



Littelmann Paths for Affine Lie Algebras

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ABSTRACT. We give a new combinatorial model for the crystal graphs of an affine Lie algebra $\widehat{\mathfrak{g}}$, unifying Littelmann's path model with the Kyoto path model. The vertices of the crystal graph are represented by certain infinitely looping paths which we call skeins.

We apply this model to the case when the corresponding finite-dimensional algebra \mathfrak{g} has a minuscule representation (classical type and E_6, E_7). We prove that the basic level-one representation of $\widehat{\mathfrak{g}}$, when considered as a representation of \mathfrak{g} , is an infinite tensor product of fundamental representations of \mathfrak{g} .

This is the infinite limit of a finer result: that the finite-dimensional Demazure submodules of the basic representation are finite tensor products. The corresponding Demazure characters give generalizations of the Hall-Littlewood polynomials.

This paper is an extended abstract of [Mag].

1. Littelmann's path model

Littelmann's combinatorial model [Lit1],[Lit2],[LLM2] for the representations of a Kac-Moody algebra \mathfrak{g} is a vast generalization of Young tableaux. Littelmann's paths and path operators give a flexible construction of the crystal graphs associated to quantum \mathfrak{g} -modules by Kashiwara [K1] and Lusztig [Lus] (see also [Jos],[HK]). We briefly sketch Littelmann's theory.

For concreteness, let \mathfrak{g} be a complex simple Lie algebra. For our purposes, we define a \mathfrak{g} -crystal as a set \mathcal{B} with a *weight function*, $\text{wt} : \mathcal{B} \rightarrow \bigoplus_{i=1}^r \mathbb{Z}\varpi_i$, as well as partially defined *crystal operators* $e_1, \dots, e_r, f_1, \dots, f_r : \mathcal{B} \rightarrow \mathcal{B}$ satisfying:

$$\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i \quad \text{and} \quad e_i(b) = b' \iff f_i(b') = b.$$

Here $\varpi_1, \dots, \varpi_r$ are the fundamental weights and $\alpha_1, \dots, \alpha_r$ are the roots of \mathfrak{g} . A *dominant* element is a $b \in \mathcal{B}$ such that $e_i(b)$ is not defined for any i . We say that a crystal \mathcal{B} is a *model* for a \mathfrak{g} -module V if the formal character of \mathcal{B} is equal to the character of V , and the dominant elements of \mathcal{B} correspond to the highest-weight vectors of V . That is:

$$\text{char}(V) = \sum_{b \in \mathcal{B}} e^{\text{wt}(b)} \quad \text{and} \quad V \cong \bigoplus_{b \in \text{dom}} V(\text{wt}(b)),$$

where the second sum is over the dominant elements of \mathcal{B} . Clearly, a \mathfrak{g} -module V is determined up to isomorphism by any model \mathcal{B} .

We construct such \mathfrak{g} -crystals \mathcal{B} consisting of polygonal paths in the vector space of weights, $\mathfrak{h}_{\mathbb{R}}^* := \bigoplus_{i=1}^r \mathbb{R}\varpi_i$. Specifically:

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- The *elements* of \mathcal{B} are certain continuous piecewise-linear mappings $\pi : [0, 1] \rightarrow \mathfrak{h}_{\mathbb{R}}^*$, up to reparametrization, with initial point $\pi(0) = 0$. We use the notation $\pi = (v_1 \star v_2 \star \cdots \star v_k)$, where $v_1, \dots, v_k \in \mathfrak{h}_{\mathbb{R}}^*$ are vectors, to denote the polygonal path starting at 0 and moving linearly to v_1 , then to $v_1 + v_2$, etc.
- The *weight* of a path is its endpoint:

