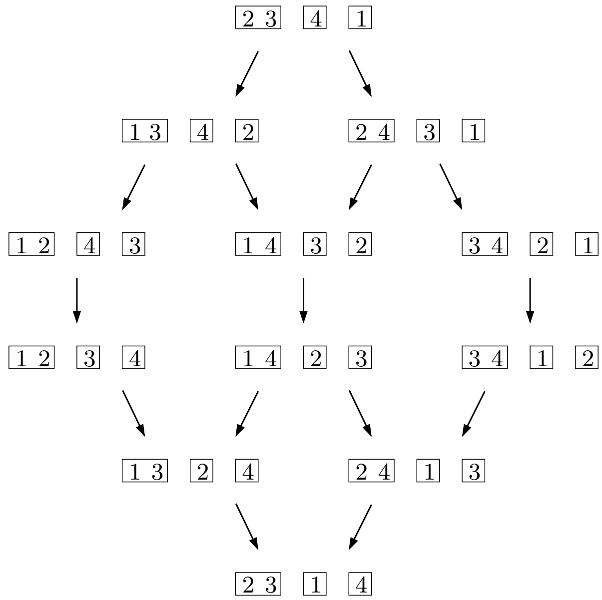


Figure 11

are given by switchs between consecutive numbers in the $n$-tuples of Young diagram corresponding to the standard ribbon tableaux.

## 4. Flips, pseudo-flips and the Yang-Baxter relation

In a first part we define flips and pseudo-flips, which are local actions given by the action of the generators of $H_m(0)$. This definition allows us in a second subpart to classify and count the possible flips. We then use the classical Yang-Baxter relations to give invariants on minimum flip paths.

### 4.1. Local action of the $T_i$.

We have in the previous section defined a bipartite graph whose vertices set is $STab_n(\lambda \setminus \lambda_{(n)})$, and whose edges are given by the covering relation of one the lattice defined by the actions of $H_m(0)$, the graph itself being independant of the chosen action of $H_m(0)$. It is natural to study this covering relation, more specifically the action of a $T_i$ on the ribbon tableaux themselves.

This covering relation is given by the action of a $T_i$, that is the switching of $i$ and $i+1$ in the $n$-tuple of Young tableaux. A standard ribbon tableau can be constructed from the corresponding Young tableaux by adding ribbons with increasing or decreasing number, or both. Thus if we have two standard $n$-tuples of Young tableaux, $(t_j)_{0 \geqslant j \geqslant k-1}$ and $(t'_j)_{0 \geqslant j \geqslant k-1}$ which can be obtained from the other by switching $i$ and $i+1$, we can construct the associated ribbon tableaux by placing the ribbons 1 to $i-1$ and then the ribbons $n$ downto $i+2$ on the border of $\lambda$, and these will be the same in the two ribbons tableaux.

So the action of $T_i$ is a local one, which only acts on the ribbons $i$ and $i+1$, leaving all the others unchanged. Its effects depend on the difference $\Delta d$ between the diagonals of the two ribbons, $d_i$ and $d_{i+1}$, and whether it's bigger or lesser than $n$. $\Delta_d$ cannot be equal to $n$, because then $i$ and $i+1$ would be in the same $t_j$, and they would be on adjacent diagonals, thus they would be adjacents, in which case they could not be switched.

If $\Delta d > n$, then the two ribbons are disjoint, so switching $i$ and $i+1$ in $t$ amounts to switching the numbers of the ribbons. We call this transformation a *n-pseudo-flip* (or just pseudo-flip) of *genus* $\Delta d$. A pseudo-flip changes the standard ribbon tableau, but leaves the underlying ribbon tiling invariant.

If $\Delta d < n$, then the two ribbons overlap, and by switching $i$ and $i+1$ we change the ribbon which overlaps the other, and then we have a *n-flip* of *genus* $\Delta d$.

