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A Hopf Algebra of Parking Functions

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Abstract. If the moments of a probability measure on $\mathbb{R}$ are interpreted as a specialization of complete homogeneous symmetric functions, its free cumulants are, up to sign, the corresponding specializations of a sequence of Schur positive symmetric functions (f_n). We prove that (f_n) is the Frobenius characteristic of the natural permutation representation of $\mathfrak{S}_n$ on the set of prime parking functions. This observation leads us to the construction of a Hopf algebra of parking functions, which we study in some detail.

Résumé. Si on interprète les moments d'une mesure de probabilité sur $\mathbb{R}$ comme une spécialisation de fonctions symétriques complètes, ses cumulants libres sont, au signe près, les spécialisations correspondantes d'une suite de functions symétriques (f_n) Schur-positives. Nous montrons que (f_n) est la caractéristique de Frobenius d'une représentation permutationnelle naturelle de $\mathfrak{S}_n$ sur l'ensemble des fonctions de parking primitives. Cette observation nous conduit à construire une algèbre de Hopf des fonctions de parking que nous étudions ensuite en détail.

1. Introduction

The free cumulants R_n of a probability measure μ on $\mathbb{R}$ are defined (see *e.g.,* [**20**]) by means of the generating series of its moments M_n

$$G_\mu(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x} = z^{-1} + \sum_{n\geq 1} M_n z^{-n-1} \tag{1.1}$$

as the coefficients of its compositional inverse

$$K_\mu(z) := G_\mu(z)^{\langle -1\rangle} = z^{-1} + \sum_{n\geq 1} R_n z^{n-1}\,. \tag{1.2}$$

It is in general instructive to interpret the coefficients of a formal power series as the specializations of the elements of some generating family of the algebra of symmetric functions. In this context, it is the interpretation

$$M_n = \phi(h_n) = h_n(A) \tag{1.3}$$

which is relevant. Indeed, the process of functional inversion (Lagrange inversion) admits a simple expression within this formalism (see [**14**], ex. 24 p. 35). If the symmetric functions h_n^* are defined by the equations

$$u = tH(t) \iff t = uH^*(u) \tag{1.4}$$

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where $H(t) := \sum_{n\geq 0} h_n t^n$, $H^*(u) := \sum_{n\geq 0} h_n^* u^n$, then, using the λ-ring notation,

$$h_n^*(X) = \frac{1}{n+1}(-1)^n e_n((n+1)X) := \frac{1}{n+1}[t^n]E(-t)^{n+1} \tag{1.5}$$

where $E(t)$ is defined by $E(t)H(t) = 1$. This defines an involution $f \mapsto f^*$ of the ring of symmetric functions.

Now, if one sets $M_n = h_n(A)$ as above, then

$$G_\mu(z) = z^{-1}H(z^{-1}) = u \iff z = K_\mu(u) = \frac{1}{u}E^*(-u) = u^{-1} + \sum_{n\geq 1}(-1)^n e_n^* u^{n-1}\,. \tag{1.6}$$

Hence,

$$R_n = (-1)^n e_n^*(A)\,. \tag{1.7}$$

It follows immediately from the explicit formula (see [**14**] p. 35)

$$-e_n^* = \frac{1}{n-1}\sum_{\lambda\vdash n}\binom{n-1}{l(\lambda)}\binom{l(\lambda)}{m_1, m_2, \ldots, m_n} e_\lambda \tag{1.8}$$

(where $\lambda = 1^{m_1}2^{m_2}\cdots n^{m_n}$) that $-e_n^*$ is Schur positive. Clearly, $-e_n^*$ is the Frobenius characteristic of a permutation representation Π_n, twisted by the sign character. Let us set

$$(-1)^{(n-1)}R_n = -e_n^* =: \omega(f_n) \tag{1.9}$$

so that f_n is the character of Π_n. We start with a construction of this representation in terms of parking functions. This leads us to the definition of a Hopf algebra of parking functions that generalizes the constructions of [**15, 3**]. We expect that this combinatorics can be generalized to other root systems, at least for type B (see, *e.g.*, [**2**]).

We note that our construction of Π_n is merely a variation about previously known results (see in particular [**12, 17**]). However, since this is this precise version that led us to the Hopf algebra of parking functions and some of its properties, we decided to present it in detail.

Although many definitions will be recalled, we shall assume that the reader is familiar with the notation of [**5, 3**].

Acknowledgements.- The problem of constructing the representation Π_n was suggested by S. Kerov during his stay in Marne-la-Vallée in 1996. The question was forgotten for a long time without any attempt of solution, and rediscovered recently on the occasion of talks by S. Ferrières and P. Biane. Thanks also to P. Biane for providing the reference [**17**].

2. Parking functions

2.1. Parking functions. A *parking function* on $[n] = \{1, 2, \ldots, n\}$ is a word $\mathbf{a} = a_1a_2\cdots a_n$ of length n on $[n]$ whose nondecreasing rearrangement $\mathbf{a}^\uparrow = a'_1a'_2\cdots a'_n$ satisfies $a'_i \leq i$ for all i. Let PF_n be the set of such words. It is well-known that $|\mathrm{PF}_n| = (n+1)^{n-1}$, and that the permutation representation of $\mathfrak{S}_n$ naturally supported by PF_n has Frobenius characteristic $(-1)^n\omega(h_n^*)$ (see [**8**]).