$$\text{wt}(\pi) := \pi(1) = v_1 + \cdots + v_k.$$

- The *crystal lowering operator* f_i is defined as follows (and there is a similar definition of the raising operator e_i). Let \star denote the natural associative operation of concatenation of paths, and let any linear map $w : \mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathfrak{h}_{\mathbb{R}}^*$ act pointwise on paths: $w(\pi) := (w(v_1) \star \cdots \star w(v_k))$. We will divide a path π into three well-defined sub-paths, $\pi = \pi_1 \star \pi_2 \star \pi_3$, and reflect the middle piece by the simple reflection s_i :

$$f_i \pi := \pi_1 \star s_i \pi_2 \star \pi_3.$$

The pieces π_1, π_2, π_3 are determined according to the behavior of the i -height function $h_i(t) = h_i^\pi(t) := \langle \pi(t), \alpha_i^\vee \rangle$. As the point $\pi(t)$ moves along the path from $\pi(0) = 0$ to $\pi(1) = \text{wt}(\pi)$, this function may attain its minimum value $h_i(t) = M$ several times. If, after the *last* minimum point, $h_i(t)$ never rises to the value $M+1$, then $f_i \pi$ is *undefined*. Otherwise, we define π_2 as the last sub-path of π on which $M \leq h_i(t) \leq M+1$, and π_1, π_3 as the remaining initial and final pieces of π .

A key advantage of the path model is that the crystal operators, while complicated, are universally defined for all paths. Hence a path crystal is completely specified by giving its set of paths \mathcal{B} .

Also, the dominant elements have a neat pictorial characterization, as the paths π which never leave the fundamental Weyl chamber: that is, $h_i^\pi(t) \geq 0$ for all $t \in [0, 1]$ and all $i = 1, \dots, r$. For simplicity we restrict ourselves to *integral* dominant paths, meaning that all the steps are integral weights: $v_1, \dots, v_k \in \oplus_{i=1}^r \mathbb{Z} \varpi_i$. (For arbitrary dominant paths, see [Lit2].)

Littelmann's Character Theorem [Lit2] states that if π is any integral dominant path with weight λ , then the set of paths $\mathcal{B}(\pi)$ generated from π by f_1, \dots, f_r is a model for the irreducible \mathfrak{g} -module $V(\lambda)$. (This $\mathcal{B}(\pi)$ is also closed under e_1, \dots, e_r .) Note that we can choose *any* integral path π which stays within the Weyl chamber and ends at λ , and each such choice gives a different (but isomorphic) path crystal modelling $V(\lambda)$. In principle, any reasonable indexing set for a basis of $V(\lambda)$ should be in natural bijection with $\mathcal{B}(\pi)$ for some choice of π . For example, classical Young tableaux correspond to choosing the steps v_j to be coordinate vectors in $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^n$.

Furthermore, we have Littelmann's Product Theorem [Lit2]: if π_1, \dots, π_m are dominant integral paths of respective weight $\lambda_1, \dots, \lambda_m$, then $\mathcal{B}(\pi_1) \star \cdots \star \mathcal{B}(\pi_m)$, the set of all concatenations, is a model for the tensor product $V(\lambda_1) \otimes \cdots \otimes V(\lambda_m)$.

Everything we have said also holds for the corresponding affine algebra [Kac, Ch. 6 and 7]:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

provided we replace the roots $\alpha_1, \dots, \alpha_r$ of \mathfrak{g} by the roots $\alpha_0, \alpha_1, \dots, \alpha_r$ of $\widehat{\mathfrak{g}}$; and the weights $\varpi_1, \dots, \varpi_r$ of \mathfrak{g} by the weights $\Lambda_0, \Lambda_1, \dots, \Lambda_r$ of $\widehat{\mathfrak{g}}$. We also replace the vector space $\mathfrak{h}_{\mathbb{R}}^*$ by $\widehat{\mathfrak{h}}_{\mathbb{R}}^* := \oplus_{i=0}^r \mathbb{R} \Lambda_i \oplus \mathbb{R} \delta$, where δ is the non-divisible positive imaginary root of $\widehat{\mathfrak{g}}$. (Indeed, Littelmann's theory works uniformly for all symmetrizable Kac-Moody algebras.) We denote representations and path crystals of \mathfrak{g} as $V(\lambda)$ and \mathcal{B} , and the corresponding objects for $\widehat{\mathfrak{g}}$ as $\widehat{V}(\Lambda)$ and $\widehat{\mathcal{B}}$.