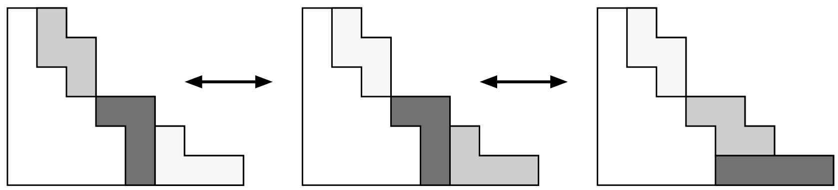

FIGURE 12. A pseudo-flip and a flip of 4-ribbons

Figure 12 gives examples of flip and pseudo-flip for 4-ribbons. If we only consider the underlaying ribbon tiling, a flip between standard ribbon tableaux defines a flip between two tilings. This is the same flip as was defined in [**She**], even tough it was apparently differently defined. For $n = 2$, we find the usual domino flip.

**4.2. Geometric classification and enumeration of flips.**

The genus of a flip describes the geometry of this flip  it is the difference between the diagnoals of the two ribbons involved, and so it tells the length along which the two ribbons overlap. In a flip of genus $d$ between $n$-ribbons, the ribbons overlap on $n - d + 1$ cells. This gives a geometric classification of flips, and will allow us to count the possible flips :

THEOREM 4.1. *There is* $(2^{n-1} - 1)2^{n-2}$ *different possible geometries for flips of* $n$*-ribbons.*

PROOF. We will call $F_d(n)$ the number of geometrically differents $n$-flips of genus $d$, and $F(n)$ the number of all $n$-flips.

When $d > 1$, as only these $n-d+1$ cells are involved in the flip, the geometry of this flip is the same as for a flip of genus 1 between ribbons of length $n-d+1$. So an $n$-flip of genus $d$ geometrically consis tof two parts : First, the overlapping part, where the flip occur, which is a flip of genus 1 of $(n-d+1)$-ribbons, and then the remaining part of the ribbons, which can be seen as two $d-1$ ribbons starting from the overlapping part. From this we can derive the relation $F_d(n) = 2^{2(d-1)} F_1(n - d + 1)$.

In order to compute $F_1(n)$ let us see what exactly is a flip of genus 1, which we will call a maximal flip.

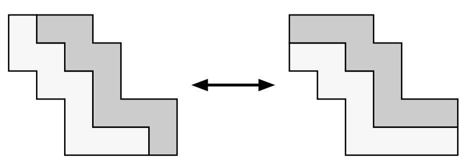

FIGURE 13

In such a flip, the ribbons' heads are adjacent, and the ribbons overlap on their whole length. In one of the two configurations involved in the flip, the ribbon's head will be side to side, and so the one with the greater diagonal (that is the one whose head is on the right) will have a "up" as its first step, as can be seen in figure 13. Now, if we take such a ribbon with an upper first step, we can put a ribbon immediatly to its left, such that these two ribbons overlap on their whole length, and it is possible to do a maximal flip on these ribbons. So a flip of genus 1 can be coded by a ribbon whose first step is imposed, thus by a word of

$\{0,1\}^{n-1}$. This word can also be seen as coding the intersection of the common boundaries of the ribbons in the two configurations of the flip, as shown in figure 14

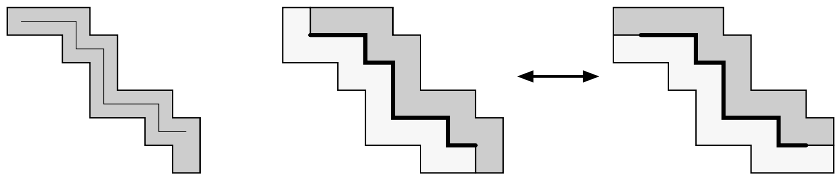

FIGURE 14. A 12-flip of genus 1, whose coding is $(0,1,0,0,1,1,0,1,0,0)$

This gives us $F_1(n) = 2^{n-2}$, and so :

$$F_d(n) = 2^{2(d-1)}2^{n-d-1} = 2^{n+d-3}$$

$$F(n) = 2^{n-3}\sum_{d=1}^{k-1} 2^d = (2^{n-1}-1)2^{n-2}$$

□

For $n = 3$, we have $F_1(3) = 2$, $F_2(3) = 4$ and $F(3) = 6$. These six flips of 3-ribbons are given in figure 15 and 16.