2.2. Prime parking functions. Gessel introduced in 1997 (see [**22**]) the notion of *prime parking function.* One says that $\mathbf{a}$ has a *breakpoint* at b if $|\{\mathbf{a}_i \leq b\}| = b$. Then, $\mathbf{a} \in \mathrm{PF}_n$ is said to be prime if its only breakpoint is $b = n$.

Let $\mathrm{PPF}_n \subset \mathrm{PF}_n$ be the set of prime parking functions on $[n]$. It can easily be shown that $|\mathrm{PPF}_n| = (n-1)^{n-1}$ (see [**22, 10**]).

2.3. Operations on parking functions. For a word w on the alphabet $1, 2, \ldots$, denote by $w[k]$ the word obtained by replacing each letter i by $i+k$. If u and v are two words, with u of length k, one defines the *shifted concatenation*

$$u \bullet v = u \cdot (v[k]) \tag{2.1}$$

and the *shifted shuffle*

$$u \Cup v = u \sqcup\!\sqcup (v[k])\,. \tag{2.2}$$

It is immediate to see that the set of permutations is closed under both operations, and that the subalgebra spanned by those elements is isomorphic to the convolution algebra of symmetric groups (see [**15**]) or to Free Quasi-Symmetric Functions (see [**3**]).

It is equally immediate to see that the set of all parking functions is closed under these operations and that the prime parking functions exactly are the parking functions that do not occur in any nontrivial shifted shuffle of parking functions. These properties allow us to define a Hopf algebra of parking functions (see Section 3).

Let us now move to representation theory.

2.4. The module of prime parking functions. Recall that the expression of complete symmetric functions in the basis e_λ is the commutative image of the formula

$$(-1)^n S_n = \sum_{I \vDash n} (-1)^{l(I)} \Lambda^I \tag{2.3}$$

which, applied to h_n^*, gives

$$\mathrm{ch}(\mathrm{PF}_n) = (-1)^n \omega(h_n^*) = \sum_{I \vDash n} f_{i_1} \cdot f_{i_2} \cdots f_{i_r}\,. \tag{2.4}$$

Now, let us interpret this last formula. Parking functions can be classified according to the factorization of their nondecreasing reorderings $\mathbf{a}^\uparrow$ with respect to the operation of shifted concatenation. That is, if

$$\mathbf{a}^\uparrow = w_1 \bullet w_2 \bullet \cdots \bullet w_r \tag{2.5}$$

is the unique maximal factorization of $\mathbf{a}^\uparrow$, each w_i is a nondecreasing prime parking function. Let us define $i_k = |w_k|$ and let $I = (i_1, \ldots, i_r)$. We shall say that $\mathbf{a}$ is of *type* I and denote by PPF_I the set of parking functions of type I.

Then, the set PPF_n of prime parking functions of size n obviously is a sub-permutation representation of PF_n, and it remains to compute its Frobenius characteristic. We prove that it is f_n, so that Π_n can be identified with PPF_n. It is sufficient to show that the number of prime parking functions whose reordered evaluation is a given partition λ is equal to $\frac{1}{n-1}\binom{n-1}{l(\lambda)}\binom{l(\lambda)}{m_1,m_2,\ldots,m_n}$ where $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$. Indeed, this number corresponds to the number of ways of putting the λ_i over $n-1$ places in a circle ; there is one circular word associated with each circle whose reading is a prime parking function (see [**4**]). It then easily comes that

$$\mathrm{ch}(\mathrm{PPF}_n) = f_n\,, \tag{2.6}$$

so that Π_n can be identified with PPF_n, as claimed before.

As a consequence, the set PPF_I of parking functions of type I is a sub-permutation representation of PF_n too, and its Frobenius characteristic is

$$\mathrm{ch}(\mathrm{PPF}_I) = f_{i_1} \ldots f_{i_r}\,. \tag{2.7}$$

Summing over all compositions I of n finally gives the right interpretation of Equation (2.4). A more transparent proof is given in Section 3.8.

3. A Hopf algebra of parking functions

3.1. The algebra PQSym. We can embed the algebra of *Free Quasi-Symmetric functions* **FQSym** of [**3**] inside the algebra spanned by the elements $\mathbf{F_a}$ ($\mathbf{a} \in \mathrm{PF}$), whose multiplication rule is defined by

$$\mathbf{F}_{\mathbf{a}'}\mathbf{F}_{\mathbf{a}''} := \sum_{\mathbf{a}\in\mathbf{a}'\shuffle\mathbf{a}''} \mathbf{F_a}\,. \tag{3.1}$$

We shall call this algebra **PQSym** (Parking Quasi-Symmetric functions).