We can also model the affine Demazure module $\widehat{V}_z(\Lambda) := U(\widehat{\mathfrak{n}}_+) \cdot v_{z\Lambda}$, where $\widehat{\mathfrak{n}}_+$ is the algebra spanned by the positive weight-spaces of $\widehat{\mathfrak{g}}$, $z \in \widehat{W}$ is a Weyl group element, and $v_{z\Lambda}$ is a non-zero vector of extremal weight $z\Lambda$ in $\widehat{V}(\Lambda)$. Demazure modules are always finite-dimensional vector spaces. If $z = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition and π is an integral dominant path of weight Λ , we define the *Demazure path crystal*:

$$\widehat{\mathcal{B}}_z(\pi) := \{f_{i_1}^{k_1} \cdots f_{i_m}^{k_m} \pi \mid k_1, \dots, k_m \geq 0\}.$$

Because of the local nilpotence of the lowering operators, this is always a finite set.

Then the formal character of $\widehat{\mathcal{B}}_z(\pi)$ is equal to the character of $\widehat{V}_z(\Lambda)$, and π is the unique dominant path [Lit1]. Now suppose $z = t_{-\lambda^\vee}$, an anti-dominant translation in \widehat{W} , so that $\widehat{V}_{\lambda^\vee}(\Lambda) := \widehat{V}_z(\Lambda)$ is a \mathfrak{g} -submodule of $\widehat{V}(\Lambda)$; and consider $\widehat{\mathcal{B}}_{\lambda^\vee}(\pi) := \widehat{\mathcal{B}}_z(\pi)$ as a \mathfrak{g} -crystal by forgetting the action of f_0, e_0 and projecting the

affine weight function to $\mathfrak{h}_{\mathbb{R}}^*$. Then Littelmann's Restriction Theorem [Lit2] implies that the \mathfrak{g} -crystal $\hat{\mathcal{B}}_{\lambda^\vee}(\pi)$ is a model for the \mathfrak{g} -module $\hat{V}_{\lambda^\vee}(\Lambda)$.

2. The Skein model

For the case of an affine algebra $\hat{\mathfrak{g}}$, we introduce a generalization of Littelmann's model by allowing certain infinite paths.

Let us introduce a notation for a path π which emphasizes the vector steps going toward the endpoint $\Lambda = \text{wt}(\pi)$ rather than away from the starting point 0. Define

$$\pi = (\star v_k \star \cdots \star v_1 \vdash \Lambda) := (v' \star v_k \star \cdots \star v_1),$$

the path with endpoint Λ , last step v_1 , etc, and first step $v' := \Lambda - (v_k + \cdots + v_1)$, a makeweight to assure that the steps add up to Λ .

A *skein* is an infinite list:

$$\pi = (\cdots \star v_2 \star v_1 \vdash \Lambda),$$

where $\Lambda \in \oplus_{i=0}^r \mathbb{Z}\Lambda_i$ and $v_j \in \mathfrak{h}_{\mathbb{R}}^*$ (not $\hat{\mathfrak{h}}_{\mathbb{R}}^*$), subject to conditions (i) and (ii) below. For $i = 0, \dots, r$ and $k > 0$, define:

$$h_i[k] := \langle \Lambda - (v_1 + \cdots + v_k), \alpha_i^\vee \rangle.$$

We require:

- (i) For each i and all $k \gg 0$, we have $h_i[k] \geq 0$.
- (ii) For each i , there are infinitely many k such that $h_i[k] = 0$.

We think of the skein π as a ‘‘projective limit’’ of the paths

$$\pi[k] := (\star v_k \star \cdots \star v_1 \vdash \Lambda) \quad \text{as } k \rightarrow \infty.$$

The conditions on π assure that only a finite number of steps of π lie outside the fundamental chamber \hat{C} , and that π touches each wall of \hat{C} infinitely many times. Note that π stays always at the level $\ell = \langle \Lambda, K \rangle$.