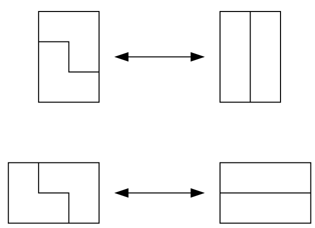

FIGURE 15. The 3-flips with genus 1

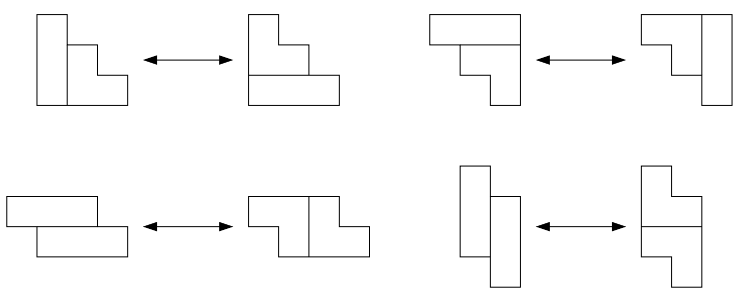

FIGURE 16. The 3-flips with genus 2

Remark that taking $n = 2$, we have $F(2) = 1$, so this is consistent with the trivial fact on domino tilings that there exists only one kind of flip.

### 4.3. Minimal flip paths and Yang-Baxter relations.

We will now prove the following result :

THEOREM 4.2. *Given $t$ and $t'$ two standard ribbon tableaux of same shape, the set of the genus (with multiplicity) of the flips and pseudo-flips in a path between $t$ and $t'$ is invariant for minimal length paths.*

PROOF. Let's consider two standard ribbons tableaux $t$ and $t'$ of same shape $\lambda \setminus \lambda_{(n)}$, and minimal length paths of flips and pseudo-flips between these paths. By chosing a way to rearrange $\lambda^{(n)}$ in a skew partition, we can see $t$ and $t'$ as the permutations $\sigma_t$ and $\sigma_{t'}$ corresponding to the line-readings of the skew Young tableaux associated. Flips and pseudo-flips paths then are chains of elementary transpoition from $\sigma_t$ to $\sigma'_t$. As these paths are minimal, the corresponding chains of transpositions are congruent by the braid relation, and so we can use following result for Coxeter groups :

An elementary transposition, $\sigma_i$, switches $i$ and $i+1$ in $\sigma$. We will associate to this transposition a factor $(x_{\sigma^{-1}(i+1)} - x_{\sigma^{-1}(i)})$, where $x_1, \ldots, x_n$ is a set of independant variables. This gives the places of the switched terms in $\sigma$. For a chain of transposition $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_m}$ from $\sigma_t$ to $\sigma_{t'}$, let us take the product

$$(x_{\sigma_t{}^{-1}(i_1+1)} - x_{\sigma_t{}^{-1}(i_1)})(x_{(\sigma_t\sigma_{i_1})^{-1}(i_2)} - x_{(\sigma_t\sigma_{i_1})^{-1}(i_2+1)})\ldots$$

which gives all the places of the switched elements in the path from $\sigma_t$ to $\sigma'_t$. Then this product is invariant for all chains of transposition from $t$ to $t'$ which are congruent by the braid relation.

Now let us specialize the $x_j$ into another set $y_l$ by setting $x_j = y_{d(j)}$ where $d(j)$ is the diagonal of a ribbon corresponding to the element at place $j$ in $\sigma_t$. $(x_{j_1} - x_{j_2})$ becomes $(y_{l_1} - y_{l_2})$ where $l_1$ and $l_2$ are the diagonals of the ribbon involved in the flip or pseudo-flip. $(y_{l_1} - yl_2)$ then gives the genus of the flip or pseudo-flip, and so the multiset of genus is invariant for all minimal paths between $t$ and $t'$. □

Two such minimal paths are given in figure 17 for 4-ribbons.