For example,

$$\mathbf{F}_{12}\mathbf{F}_{11} = \mathbf{F}_{1233} + \mathbf{F}_{1323} + \mathbf{F}_{1332} + \mathbf{F}_{3123} + \mathbf{F}_{3132} + \mathbf{F}_{3312}\,. \tag{3.2}$$

3.2. The coalgebra PQSym. There is a comultiplication on **PQSym** that naturally extends the comultiplication of **FQSym**. Recall (see [**15, 3**]) that if σ is a permutation,

$$\Delta\mathbf{F}_\sigma = \sum_{u\cdot v=\sigma} \mathbf{F}_{\mathrm{Std}(u)} \otimes \mathbf{F}_{\mathrm{Std}(v)}\,, \tag{3.3}$$

where Std denotes the usual notion of standardization of a word.

Given a word w, it is possible to define a notion of *parkization* $\mathrm{Park}(w)$, a parking function that coincides with $\mathrm{Std}(w)$ when w is a word without repetition.

For $w = w_1w_2\cdots w_n$ on $\{1,2,\ldots\}$, let us define

$$d(w) := \min\{i | \#\{w_j \le i\} < i\}\,. \tag{3.4}$$

If $d(w) = n+1$, then w is a parking function and the algorithm terminates, returning w. Otherwise, let w' be the word obtained by decrementing all the elements of w greater than $d(w)$. Then $\mathrm{Park}(w) := \mathrm{Park}(w')$. Since w' is smaller than w in the lexicographic order, the algorithm terminates and always returns a parking function.

For example, let $w = (3,5,1,1,11,8,8,2)$. Then $d(w) = 6$ and $w' = (3,5,1,1,10,7,7,2)$. Then $d(w') = 6$ and $w'' = (3,5,1,1,9,6,6,2)$. Finally, $d(w'') = 8$ and $w''' = (3,5,1,1,8,6,6,2)$, that is a parking function. Thus, $\mathrm{Park}(w) = (3,5,1,1,8,6,6,2)$.

Now, the comultiplication on **PQSym** in defined as

$$\Delta\mathbf{F_a} := \sum_{u\cdot v=\mathbf{a}} \mathbf{F}_{\mathrm{Park}(u)} \otimes \mathbf{F}_{\mathrm{Park}(v)}\,, \tag{3.5}$$

For example,

$$\Delta\mathbf{F}_{3132} = 1 \otimes \mathbf{F}_{3132} + \mathbf{F}_1 \otimes \mathbf{F}_{132} + \mathbf{F}_{21} \otimes \mathbf{F}_{21} + \mathbf{F}_{212} \otimes \mathbf{F}_1 + \mathbf{F}_{3132} \otimes 1\,. \tag{3.6}$$

One can easily check that the product and the comultiplication of **PQSym** are compatible, so that **PQSym** is endowed with a bialgebra structure.

3.3. The Hopf algebra PQSym. Since **PQSym** is endowed with a bialgebra structure naturally graded by the size of parking functions, one defines the antipode as the inverse of the identity for the convolution product and then endow **PQSym** with a Hopf algebra structure.

The formula for the antipode can be written on the basis of $\mathbf{F_a}$ functions, as

$$\nu(\mathbf{F_a}) = \sum_{r;u_1\cdots u_r=\mathbf{a};|u_i|\ge 1} (-1)^r\, \mathbf{F}_{\mathrm{Park}(u_1)}\mathbf{F}_{\mathrm{Park}(u_2)}\cdots\mathbf{F}_{\mathrm{Park}(u_r)} \tag{3.7}$$

For example,

$$\nu(\mathbf{F}_{122}) = -\mathbf{F}_{122} + \mathbf{F}_1\mathbf{F}_{11} + \mathbf{F}_{12}\mathbf{F}_1 - \mathbf{F}_1^3 = \mathbf{F}_{212} + \mathbf{F}_{221} - \mathbf{F}_{213} - \mathbf{F}_{231} - \mathbf{F}_{321}\,. \tag{3.8}$$

3.4. The graded dual $\mathbf{PQSym}^*$. Let $\mathbf{G_a} = \mathbf{F_a^*} \in \mathbf{PQSym}^*$ be the dual basis of $(\mathbf{F_a})$. If $\langle\, , \,\rangle$ denotes the duality bracket, the product on $\mathbf{PQSym}^*$ is given by

$$\mathbf{G_{a'}}\mathbf{G_{a''}} = \sum_{\mathbf{a}} \langle\, \mathbf{G_{a'}} \otimes \mathbf{G_{a''}}, \Delta\mathbf{F_a} \,\rangle\, \mathbf{G_a} = \sum_{\mathbf{a}\in\mathbf{a'}*\mathbf{a''}} \mathbf{G_a}\,, \tag{3.9}$$

where the *convolution* $\mathbf{a'}*\mathbf{a''}$ of two parking functions is defined as

$$\mathbf{a'}*\mathbf{a''} = \sum_{u,v;\mathbf{a}=u\cdot v, \mathrm{Park}(u)=\mathbf{a'}, \mathrm{Park}(v)=\mathbf{a''}} \mathbf{a}\,. \tag{3.10}$$

For example,

$$\begin{aligned} \mathbf{G}_{12}\mathbf{G}_{11} = \mathbf{G}_{1211} + \mathbf{G}_{1222} + \mathbf{G}_{1233} + \mathbf{G}_{1311} + \mathbf{G}_{1322} \\ + \mathbf{G}_{1411} + \mathbf{G}_{1422} + \mathbf{G}_{2311} + \mathbf{G}_{2411} + \mathbf{G}_{3411}\,. \end{aligned} \tag{3.11}$$