LEMMA 2.1. *For a skein π and $i = 0, \dots, r$, one of the following is true:*

- (i) $f_i(\pi[k])$ is undefined for all $k \gg 0$;
- (ii) there is a unique skein π' such that $\pi'[k] = f_i(\pi[k])$ for all $k \gg 0$.

In the second case, we define $f_i\pi := \pi'$.

PROOF. Recall that a path π is i -neutral if $h_i^\pi(t) \geq 0$ for all t and $h_i^\pi(1) = 0$. For a fixed i , divide π into a concatenation: $\pi = (\cdots \star \pi_2 \star \pi_1 \star \pi_0 \vdash \Lambda)$, where each π_j is an i -neutral finite path except for π_0 , which is an arbitrary finite path. Now it is clear that if $f_i(\pi_0)$ is undefined, then (i) holds. Otherwise (ii) holds and

$$f_i\pi = (\cdots \star \pi_2 \star \pi_1 \star f_i(\pi_0) \vdash \Lambda - \alpha_i).$$

□

We can immediately carry over the definitions of the path model to skeins, including that of (Demazure) path crystals. For example, we say that π is an integral dominant skein if $\pi[k]$ is integral dominant for $k \gg 0$, and hence for all k . There exist integral dominant skeins of level $\ell = 1$ only when \mathfrak{g} has a minuscule coweight. We cannot concatenate two skeins, but we can concatenate a skein π_1 and a path π_0 : that is, $\pi_1 \star \pi_0 := (\pi_1 \star \pi_0 \vdash \text{wt}(\pi_1) + \text{wt}(\pi_0))$.

PROPOSITION 2.1. *For an integral dominant skein π of weight Λ , the crystal $\hat{\mathcal{B}}(\pi)$ is a model for $\hat{V}(\Lambda)$, and $\hat{\mathcal{B}}_z(\pi)$ is a model for the Demazure module $\hat{V}_z(\Lambda)$.*

PROOF. Given an integral dominant skein π and a Weyl group element $z \in \widetilde{W}$, we can divide $\pi = \pi_1 \star \pi_0$ in such a way that the Demazure operator \hat{B}_z acts on π by reflecting intervals in π_0 rather than π_1 . This gives an isomorphism between the Demazure crystals generated by the path $\text{wt}(\pi_1) \star \pi_0$ and by the skein π :

$$\begin{array}{ccc} \hat{B}_z(\text{wt}(\pi_1) \star \pi_0) & \xrightarrow{\sim} & \hat{B}_z(\pi_1 \star \pi_0) = \hat{B}_z(\pi) \\ \text{wt}(\pi_1) \star \pi' & \mapsto & \pi_1 \star \pi' \end{array} .$$

This proves the assertion about Demazure modules.

Now, given an infinite chain of Weyl group elements $z_1 < z_2 < \dots$, we have the morphisms of $\widehat{\mathfrak{g}}$ -crystals:

$$\begin{array}{ccccc} \hat{B}_{z_1}(\Lambda) & \xleftarrow{\sim} & \hat{B}_{z_1}(\text{wt}(\pi_1) \star \pi_0) & \xrightarrow{\sim} & \hat{B}_{z_1}(\pi) \\ \cap & & & & \cap \\ \hat{B}_{z_2}(\Lambda) & \xleftarrow{\sim} & \hat{B}_{z_2}(\text{wt}(\pi'_1) \star \pi'_0) & \xrightarrow{\sim} & \hat{B}_{z_2}(\pi) \\ \cap & & & & \cap \\ \vdots & & \vdots & & \vdots \\ \hat{B}(\Lambda) & & & & \hat{B}(\pi) \end{array}$$