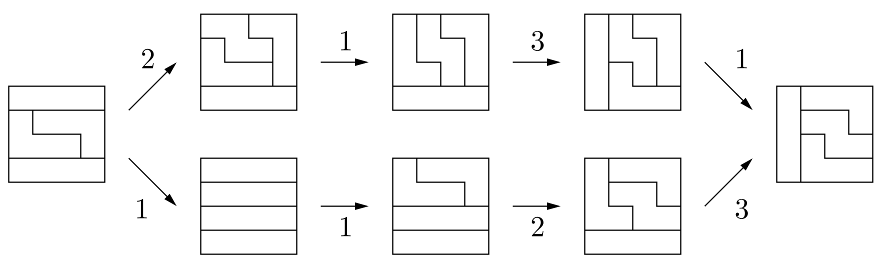


FIGURE 17. Two minimal flip paths between two 4-tilings, with the genuses of the flips shown

## 5. Conclusion and perspective

Up to this day, ribbon tilings and ribbon tableaux have been considered two completly different kinds of combinatorics objects, studied by different people with different methods. This paper is intended as an attempt to link these domains, altough the link is somewhat tenuous.

One important question relative to this topic is to know whether the different lattice structures thus defined on $STab_n(\lambda \setminus \lambda_{(n)})$ induce lattice structures on the set of $n$-ribbon tillings of $\lambda \setminus \lambda_{(n)}$. Actually, the present study was motivated by this problem. Alas several unsolved related questions remain unsolved.

In particular, we meet the following problem : given two standard $n$-tuples of Young tableaux of same shape, is there a simple way to determine if the associated ribbon tableaux have the same underlying ribbon tilings, without having to compute these ribbon tableaux ?

## References

[Bjö84] A. Björner, *Orderings of coxeter groups*, Combinatorics and Algebra (Providence, R. I.) (C. Greene, ed.), vol. 34, Am. Math. Society, 1984, pp. 175–195.

[BW88] A. Björner and M. Wachs, *Generalized quotients in coxeter groups*, Trans. of the Am. Math. Society (1988).

[CL95] C. Carré and B. Leclerc, *Splitting the square of a schur function into its symetric and antisymetric parts*, Journal of algebraic combinatorics **4** (1995), 201–231.

[dBR61] Gilbert de B. Robinson, *Representation theory of the symetric group*, 1961.

[DP90] P. A. Davey and H. A. Priestley, *An introduction to lattice and orders*, Cambridge University Press, 1990.

[GJ81] A. Kerber G. James, *The representation theory of the symetric group*, 1981.

[GR63] G. Guilbaud and P. Rosenstiehl, *Analyse algébrique d'un scrutin*, Math. Sci. Hum (1963), 9–33.

[KT97] D. Krob and J.-Y. Thibon, *Noncommutative symetric functions IV: Quantum linear groups and hecke algebras at* $q = 0$, Journal of algebraic cominatorics **6** (1997), 339–376.

[Las] A. Lascoux, *Ordering the affine symmetric group*, `http://www-igm.univ-mlv.fr/~al/ARTICLES/OrderingAffine.ps.gz`.

[LLT96] A. Lascoux, B. Leclerc, and Jean-Yves Thibon, *Hecke algebras at roots of unity and canonical bases of quantum affine algebras*, Communications in Math. Physics **181** (1996), 205–263.

[LLT97] A. Lascoux, B. Leclerc, and J.-Y. Thibon, *Ribbon tableaux, hall-littlewood functions, quantum, affine algebras, and unipotent varieties*, Journal of math. physics **38 (2)** (1997).

[LT00] B. Leclerc and J.-Y. Thibon, *Littlewood-richardson coefficients and kazhdan-lusztig polynomials*, Advanced Studies in Pure Mathematics **28** (2000), 155–200.

[MP] C. Moore and I. Pak, *Ribbon tile invariants from signed area*, Journal of combinatorial theory **Series A**.

[Pak90] I. Pak, *Ribbon tile invariants*, Trans. of the Am. Math. Society (1990).

[Rém99] É. Rémila, *On the lattice structure of the set of tilings of a simply connected figure with dominoes*, Proccedings of the 3rd International Conference on Orders, Algorithms and Applications (ORDAL), 1999.

[She] S. Sheffield, *Ribbon tilings and multidimensional height functions*, `arxiv.org:math.CO/0107095`.

[SW85] D. W. Stanton and D. E. White, *A Schensted algorithm for rim hook tableaux*, Journal of combinatorial theory **Series A** (1985), 211–247.

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