When restricted to permutations, it coincides with the convolution of [**19, 15**]. Remark that in particular,

$$\mathbf{G}_1^n = \sum_{\mathbf{a}\in\mathrm{PF}_n} \mathbf{G_a}\,. \tag{3.12}$$

Using the duality bracket once more, one easily gets the formula for the comultiplication of $\mathbf{G_a}$ as

$$\Delta\mathbf{G_a} := \sum_{u,v;\mathbf{a}\in u \Cup v} \mathbf{G}_{\mathrm{Park}(u)} \otimes \mathbf{G}_{\mathrm{Park}(v)}\,. \tag{3.13}$$

There also exists a direct way to define the comultiplication of $\mathbf{G_a}$ using the breakpoints of Gessel (see [**22**]). In particular, the number of terms in the coproduct is equal to the number of breakpoints of the parking function plus one.

For example,

$$\begin{aligned} \Delta\mathbf{G}_{41252} = 1 \otimes \mathbf{G}_{41252} + \mathbf{G}_1 \otimes \mathbf{G}_{3141} + \mathbf{G}_{122} \otimes \mathbf{G}_{12} \\ + \mathbf{G}_{4122} \otimes \mathbf{G}_1 + \mathbf{G}_{41252} \otimes 1\,, \end{aligned} \tag{3.14}$$

whereas 41252 has 4 breakpoints : 1, 3, 4, and 5.

3.5. Algebraic structure. Let us say that a word w over $\mathbb{N}^*$ is *connected* if it cannot be written as a shifted concatenation $w = u \bullet v$, and *anti-connected* if its mirror image $\overline{w}$ is connected.

Then, **PQSym** is free over the set

$$\{\mathbf{F_c} \,|\, \mathbf{c} \in \mathrm{PF},\ \text{connected}\} \tag{3.15}$$

ans $\mathbf{PQSym}^*$ is free over the set

$$\{\mathbf{G_d} \,|\, \mathbf{d} \in \mathrm{PF},\ \text{anti-connected}\} \tag{3.16}$$

This property proves that **PQSym** and $\mathbf{PQSym}^*$ are isomorphic as algebras. Moreover, it is possible to build an isomorphism φ between **PQSym** and $\mathbf{PQSym}^*$ that is compatible with the product and the comultiplication. So **PQSym** is isomorphic to $\mathbf{PQSym}^*$ as a *Hopf algebra.*

When restricted to **FQSym**, the isomorphism φ is defined by

$$\varphi(\mathbf{F}_\sigma) := \sum_{\mathbf{a},\mathrm{Std}(\mathbf{a})=\sigma^{-1}} \mathbf{G_a}\,. \tag{3.17}$$

The ordinary generating function for the numbers c_n of connected parking functions is

$$\begin{aligned}\sum_{n\geq 1} c_n t^n &= 1 - \left(\sum_{n\geq 0} (n+1)^{(n-1)} t^n\right)^{-1} \\ &= t + 2\,t^2 + 11\,t^3 + 92\,t^4 + 1014\,t^5 + 13795\,t^6 + 223061\,t^7 + 4180785\,t^8 \\ &\quad + 89191196\,t^9 + 2135610879\,t^{10} + 56749806356\,t^{11} + 1658094051392\,t^{12} \\ &\quad + O\left(t^{13}\right) .\end{aligned} \tag{3.18}$$

3.6. Multiplicative Bases. Let $\mathbf{a} = \mathbf{a}_1 \bullet \mathbf{a}_2 \bullet \cdots \bullet \mathbf{a}_r$ be the maximal factorization of $\mathbf{a}$ into connected parking functions. We set

$$\mathbf{F}^{\mathbf{a}} = \mathbf{F}_{\mathbf{a}_1} \cdot \mathbf{F}_{\mathbf{a}_2} \cdots \mathbf{F}_{\mathbf{a}_r} , \tag{3.19}$$

and

$$\mathbf{G}^{\overline{\mathbf{a}}} = \mathbf{G}_{\overline{\mathbf{a}_r}} \cdots \mathbf{G}_{\overline{\mathbf{a}_1}} . \tag{3.20}$$

By a triangular argument, one can easily see that $(\mathbf{F}^{\mathbf{a}})$ (resp. $(\mathbf{G}^{\overline{\mathbf{a}}})$), where $\mathbf{a}$ runs over the connected parking functions, is a multiplicative basis of **PQSym** (resp. **PQSym***).

Now, if $\mathbf{S}_{\mathbf{a}}$ (resp. $\mathbf{T}_{\mathbf{a}}$) is the dual basis of $\mathbf{F}^{\mathbf{a}}$ (resp. $\mathbf{G}^{\overline{\mathbf{a}}}$) then

$$\{\mathbf{S}_{\mathbf{c}} \,|\, \mathbf{c} \text{ connected}\} \text{ and } \{\mathbf{T}_{\mathbf{c}} \,|\, \mathbf{c} \text{ connected}\} \tag{3.21}$$

are bases of the primitive Lie algebras **LPQ*** (resp. **LPQ**) of **PQSym*** (resp. **PQSym**).