Here $\hat{B}_z(\Lambda)$ denotes the canonical path crystal of Lakshmibai-Seshadri paths, generated from the straight-line path (Λ) . Since the $\widehat{\mathfrak{g}}$ crystals at the bottom are the unions of their Demazure crystals, they are isomorphic: $\hat{B}(\Lambda) \cong \hat{B}(\pi)$. \square

3. Product theorems

As before, we let $\widehat{\mathfrak{g}}$ be the untwisted affine Kac-Moody algebra corresponding to the complex simple algebra \mathfrak{g} . The basic representation $\hat{V}(\Lambda_0)$, the fundamental representation corresponding to the distinguished node of the extended Dynkin diagram, is the simplest and most important $\widehat{\mathfrak{g}}$ -module (cf. [Kac, Ch. 14],[PS, Ch. 10]).

One of its remarkable properties is the Tensor Product Phenomenon. In many cases, the Demazure modules $\hat{V}_z(\Lambda_0) \subset \hat{V}(\Lambda_0)$ are representations of the finite-dimensional algebra \mathfrak{g} , and they factor into a tensor product of many small \mathfrak{g} -modules. Hence the full $\hat{V}(\Lambda_0)$ could be constructed by extending the \mathfrak{g} -structure on the semi-infinite tensor power $V \otimes V \otimes \dots$ of a small \mathfrak{g} -module V .

The Kyoto school of Jimbo, Kashiwara, et al. has established this phenomenon in many cases (and for a large class of $\widehat{\mathfrak{g}}$ -modules $\hat{V}(\Lambda)$) via the theory of perfect crystals [KKMMNN], [KMOTU1], [KMOTU2], [HK], [K2] a development of their earlier theory of semi-infinite paths [DJKMO]. See especially [HKKOT]. Pappas and Rapoport [PR] have given a geometric version of the phenomenon for type A : they construct a flat deformation of Schubert varieties of the affine Grassmannian into a product of finite Grassmannians.

We extend the Tensor Product Phenomenon for $\hat{V}(\Lambda_0)$ to the non-classical types E_6 and E_7 by a uniform method which applies whenever \mathfrak{g} possesses a minuscule representation, or more precisely a minuscule coweight. We shall rely on a key property of such coweights which may be taken as the definition. Let \hat{X} be the extended Dynkin diagram (the diagram of $\widehat{\mathfrak{g}}$). A coweight ϖ^\vee of \mathfrak{g} is *minuscule* if and only if it is a fundamental coweight $\varpi^\vee = \varpi_i^\vee$ and there exists an automorphism σ of \hat{X} taking the node i to the distinguished node 0. Such automorphisms exist in types A, B, C, D, E_6, E_7 .

We let $V(\lambda)$ denote the irreducible \mathfrak{g} -module with highest weight λ , and $V(\lambda)^*$ its dual module. Our main representation-theoretic result is:

THEOREM 3.1. *Let λ^\vee be an element of the coroot lattice of \mathfrak{g} which is a sum:*

$$\lambda^\vee = \lambda_1^\vee + \dots + \lambda_m^\vee,$$

where $\lambda_1^\vee, \dots, \lambda_m^\vee$ are minuscule fundamental coweights (not necessarily distinct), with corresponding fundamental weights $\lambda_1, \dots, \lambda_m$. Let $\hat{V}_{\lambda^\vee}(\Lambda_0) \subset \hat{V}(\Lambda_0)$ be the Demazure module corresponding to the anti-dominant translation $t_{-\lambda^\vee}$ in the affine Weyl group.

Then there is an isomorphism of \mathfrak{g} -modules:

$$\hat{V}_{\lambda^\vee}(\Lambda_0) \cong V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_m)^*.$$

Now fix a minuscule coweight ϖ^\vee and its corresponding fundamental weight ϖ . Let N be the smallest positive integer such that $N\varpi^\vee$ lies in the coroot lattice of \mathfrak{g} . Then we have the following characterization of the basic irreducible $\hat{\mathfrak{g}}$ -module:

THEOREM 3.2. *The tensor power $V_N := V(\varpi)^{\otimes N}$ possesses non-zero \mathfrak{g} -invariant vectors. Fix such a vector v_N , and define the \mathfrak{g} -module $V^{\otimes \infty}$ as the direct limit of the sequence:*

$$V_N \hookrightarrow V_N^{\otimes 2} \hookrightarrow V_N^{\otimes 3} \hookrightarrow \cdots$$

where each inclusion is defined by: $v \mapsto v_N \otimes v$.