We conjecture, as in [**3**], that both Lie algebras are free, on generators whose degree generating function is

$$\begin{aligned}1 - \prod_{n\geq 1} (1-t^n)^{c_n} &= 1 - (1-t)(1-t^2)^2(1-t^3)^{11}\cdots \\ &= t + 2\,t^2 + 9\,t^3 + 80\,t^4 + 901\,t^5 + 12564\,t^6 + 206476\,t^7 \\ &\quad + 3918025\,t^8 + 84365187\,t^9 + 2034559143\,t^{10} + O\left(t^{11}\right) .\end{aligned} \tag{3.22}$$

3.7. Catalan Hopf algebra (non-crossing partitions).

3.7.1. *The Hopf algebra* **CQSym**. Parking functions are known to be related to non-crossing partitions (see [**2, 21, 22**]). There is a simple bijection between non-decreasing parking functions and non-crossing partitions. Starting with a non-crossing partition, *e.g.*,

$$\pi = 13|2|45 , \tag{3.23}$$

one replaces all the letters of each block by its minimum, and reorders them as a non-decreasing word

$$13|2|45 \to 11244 \tag{3.24}$$

which is a parking function. In the sequel, we identify non-decreasing parking functions and non-crossing partitions via this bijection.

For a general $\mathbf{a} \in \mathrm{PF}_n$, let $\mathrm{NC}(\mathbf{a})$ be the non-crossing partition corresponding to $\mathbf{a}^{\uparrow}$ by the inverse bijection, *e.g.*, $\mathrm{NC}(42141) = \pi$ as above. Then, the elements of **PQSym**

$$\mathbf{P}^{\pi} := \sum_{\mathbf{a};\mathrm{NC}(\mathbf{a})=\pi} \mathbf{F}_{\mathbf{a}} \tag{3.25}$$

span a sub-algebra of **PQSym**, isomorphic to the algebra of the free semigroup of non-crossing partitions under the operation of concatenation of diagrams,

$$\mathbf{P}^{\pi'}\mathbf{P}^{\pi''} = \mathbf{P}^{\pi'\bullet\pi''} , \tag{3.26}$$

that is equivalent to shifted concatenation on words. Notice that $\mathbf{P}^{\pi}$ is the sum of all permutations of the non-decreasing word corresponding to the given non-crossing partition. We call this algebra the *Catalan subalgebra* of **PQSym** and denote it by **CQSym**. The comultiplication is given on the basis $\mathbf{P}^\pi$ by

$$\Delta \mathbf{P}^\pi = \sum_{u,v;(u.v)^\uparrow=\pi} \mathbf{P}^{\mathrm{Park}(u)} \otimes \mathbf{P}^{\mathrm{Park}(v)}\,, \tag{3.27}$$

where u and v run over the set of non-decreasing words.

For example, one has

$$\begin{aligned}\Delta \mathbf{P}^{1124} &= 1 \otimes \mathbf{P}^{1124} + \mathbf{P}^1 \otimes \left(\mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{123}\right) + \mathbf{P}^{11} \otimes \mathbf{P}^{12} \\ &+ \mathbf{P}^{12} \otimes \left(\mathbf{P}^{11} + 2\mathbf{P}^{12}\right) + \left(\mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{123}\right) \otimes \mathbf{P}^1 + \mathbf{P}^{1124} \otimes 1\,.\end{aligned} \tag{3.28}$$

One can easily check that the product and the comultiplication of **CQSym** are compatible, so that **CQSym** is endowed with a graded bialgebra structure, and therefore, with a Hopf algebra structure. Formula (3.27) immediately proves that the coalgebra **CQSym** is co-commutative.

3.7.2. *The dual Hopf algebra* $\mathbf{CQSym}^*$. Let us denote by $\mathcal{M}_\pi$ the dual basis of $\mathbf{P}^\pi$ in the *commutative* algebra $\mathbf{CQSym}^*$. Remark that $\mathbf{CQSym}^*$ is the quotient of $\mathbf{PQSym}^*$ by the relations $\mathbf{G}_a \equiv \mathbf{G}_b$ if $a^\uparrow = b^\uparrow$. It is then immediate (see Equation (3.9)) that the multiplication is this basis is given by

$$\mathcal{M}_{\pi'}\mathcal{M}_{\pi''} = \sum_{\pi;\pi\in\pi' * \pi''} \mathcal{M}_{\pi^\uparrow}\,. \tag{3.29}$$

For example,

$$\begin{aligned}\mathcal{M}_{12}\mathcal{M}_{11} &= \mathcal{M}_{1112} + \mathcal{M}_{1113} + \mathcal{M}_{1114} + \mathcal{M}_{1123} + \mathcal{M}_{1124} \\ &+ \mathcal{M}_{1134} + \mathcal{M}_{1222} + \mathcal{M}_{1223} + \mathcal{M}_{1224} + \mathcal{M}_{1233}\,.\end{aligned} \tag{3.30}$$

This algebra can be embedded in the polynomial algebra $\mathbb{C}[x_1, x_2, \ldots]$ by

$$\mathcal{M}_\pi = \sum_{\mathbf{a}(w)=\pi} \underline{w}\,, \tag{3.31}$$

where $\underline{w}$ is the commutative image of w (*i.e.*, $i \mapsto x_i$).

For example,

$$\mathcal{M}_{111} = \sum_i x_i^3\,. \tag{3.32}$$

$$\mathcal{M}_{112} = \sum_i x_i^2 x_{i+1}\,. \tag{3.33}$$

$$\mathcal{M}_{113} = \sum_{i,j;j\geq i+2} x_i^2 x_j\,. \tag{3.34}$$

$$\mathcal{M}_{122} = \sum_{i,j;i<j} x_i x_j^2\,. \tag{3.35}$$

$$\mathcal{M}_{123} = \sum_{i,j,k;i<j<k} x_i x_j x_k\,. \tag{3.36}$$

Notice that $\mathcal{M}_{111} = M_3$; $\mathcal{M}_{112} + \mathcal{M}_{113} = M_{21}$; $\mathcal{M}_{122} = M_{12}$ and $\mathcal{M}_{123} = M_{111}$. In general, if $\pi = \pi_1 \bullet \cdots \bullet \pi_r$ is the factorization of π in connected parking functions, let $i_k := |\pi_k|$ and $c(\pi) := (i_1, \cdots, i_k)$ a composition of n. Then

$$\gamma(M_I) := \sum_{c(\pi)=I} \mathcal{M}_\pi \tag{3.37}$$

gives an embedding of *QSym* into $\mathbf{CQSym}^*$.

3.7.3. *Catalan Ribbon functions.* In the classical case, the non-commutative complete fonctions split into a sum of ribbon Schur functions, using a simple order on compositions. To get an analogous construction in our case, we define a partial order on non-decreasing parking functions.

Let π be a non-decreasing parking function and $\mathrm{Ev}(\pi)$ be its evaluation vector. The successors of π are the non-decreasing parking functions whose evaluations are given by the following algorithm: given two non-zero elements of $\mathrm{Ev}(\pi)$ with only zeroes between them, replace the left one by the sum of both and the right one by 0.

For example, the successors of 113346 are 111146, 113336, and 113344.

By transitive closure, the successor map gives rise to a partial order on non-decreasing parking functions. We will write $\pi \preceq \pi'$ if π' is obtained from π by successive applications of successor maps.

Now, define the Catalan Ribbon functions by

$$\mathbf{P}^{\pi} =: \sum_{\pi' \succeq \pi} \mathbf{R}_{\pi'} \,. \tag{3.38}$$

This last equation completely defines the $\mathbf{R}_{\pi}$.

The product of two $\mathbf{R}$ functions is then

$$\mathbf{R}_{\pi'}\mathbf{R}_{\pi''} = \mathbf{R}_{\pi' \bullet \pi''} + \mathbf{R}_{\pi' \triangleright \pi''} \,, \tag{3.39}$$

where $\triangleright$ is the shifted concatenation defined by shifting all elements of π'' by the difference between the greatest and the smallest element of π'.

For example,

$$\mathbf{R}_{11224}\mathbf{R}_{113} = \mathbf{R}_{11224668} + \mathbf{R}_{11224446} \,. \tag{3.40}$$

3.8. Compositions. Recall that non-crossing partitions can be classified according to the factorization $\pi = \pi_1 \bullet \cdots \bullet \pi_r$ into irreducible non-crossing partitions. We set

$$\mathbf{V}^{I} := \sum_{c(\pi)=I} \mathbf{P}^{\pi} \tag{3.41}$$

as an element of **PQSym**. If one defines $\mathbf{V}_n = \mathbf{V}^{(n)}$, we have

$$\mathbf{V}_n = \sum_{\mathbf{a} \in \mathrm{PPF}_n} \mathbf{F}_{\mathbf{a}} \tag{3.42}$$

and

$$\mathbf{V}^{I} = \mathbf{V}_{i_1} \cdots \mathbf{V}_{i_r} = \sum_{\mathbf{a} \in \mathrm{PPF}_I} \mathbf{F}_{\mathbf{a}} \,. \tag{3.43}$$

At this point, it is useful to observe that if $C(w)$ denotes the descent composition of a word w, the map

$$\eta : \mathbf{F}_{\mathbf{a}} \mapsto F_{C(\mathbf{a})} \,, \tag{3.44}$$

which is a Hopf algebra morphism $\mathbf{PQSym} \to QSym$, maps $\mathbf{V}^I$ to the Frobenius characteristic of the underlying permutation representation of $\mathfrak{S}_n$ on PPF_I.

$$\eta(\mathbf{V}^I) = \sum_{\mathbf{a} \in \mathrm{PPF}_I} \mathbf{F}_{C(\mathbf{a})} = \mathrm{ch}(\mathrm{PPF}_I) \,. \tag{3.45}$$