Then $\hat{V}(\Lambda_0)$ is isomorphic as a \mathfrak{g} -module to $V^{\otimes \infty}$.

It would be interesting to define the action of the full algebra $\hat{\mathfrak{g}}$ on $V^{\otimes \infty}$, and thus give a uniform “path construction” of the basic representation (cf. [DJKMO]): that is, to define the raising and lowering operators E_0, F_0 , as well as the energy operator d . Combinatorial definitions of the energy for \mathfrak{g} of classical type produce generalizations of the Hall-Littlewood and Kostka-Foulkes polynomials (c.f. [Oka]), with connections to Macdonald polynomials [San], [Ion].

4. Crystal theorems

We prove Theorem 3 by reducing it to an identity of paths: we construct a path crystal for the affine Demazure module which is at the same time a path crystal for the tensor product.

For λ a dominant weight, define its dual weight λ^* by the dual \mathfrak{g} -module: $V(\lambda^*) = V(\lambda)^*$.

THEOREM 4.1. *Let λ^\vee be as in Theorem 3, and let $\mathcal{B}(\lambda)$ denote the path crystal generated by the straight-line path (λ) . Then the set of concatenated paths $\Lambda_0 \star \mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$ is a path crystal for the Demazure module $\hat{V}_{\lambda^\vee}(\Lambda_0)$. In fact, there is a unique $\hat{\mathfrak{g}}$ -dominant path π with weight Λ_0 such that:*

$$\hat{\mathcal{B}}_{\lambda^\vee}(\pi) = \Lambda_0 \star \mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*) \pmod{\mathbb{R}\delta}.$$

This is to be understood as an equality of sets of paths in $\hat{\mathfrak{h}}_{\mathbb{R}}^*$ mod $\mathbb{R}\delta$, and hence an isomorphism of $\hat{\mathfrak{g}}$ -crystals.

PROOF. Let σ_j be the automorphism of the diagram \hat{X} corresponding to the minuscule coweight λ_j^\vee for $j = 1, \dots, m$. This also defines an automorphism of $\hat{\mathfrak{h}}^*$ by $\sigma(\Lambda_i) = \Lambda_{\sigma(i)}$. We define π_m inductively as the last of a sequence of paths $\pi_0, \pi_1, \dots, \pi_m$:

$$\pi_0 := \Lambda_0, \quad \pi_j := \sigma_j^{-1}(\pi_{j-1} \star \lambda_j^*).$$

We may picture the path π_m as jumping from 0 up to level Λ_0 , winding horizontally around the fundamental alcove $A \subset \mathfrak{h}_{\mathbb{R}}^* + \Lambda_0$, and ending at Λ_0 .

We prove the Theorem by showing that the Demazure operator $\hat{\mathcal{B}}_{\lambda^\vee} = \hat{\mathcal{B}}_{\lambda_1^\vee} \hat{\mathcal{B}}_{\lambda_2^\vee} \cdots \hat{\mathcal{B}}_{\lambda_m^\vee}$ “unwinds” π_m starting from its endpoint. The dual weights enter because $\lambda_j^* = -\sigma_j(\lambda_j)$.

The key fact is that the linear mapping σ_i preserves the set of paths $\mathcal{B}(\lambda_j^*)$ for all i, j . This is obvious if $V(\lambda_j^*)$ is a minuscule representation, but the general case requires some work using results of Stembridge [Ste]. \square

Theorem 3 now follows immediately. Indeed, $s_i \Lambda_0 = \Lambda_0$ for $i = 1, \dots, r$, so $f_i(\Lambda_0 \star \pi') = \Lambda_0 \star f_i(\pi')$ for any path π' . Thus the right-hand side of the equation in the Theorem is isomorphic as a \mathfrak{g} -crystal to $\mathcal{B}(\lambda_1^*) \star \cdots \star \mathcal{B}(\lambda_m^*)$, which models $V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_r)^*$. See [GM] for methods of enumerating the paths in this crystal (and hence computing the dimension of the corresponding representation).

Theorem 4 follows as a corollary. We describe the crystal graph of the semi-infinite tensor product by the appropriate skein-crystal. We thus recover the Kyoto path model for classical \mathfrak{g} , and our results are equally valid for E_6, E_7 .

THEOREM 4.2. *Let ϖ^\vee, N be as in Theorem 4. Define the m -fold concatenation $\mathcal{B}_m = \mathcal{B}(\varpi^*) \star \cdots \star \mathcal{B}(\varpi^*)$. Then $\Lambda_0 \star \mathcal{B}_N$ contains a unique $\widehat{\mathfrak{g}}$ -dominant path $\Lambda_0 \star \pi_N$.*

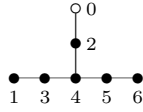
Define the skein $\pi := (\cdots \star \pi_N \star \pi_N \star \pi_N \vdash \Lambda_0)$, which satisfies $\pi \star \pi_N = \pi$. Then the $\widehat{\mathfrak{g}}$ -crystal of $\widehat{V}(\Lambda_0)$ is given by the skein-crystal:

$$\widehat{\mathcal{B}}(\pi) = \bigcup_{m \geq 1} \pi \star \mathcal{B}_m.$$

That is, $\widehat{\mathcal{B}}(\pi)$ is the set of all semi-infinite paths which are equal to π except for a finite length near the end, and all of whose vector steps lie in $\mathcal{B}(\varpi^*)$.

5. Example: E_6

Referring to Bourbaki [**Bour**], we write the extended Dynkin diagram $\widehat{X} = \widehat{E}_6$:



The simple roots are defined inside \mathbb{R}^6 with standard basis $\epsilon_1, \dots, \epsilon_6$. (Our ϵ_6 is $\frac{1}{\sqrt{3}}(-\epsilon_6 - \epsilon_7 + \epsilon_8)$ in Bourbaki's notation.) They are:

$$\begin{aligned} \alpha_1 &= \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{2}\epsilon_6, & \alpha_2 &= \epsilon_1 + \epsilon_2, \\ \alpha_3 &= \epsilon_2 - \epsilon_1, & \alpha_4 &= \epsilon_3 - \epsilon_2, & \alpha_5 &= \epsilon_4 - \epsilon_3, & \alpha_6 &= \epsilon_5 - \epsilon_4. \end{aligned}$$

Since E_6 is simply laced, the coroots and coweights may be identified with the roots and weights, with the natural pairing given by the standard dot product on \mathbb{R}^6 .

We focus on the minuscule coweight ϖ_1^\vee corresponding to the diagram automorphism σ with $\sigma(1) = 0$ and $\sigma(0) = 6$. In this case, the corresponding fundamental representation $V(\varpi_1)$ is also minuscule, meaning that all of its weights are extremal weights $\lambda \in W(E_6) \cdot \varpi_1$. The roots $\alpha_2, \dots, \alpha_6$ generate the root subsystem $D_5 \subset E_6$, and the reflection subgroup $W(D_5) = \text{Stab}_{W(E_6)}(\varpi_1)$ acts by permuting $\epsilon_1, \dots, \epsilon_5$ (the subgroup $W(A_4) = S_5$) and by changing an even number of signs $\pm\epsilon_1, \dots, \pm\epsilon_5$. We have $\dim V(\varpi_1) = |W(E_6)/W(D_5)| = 27$. The weights are:

$$\begin{aligned} \varpi_1 &= \frac{2\sqrt{3}}{3}\epsilon_6, \\ S_5 \cdot \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6}\epsilon_6, \\ S_5 \cdot \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6}\epsilon_6, \\ -\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6}\epsilon_6, \\ \pm S_5 \cdot \epsilon_1 - \frac{\sqrt{3}}{3}\epsilon_6. \end{aligned}$$

The lowest weight is $-\varpi_6 = -\epsilon_5 - \frac{\sqrt{3}}{3}\epsilon_6$, so that $V(\varpi_1)^* = V(\varpi_6)$ and $\varpi_1^* = \varpi_6$.