As a consequence, the number of parking functions of type I with descent set J is equal to the scalar product of symmetric functions

$$\langle r_J, f^I \rangle \tag{3.46}$$

where $f^I = f_{i_1} \cdots f_{i_r} = \mathrm{ch}(\mathrm{PPF}_I)$ and r_J is the ribbon Schur function. This extends Prop. 3.2.(a) of [**21**]. Remark that in particular,

$$\mathbf{F}_{\mathrm{PF}_n} := \sum_{\mathbf{a} \in \mathrm{PF}_n} \mathbf{F}_{\mathbf{a}} = \sum_{I \vDash n} \mathbf{V}^I \,, \tag{3.47}$$

a realisation of Equation (2.4) as an identity in **PQSym**. By inversion, one obtains

$$\mathbf{F}_{\mathrm{PPF}_n} = \sum_{I \vDash n} (-1)^{n-l(I)} \mathbf{F}_{\mathrm{PF}_I}\,, \tag{3.48}$$

where

$$\mathrm{PF}_I := \mathrm{PF}_{i_1} \Cup \mathrm{PF}_{i_2} \Cup \cdots \Cup \mathrm{PF}_{i_r}\,. \tag{3.49}$$

These identities are easily visualized on the encoding of parking functions with skew Young diagrams as in [**17**] or in [**7**].

The transpose γ^* of the map γ defined in Equation (3.37), is the map

$$\begin{aligned} \mathbf{ch} :& \mathbf{CQSym}^* \to \mathbf{Sym} \\ & \mathbf{P}^\pi \mapsto S^{c(\pi)}\,. \end{aligned} \tag{3.50}$$

which sends $\mathbf{P}^\pi$ to the characteristic non-commutative symmetric function of the natural projective $H_n(0)$-module with basis $\{\mathbf{a} \in \mathrm{PF}_n | NC(\mathbf{a}) = \pi\}$.

Then,

$$g := \sum_{n \geq 0} g_n := \sum_{n \geq 0} \mathbf{ch}(\mathbf{F}_{PF_n}) = \sum_I \mathbf{ch}(V^I). \tag{3.51}$$

is the series obtained by applying the non-commutative Lagrange inversion formula of [**6, 18**] to the generating series of complete functions, *i.e.*, g is the unique solution of the equation

$$g = 1 + S_1 g + S_2 g^2 + \cdots = \sum_{n \geq 0} S_n g^n\,. \tag{3.52}$$

3.9. Schröder Hopf algebra (planar trees). Let $\equiv$ denote the hypoplactic congruence (see [**11, 16**]), and denote by $\mathsf{P}(w)$ the hypoplactic P-symbol of a word w (its quasi-ribbon). P-symbols of parking functions are called *parking quasi-ribbons.*

With a parking quasi-ribbon $\mathbf{q}$, we associate the element

$$\mathbf{P}_{\mathbf{q}} := \sum_{P(\mathbf{a}) = \mathbf{q}} \mathbf{F}_{\mathbf{a}}\,. \tag{3.53}$$

Then, the $\mathbf{P}_{\mathbf{q}}$ form the basis of a Hopf sub-algebra of **PQSym**, denoted by **SQSym**. Its dual **SQSym*** is the quotient **PQSym**$/\mathcal{J}$ where $\mathcal{J}$ is the two-sided ideal generated by

$$\{\mathbf{G}_{\mathbf{a}} - \mathbf{G}_{\mathbf{a}'} | \mathbf{a} \equiv \mathbf{a}'\}\,. \tag{3.54}$$

If $\overline{\mathbf{G}_{\mathbf{a}}}$ denoted the equivalence class of $\mathbf{G}_{\mathbf{a}}$ modulo $\mathcal{J}$, the dual basis of $(\mathbf{P}_{\mathbf{q}})$ is

$$\mathbf{Q}_{\mathbf{q}} := \overline{\mathbf{G}_{\mathbf{a}}}\,, \tag{3.55}$$

where $\mathbf{a}$ is any parking function such that $\mathbf{a} \equiv \mathbf{q}$.

The dimension of the component of degree n of **SQSym** and **SQSym*** is the little Schröder number (or super-Catalan) s_n : their Hilbert series is

$$\sum_{n \geq 0} s_n t^n = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t} = 1 + t + 3t^2 + 11t^3 + 45t^4 + \cdots \tag{3.56}$$

Indeed,

$$\begin{aligned} \dim(\mathbf{SQSym}_n) &= \left\langle \sum_{I \vDash n} F_I, \mathbf{ch}(\mathbf{F}_{\mathrm{PF}_n}) \right\rangle = \left\langle \frac{1}{2} \sum_{k=0}^{n} e_k h_{n-k}, \frac{1}{n+1} h_n((n+1)X) \right\rangle \\ &= \frac{1}{2n+2} \sum_{k=0}^{n} \binom{n+1}{k} \binom{2n-k}{n-k} = s_n\,. \end{aligned} \tag{3.57}$$

The embedding of Formula (3.17) induces an embedding

$$\mathit{QSym} \simeq \mathbf{FQSym}^* / (\mathcal{J} \cap \mathbf{FQSym}^*) \to \mathbf{PQSym}^* / \mathcal{J} = \mathbf{SQSym}^*\,. \tag{3.58}$$

It is likely that **SQSym** is isomorphic to the free dendriform trialgebra of [**13**] as an algebra, but not as a coalgebra.