The simplest path crystal for $V(\varpi_1^*)$ is the set of 27 straight-line paths from 0 to the negatives of the above extremal weights:

$$\mathcal{B}(\varpi_1^*) = \{ (v) \mid v \in -W(E_6) \cdot \varpi_1 \}$$

We have $3\varpi_1^\vee \in \oplus_{i=1}^6 \mathbb{R}\alpha_i^\vee$ the coroot lattice, so that $N = 3$ in Theorem 4, and this N is also the order of the automorphism σ . The path crystal $\mathcal{B}_3 := \mathcal{B}(\varpi_1^*) \star \mathcal{B}(\varpi_1^*) \star \mathcal{B}(\varpi_1^*)$, the set of all 3-step walks with steps chosen from the 27 weights of $V(\varpi_1^*)$, is a model for $V(\varpi_1^*)^{\otimes 3}$.

By Theorem 5, $\Lambda_0 \star \mathcal{B}_3$ contains a unique $\widehat{\mathfrak{g}}$ -dominant path $\Lambda_0 \star \pi_3$, where

$$\pi_3 := (\varpi_6) \star (\varpi_1 - \varpi_6) \star (-\varpi_1).$$

In this case, π_3 has the even stronger property that it is the unique \mathfrak{g} -dominant path of weight 0, so that it corresponds to the one-dimensional space of \mathfrak{g} -invariant vectors in $V(\varpi_1^*)^{\otimes 3}$.

Now Theorem 5 states that the affine Demazure module $\widehat{V}_{3m\varpi_1^*}(\Lambda_0)$ is modelled by the $\widehat{\mathfrak{g}}$ -path crystal:

$$\mathcal{B}_{3m} = \{(\Lambda_0 \star v_1 \star \cdots \star v_{3m}) \mid v_j \in -W(E_6) \cdot \varpi_1\},$$

the set of all $3m$ -step walks in $\Lambda_0 \oplus \mathbb{R}^6$ starting at Λ_0 , with steps chosen from the 27 weights of $V(\varpi_1^*)$. This path crystal is generated from its unique $\widehat{\mathfrak{g}}$ -dominant path $\Lambda_0 \star \pi_3 \star \cdots \star \pi_3$. Considering it as a \mathfrak{g} -crystal, we have $\mathcal{B}_{3m} \cong \mathcal{B}_3^{*m}$, which shows that $\widehat{V}_{3m\varpi_1^*}(\Lambda_0) \cong V(\varpi_1^*)^{\otimes 3m}$ as \mathfrak{g} -modules.

By Theorem 6, the $\widehat{\mathfrak{g}}$ -crystal of the basic $\widehat{\mathfrak{g}}$ -module $\widehat{V}(\Lambda_0)$ is given by the set of all infinite walks (skeins) of the form:

$$\pi = \Lambda_0 \star \underbrace{\pi_3 \star \cdots \star \pi_3}_{\text{infinite}} \star v_1 \star \cdots \star v_{3m},$$

with $m > 0$ and $v_j \in -W(E_6) \cdot \varpi_1$. The endpoint of such a skein is $\text{wt}(\pi) := \Lambda_0 + v_1 + \cdots + v_{3m}$. The crystal operators f_i are defined just as for finite paths. Acting near the end of the skein, they unwind the coils π_3 one at a time, right-to-left. As a \mathfrak{g} -module, $\widehat{V}(\Lambda_0)$ is an infinite tensor power of $V(\varpi_1^*)$.

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