3.10. PQSym* as a combinatorial Hopf algebra. Since **FQSym** can be embedded in **PQSym**, we have a canonical Hopf embedding of **Sym** in **PQSym** given by

$$S_n \mapsto \mathbf{F}_{12\cdots n}\,. \tag{3.59}$$

With parking functions, we have other possibilities: for example,

$$j(S_n) := \mathbf{F}_{11\cdots 1} \tag{3.60}$$

is a Hopf embedding, whose dual j^* maps **PQSym*** to *QSym* and therefore endows **PQSym*** with a different structure of combinatorial Hopf algebra in the sense of [**1**].

On the dual side, the transpose η^* of the map η defined in the previous section corresponds to the Hopf embedding

$$S_n \mapsto \sum_{\mathrm{Std}(\mathbf{a})=12\cdots n} \mathbf{G}_{\mathbf{a}} \tag{3.61}$$

of **Sym** into **PQSym***, which is therefore the restriction of the self-duality isomorphism of formula (3.17) to the **Sym** subalgebra $S_n = \mathbf{F}_{12\cdots n}$ of **PQSym**.

4. Realization of PQSym

It is possible to find a realization of **PQSym** in terms of $(0,1)$-matrices, that is reminiscent of the construction of **MQSym** (see [**9, 3**]), and that coincides with it when restricted to permutation matrices, providing the natural embedding of **FQSym** in **MQSym**.

Let $\mathcal{M}_n$ be the vector space spanned by symbols X_M where M runs over $(0,1)$-matrices with n columns and an infinite number of rows, with n nonzero entries, so that at most n rows are nonzero.

Given such a matrix M, we define its *vertical packing* $P = \mathrm{vp}(M)$ as the finite matrix obtained by removing the null rows of M.

For a vertically packed matrix P, we define

$$\mathbf{M}_P = \sum_{\mathrm{vp}(M)=P} X_M\,. \tag{4.1}$$

Now, given a $(0,1)$-matrix, we define its reading $r(M)$ as the word obtained by reading its entries by rows, from left to right and top to bottom and recording the numbers of the columns of the ones. For example, the reading of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} \tag{4.2}$$

is $(2,3,1,2)$.

A matrix M is said to be of *parking type* if $r(M)$ is a parking function. Finally, for a parking function $\mathbf{a}$, we set

$$\mathbf{F}_{\mathbf{a}} := \sum_{r(P)=\mathbf{a},P \text{ vertically packed}} \mathbf{M}_P = \sum_{r(M)=\mathbf{a}} X_M\,. \tag{4.3}$$

For example,

$$\mathbf{F}_{(1,2,2)} = \mathbf{M}_{\begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}} + \mathbf{M}_{\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}}\,. \tag{4.4}$$

The multiplication on $\mathcal{M} = \oplus_n \mathcal{M}_n$ is defined by columnwise concatenation of the matrices:

$$X_M X_N = X_{M\cdot N}\,. \tag{4.5}$$

In order to explicit the product of $\mathbf{M}_P$ by $\mathbf{M}_Q$, we first need a definition. Let P and Q be two vertically packed matrices with respective heights p and q. The *augmented shuffle* of P and Q is defined as follows: let r be an integer in $[\max(p,q), p+q]$. One inserts zero rows in P and Q in all possible ways so that the resulting matrices have $p+q$ rows. Let R be the matrix obtained by concatenation of such pairs of matrices. The augmented shuffle consists in the set of such matrices R with nonzero rows. We denote this set by $\uplus(P,Q)$.

With this notation,

$$\mathbf{M}_P\mathbf{M}_Q = \sum_{R\in\uplus(P,Q)} \mathbf{M}_R\,, \tag{4.6}$$

and also

$$\mathbf{F}_{\mathbf{a}'}\mathbf{F}_{\mathbf{a}''} = \sum_{\mathbf{a}\in\mathbf{a}'\Cup\mathbf{a}''} \mathbf{F}_{\mathbf{a}}\,, \tag{4.7}$$

that is the same as Equation (3.1).

Finally, concerning the comultiplication, one has first to define the parkization $\operatorname{Park}(M)$ of a vertically packed matrix M, which consists in iteratively removing column $d(r(M))$ until M becomes a parking matrix.

The comultiplication of a matrix $\mathbf{M}_P$ is then defined as:

$$\Delta\mathbf{M}_P = \sum_{Q\cdot R=P} \mathbf{M}_{\operatorname{Park}(Q)}\otimes\mathbf{M}_{\operatorname{Park}(R)}\,, \tag{4.8}$$

It is then easy to check that

$$\Delta\mathbf{F}_{\mathbf{a}} = \sum_{u\cdot v=\mathbf{a}} \mathbf{F}_{\operatorname{Park}(u)}\otimes\mathbf{F}_{\operatorname{Park}(v)}\,, \tag{4.9}$$

which is the same as Equation (3.3).

4.1. Realization of FQSym. A parking matrix M is said to be a *word matrix* if there is exactly one 1 in each column. Then **FQSym** is the Hopf subalgebra generated by the parking word matrices.

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